Lecture 006 (November 9, 2009)

11 Cohomology

Suppose that C is a chain complex of R-modules.

1) There is an isomorphism

$$\hom(R[-n], C) \xrightarrow{\cong} Z_n(C)$$

which is defined by the assigment $f \mapsto f(1)$, where $1 \in R = R[-n]_n$ is the multiplicative identity of the ring R.

2) A chain homotopy between chain morphisms $f, g : R[-n] \to C$ can be identified with a R-module homomorphisms $\sigma : R \to C_{n+1}$ such that $\partial\sigma = f - g$. Equivalently, $f(1) - g(1) = \partial\sigma(1)$ for some element $\sigma(1) \in C_{n+1}$.

It follows that there is a commutative diagram of abelian group homomorphisms

$$\begin{tikzcd} \hom(R[-n], C) \arrow[r, "\cong"] \arrow[d] & Z_n(C) \arrow[d] \\ \pi(R[-n], C) \arrow[r, "\cong"'] & H_n(C) \end{tikzcd}$$

which is natural in chain complexes C. Here,

$$\pi(R[-n], C)$$

denotes chain homotopy classes of maps $R[-n] \to C$. Recall that the group $\pi(R[-n], C)$ can be identified with the group $[R[-n], C]$ of morphisms from $R[-n]$ to C in the derived category $\mathrm{Ho}(Ch_+(R))$.

We have therefore proved the following

Lemma 11.1. *The assignment* $[f] \mapsto [f(1)]$ *defines a natural isomorphism*

$$[R[-n], C] = \pi(R[-n], C) \xrightarrow{\cong} H_n(C).$$

Cohomology can also be defined in the derived category. Suppose that C is a complex of R-modules and that A is an R-module. The *cohomology group* $H^m(C, A)$ is defined by

$$H^m(C, A) := [C, A[-m]].$$

A *cochain complex* E of R-modules is a chain complex which is concentrated in negative degrees. In other words, $E_p = 0$ for $p > 0$. One often (in fact, almost universally) writes $E^n = E_{-n}$ for $n \geq 0$.

Example: If C is an ordinary chain complex and A is a module, then $\hom(C, A)$ is the cochain complex with

$$\hom(C, A)^n = \hom(C, A)_{-n} = \hom(C_n, A)$$

for $n \geq 0$.

It is standard to write

$$H^n \hom(C, A) := H_{-n} \hom(C, A)$$

for $n \geq 0$.

Lemma 11.2. *Suppose that C is an ordinary chain complex and that A is a module. Then there is an isomorphism*

$$H^n \hom(C, A) = H_{-n} \hom(C, A) \cong \pi(C, A[-n]).$$

Proof. A cycle f of $\hom(C, A)$ can be identified with a chain map $f : C \to A[-n]$, and the cycle f is a boundary if and only if the chain map f is chain homotopic to 0. $\square$

Corollary 11.3. *Suppose that C is an ordinary complex and that A is a module. Choose a cofibrant replacement $\epsilon : P \xrightarrow{\simeq} C$. Then there is an isomorphism*

$$H^n(C, A) \cong \pi(P, A[-n]) = H^n \hom(P, A).$$

Remark 11.4. The choice of cofibrant replacement $P \to C$ in Corollary 11.3 does not matter, and can be made functorial, all by a familiar argument.

In effect, choose a functorial fibrant replacement $\pi : P_C \to C$ as in Lecture 04. Then the lifting θ exists in the diagram

$$\begin{tikzcd} & P_C \arrow[d, "\pi"] \\ P \arrow[ur, dotted, "\theta"] \arrow[r, "\epsilon"'] & C \end{tikzcd}$$

since π is a trivial fibration and P is cofibrant. The map θ is a weak equivalence between cofibrant (and fibrant) objects is a weak equivalence, and is necessarily a chain homotopy equivalence by the Whitehead Theorem (Theorem 7.13). Finally, the functor $\hom(\ ,A)$ preserves chain homotopy equivalences (exercise).

I could have written

$$\mathrm{Ext}^n(C,A) = H^n(C,A) = [C, A[-n]]$$

and called these cohomology groups Ext *groups*, although these groups are really hyper Ext invariants.

If B is a module, it is standard to write

$$\mathrm{Ext}^n(B,A) = H^n(B[0],A) = [B[0], A[-n]],$$

and call these the Ext-groups of B with coefficients in A. The terminology will be explained later.

The Ext groups are the derived functors of homomorphisms:

Lemma 11.5. *1) Suppose that P is a projective module. Then*

$$\mathrm{Ext}^n(P, A) = 0$$

for $n > 0$.

2) Suppose that C is a chain complex and that A is a module. Then there is a natural isomorphism

$$H^0 \hom(C, A) \cong \hom(H_0(C), A).$$

Proof. For statement 1), $P[0] \to P$ is a projective resolution of P, and $P[0]_k = 0$ for $k > 0$. It follows that

$$\hom(P[0], A)_k = 0$$

for $k < 0$, so that

$$H^k \hom(P[0], A) = 0$$

in the same range.

For statement 2), choose a cofibrant replacement $P \xrightarrow{\simeq} C$ for C. The sequence

$$P_1 \xrightarrow{\partial} P_0 \to H_0(C) \to 0$$

so that the sequence

$$0 \to \hom(H_0(C), A) \to \hom(P_0, A) \xrightarrow{\partial^*} \hom(P_1, A)$$

is exact, since $\hom(\ , A)$ is right exact (exercise).

$\square$

Lemma 11.6. *1) Suppose that*

$$0 \to F \xrightarrow{j} E \xrightarrow{p} B \to 0$$

is a short exact sequence of chain complexes and that A is some module. Then there is a long exact sequence

$$\begin{aligned} 0 \to H^0(B, A) &\xrightarrow{p^*} H^0(E, A) \xrightarrow{j^*} H^0(F, A) \\ \xrightarrow{\partial} H^1(B, A) &\xrightarrow{p^*} H^1(E, A) \to \dots \end{aligned}$$

2) Suppose that

$$0 \to A \to A' \to A'' \to 0$$

is a short exact sequence of modules and that C is a chain complex. Then there is a long exact sequence

$$\begin{aligned} 0 \to H^0(C, A) &\to H^0(C, A') \to H^0(C, A'') \\ \xrightarrow{\partial} H^1(C, A) &\to H^1(C, A') \to \dots \end{aligned}$$

Proof. For statement 1), use the method of proof of Lemma 9.10 to find a weakly equivalent short exact sequence

$$0 \to F' \to P_{E'} \to P_B \to 0$$

with all objects cofibrant. Then the sequence is split exact in all degrees since all modules P_{Bn} are projective, so that the sequence of cochain complexes

$$0 \to \hom(P_B, A) \to \hom(P_{E'}, A) \to \hom(F', A) \to 0$$

is short exact. The desired long exact sequence is the long exact sequence in homology which is associated to this sequence.

For statement 2), the functor $A \mapsto \hom(P, A)$ is exact if P is projective (exercise). Thus, if $P \xrightarrow{\simeq} C$ is a cofibrant model for C, then the sequence of cochain complexes

$$0 \to \hom(P, A) \to \hom(P, A') \to \hom(P, A'') \to 0$$

is exact. The resulting long exact sequence in homology is the one we want. □

Corollary 11.7. *1) Suppose that B and A are modules. Then there is a natural isomorphism*

$$\mathrm{Ext}^0(B, A) \cong \hom(B, A).$$

2) Suppose that

$$0 \to B \xrightarrow{j} B' \xrightarrow{p} B'' \to 0$$

is a short exact sequence of modules and that A is a. Then there is a long exact sequence

$$0 \to \mathrm{Ext}^0(B'', A) \xrightarrow{p^*} \mathrm{Ext}^0(B', A) \xrightarrow{j^*} \mathrm{Ext}^0(B, A)$$
$$\xrightarrow{\partial} \mathrm{Ext}^1(B'', A) \xrightarrow{p^*} \mathrm{Ext}^1(B', A) \to \dots$$

3) Suppose that

$$0 \to A \to A' \to A'' \to 0$$

is a short exact sequence of modules and that B is a module. Then there is a long exact sequence

$$0 \to \mathrm{Ext}^0(B, A) \to \mathrm{Ext}^0(B, A') \to \mathrm{Ext}^0(B, A'')$$
$$\xrightarrow{\partial} \mathrm{Ext}^1(B, A) \to \mathrm{Ext}^1(B, A') \to \dots$$

Examples:

1) Suppose that A is an abelian group, and let

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n \to 0$$

be the standard free resolution of $\mathbb{Z}/n$. The groups $\mathrm{Ext}^*(\mathbb{Z}/n, A)$ are the homology groups of the cochain complex

$$\hom(\mathbb{Z}, A) \xrightarrow{(\times n)^*} \hom(\mathbb{Z}, A) \to 0 \to \dots$$

which complex is isomorphic to

$$A \xrightarrow{\times n} A \to 0 \to \dots$$

It follows that

$$\begin{aligned}
&\mathrm{Ext}^0(\mathbb{Z}/n, A) \cong {}_nA = \mathrm{Tor}(\mathbb{Z}/n, A),\\
&\mathrm{Ext}^1(\mathbb{Z}/n, A) \cong A/nA = A \otimes \mathbb{Z}/n,\\
&\mathrm{Ext}^k(\mathbb{Z}/n, A) = 0 \text{ if } k > 1.
\end{aligned}$$

2) Exercise: Show that $\mathrm{Ext}^k(B, A) = 0$ for $k > 1$ for all abelian groups A and B.

3) Suppose that $\mathbb{Z}$ is the trivial module over the cyclic group $\mathbb{Z}/m$, and suppose that A is an abelian group, again with the trivial $\mathbb{Z}/m$-module structure.

The sequence

$$\hom(\mathbb{Z}(\mathbb{Z}/m), A) \xrightarrow{(1-t)^*} \hom(\mathbb{Z}(\mathbb{Z}/m), A) \xrightarrow{N^*} \hom(\mathbb{Z}(\mathbb{Z}/m), A)$$

is isomorphic to the sequence

$$A \xrightarrow{0} A \xrightarrow{\times m} A$$

It follows that

$$\begin{aligned}
&\mathrm{Ext}^0(\mathbb{Z}, A) \cong A\\
&\mathrm{Ext}^{2k+1}(\mathbb{Z}, A) \cong \mathrm{Tor}(\mathbb{Z}/m, A) \text{ if } k > 0,\\
&\mathrm{Ext}^{2k}(\mathbb{Z}, A) \cong A \otimes \mathbb{Z}/m \text{ if } k > 0.
\end{aligned}$$

in the category of $\mathbb{Z}/m$-modules.

12 Injective resolutions

An *injective* R-module I is a module for which the functor $\hom(\ ,I)$ is exact. This means that, given an exact sequence

$$M \xrightarrow{g} M' \xrightarrow{f} M''$$

the sequence

$$\hom(M'', I) \xrightarrow{f^*} \hom(M', I) \xrightarrow{g^*} \hom(M'', I)$$

should be exact.

Equivalently (exercise), I is injective if and only if the dotted extension exists in every diagram of R-module homomorphisms

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & I \\ B \arrow[ur, dotted] & \end{tikzcd}$$

where i is a monomorphism.

The following result is a special case of a theorem of Grothendieck which first appeared in [2], and is proved in many places:

Theorem 12.1. *The category of R-modules has enough injectives: for every R-module A there is monomorphism $i : A \to I$ such that I is injective.*

The quickest proof of Theorem 12.1 (as stated) is an abstraction of a step of the proof in Lemma 12.3 below.

Example 12.2. Every vector space over a field k is an injective module over k, because all monomorphisms of k-vector spaces split.

Lemma 12.3. *An abelian group A is injective (ie. an injective $\mathbb{Z}$-module) if and only if A is n-divisible for all n.*

The group A is *n-divisible* if and only if for all $a \in A$ there is a $b \in A$ such that $n \cdot b = a$ (or, multiplicatively, every element has an n^{th} root).

Proof. An abelian group A is injective if and only if the morphism $A \to 0$ has the right lifting property with respect to all monomorphisms $B \subset B'$ of abelian groups. In particular, if A is injective, then $A \to 0$ has the right lifting property with respect to the multiplication by n map $\times n : \mathbb{Z} \to \mathbb{Z}$, so A is n-divisible, for all n.

Conversely, suppose that A is n-divisible for all n, and consider the poset of partial lifts

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d] & A \arrow[dd] \\ C \arrow[ur, "\theta"'] \arrow[d] & \\ B' \arrow[r] & 0 \end{tikzcd}$$

of a morphism f, where the morphisms

$$B \to C \to B'$$

are injective with a fixed composition $i : B \to B'$. This poset has maximal elements, by Zorn's Lemma.

Suppose that $x \in B' - C$, and let $C' = \langle C, x \rangle$ be the subgroup of B' which is generated by C and x. Let $x : \mathbb{Z} \to B'$ be the homomorphism which sends 1 to x. Then there is a pushout

$$\begin{tikzcd} x^{-1}(C) \arrow[r] \arrow[d, "j"'] & C \arrow[d] \\ \mathbb{Z} \arrow[r, "x"'] & C' \end{tikzcd}$$

where j is a monomorphism, and the subgroup $x^{-1}(C)$ is either 0 or of the form $n\mathbb{Z}$ for some n. In either case, there is a lift $\theta' : C' \to A$ which extends θ, since A is n-divisible for all n, and so θ is not maximal. It follows that all maximal lifts extend the homomorphism f to all of B'. $\square$

Example 12.4. Every $\mathbb{Q}$-vector space is an injective abelian group. Lemma 12.3 also tells you how to make an injective module containing an abelian group A: just put in all the n^{th} roots for all n with something like a small object argument. This is also how the proof of Theorem 12.1 works.

Maps into injectives play the same role in homological algebra as do maps out of projectives.

Corollary 12.5. *Every module A has an injective resolution, meaning an exact sequence*

$$0 \to A \xrightarrow{\sigma} I_0 \to I_{-1} \to I_{-2} \to \dots$$

for which all modules I_p are injective.

I like to write injective resolutions as maps of unbounded chain complexes $\sigma : A[0] \to I$, where I is a complex of injectives such that $I_p = 0$ for $p > 0$, and σ is a weak equivalence.

Lemma 12.6. *Suppose that $f : C \to D$ is a weak equivalence of ordinary chain complexes and that I is an injective module. Then the induced map*

$$f^* : \hom(D, I) \to \hom(C, I)$$

is a weak equivalence of cochain complexes.

Proof. If K is an acyclic complex, then the sequence

$$\ldots \xrightarrow{\partial} K_2 \xrightarrow{\partial} K_1 \xrightarrow{\partial} K_0 \to 0$$

is exact, so that the sequence

$$0 \to \hom(K_0, I) \xrightarrow{\partial^*} \hom(K_1, I) \xrightarrow{\partial^*} \ldots$$

is exact. The cochain complex $\hom(K, I)$ is therefore acyclic.

Suppose that $p : E \to D$ is a trivial fibration with acyclic kernel K. The sequence of cochain complexes

$$0 \to \hom(D, I) \xrightarrow{p^*} \hom(E, I) \to \hom(K, I) \to 0$$

is exact and so $\hom(K, I)$ is acyclic, so that p^* is a homology isomorphism.

Similarly, if $i : C \to E$ is a trivial cofibration with acyclic cokernel K', then $\hom(K', I)$ is acyclic, so that i^* is a homology isomorphism.

Every weak equivalence f has a factorization $f = p \cdot i$ where p is a trivial fibration and i is a trivial cofibration, so f^* is a homology isomorphism. $\square$

Suppose that C is an ordinary chain complex and that K is a cochain complex. There is a (third quadrant) bicomplex $\hom(C, K)$ with

$$\hom(C, K)_{p,q} = \hom(C_{-p}, K_q).$$

There is a cochain complex $\mathrm{Tot}(\hom(C, K))$ associated to this bicomplex, with

$$\mathrm{Tot}(\hom(C, K))_n = \bigoplus_{p+q=n} \hom(C_{-p}, K_q),$$

and with

$$\partial(f) = \partial^*(f) + (-1)^{-p}\partial_* f = f \cdot \partial_C + (-1)^p \partial_K \cdot f$$

for $f \in \hom(C_{-p}, K_q)$.

The following result is a generalization of Lemma 11.2 (see also [3]):

Lemma 12.7. *There is an isomorphism*

$$H_{-n} \, \mathrm{Tot}(\hom(C, K)) \cong \pi(C, K[-n]).$$

This isomorphism is natural in chain complexes C and cochain complexes K.

Proof. The group $\mathrm{Tot}(\hom(C, K))_n$ is the direct sum

$$\hom(C_n, K_0) \oplus \hom(C_{n-1}, K_{-1}) \oplus \cdots \oplus \hom(C_0, K_{-n}).$$

Write $(f_n, f_{n-1}, \dots, f_0)$ for an element of this group. Then

$$\begin{aligned} \partial(f_n, &f_{n-1}, \dots, f_0) \\ &= (f_n \partial_C, f_{n-1}\partial_C + (-1)^n \partial_K f_n, \dots, (-1)^0 \partial_K f_0). \end{aligned}$$

In particular the j-term in the $(n+2)$-tuple $\partial(f_n, \ldots, f_0)$ is

$$f_{j-1}\partial_C + (-1)^j \partial_K f_j$$

for $0 < j < n+1$.

Write

$$\tilde{f}_j = (-1)^{\sum_{i=o}^{j-1} i} f_j.$$

Then $\partial(f_n, f_{n-1}, \ldots, f_0) = 0$ if and only if the maps $\tilde{f}_j$ define a chain map $\tilde{f} : C \to K[-n]$. In effect,

$$(-1)^{\sum_{i=0}^{j-2} i} f_{j-1}\partial_C + (-1)^{\sum_{i=0}^{j-2} i}(-1)^j \partial_K f_j = \tilde{f}_{j-1}\partial_C - \partial_K \tilde{f}_j.$$

Suppose that

$$\partial(s_{n-1}, \ldots, s_0) = (f_n, \ldots, f_0),$$

and write

$$\tilde{s}_j = (-1)^{\sum_{i=o}^{j} i} s_j.$$

Then

$$\begin{aligned} \tilde{f}_j &= (-1)^{\sum_{i=0}^{j-1} i} f_j \\ &= (-1)^{\sum_{i=0}^{j-1} i} s_{j-1}\partial_C + (-1)^{\sum_{i=0}^{j-1} i}(-1)^j \partial_K s_j \\ &= \tilde{s}_{j-1}\partial_C + \partial_K \tilde{s}_j \end{aligned}$$

for all j, so the chain map defined by the maps $\tilde{f}_k$ is chain homotopic to 0.

The maps f_j and s_j can be recovered from $\tilde{f}_j$ and $\tilde{s}_j$, respectively, by multiplying by the obvious powers of -1. $\square$

Proposition 12.8. *Suppose that J is a cochain complex of injective modules, and that $f : C \to D$ is a weak equivalence of ordinary chain complexes. Then the induced map*

$$\pi(D, J[-n]) \xrightarrow{f^*} \pi(C, J[-n])$$

is an isomorphism for all $n \geq 0$.

Proof. Suppose that J has one non-trivial entry, so that $J = I[n]$, meaning that J consists of the injective module I concentrated in degree $-n$. Then there is a natural isomorphism

$$\mathrm{Tot}(\hom(C, I[n])) \cong \hom(C, I)[n]$$

(exercise). The induced map

$$f^* : \hom(D, I) \to \hom(C, I)$$

is a weak equivalence by Lemma 12.6, and so the shifted map

$$f^* : \hom(D, I)[n] \to \hom(C, I)[n]$$

is a weak equivalence.

Suppose that J has only finitely many non-zero modules. Suppose that k is the maximal degree such that $J_k \neq 0$. Then there is a short split exact sequence

$$0 \to F_{k-1}J \to J \to J_k[k] \to 0.$$

which sequence induces a natural short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tot}(\hom(C, F_{k-1}J)) \arrow[r] & \mathrm{Tot}(\hom(C, J)) \arrow[d] & \\ & & \mathrm{Tot}(\hom(C, J_k[k])) \arrow[r] & 0. \end{tikzcd}$$

Then $F_{k-1}J$ is a cochain complex of injectives with fewer non-zero modules, and we assume inductively that the map

$$f^* : \mathrm{Tot}(\hom(D, F_{k-1}J)) \to \mathrm{Tot}(\hom(C, F_{k-1}J))$$

is a weak equivalence. It follows from the argument above and a comparison of long exact sequences that the map

$$f^* : \mathrm{Tot}(\hom(D, J)) \to \mathrm{Tot}(\hom(C, J))$$

is a weak equivalence.

In general, for $k < 0$,

$$\mathrm{Tot}(\hom(C, F_k J))_p = 0$$

for $p > k$, and so

$$H_p(\mathrm{Tot}(\hom(C, F_k J))) = 0$$

for $p > k$. It follows that the map

$$H_p(\mathrm{Tot}(\hom(C, J))) \to H_p(\mathrm{Tot}(\hom(C, J/F_k J)))$$

is an isomorphism for $p > k+1$.

The complex $J/F_k J$ of injectives has only finitely many non-zero modules. Thus, given $n \leq 0$, choose k such that $n > k+1$. Then there is a commutative diagram

$$\begin{tikzcd}
H_n(\mathrm{Tot}(\hom(D, J))) \arrow[r, "\cong"] \arrow[d, "f^*"'] & H_n(\mathrm{Tot}(\hom(D, J/F_k J))) \arrow[d, "\cong"', "f^*"] \\
H_n(\mathrm{Tot}(\hom(C, J))) \arrow[r, "\cong"'] & H_n(\mathrm{Tot}(\hom(C, J/F_k J)))
\end{tikzcd}$$

Then it follows that f induces an isomorphism

$$f^* : H_n(\mathrm{Tot}(\hom(D, J))) \xrightarrow{\cong} H_n(\mathrm{Tot}(\hom(C, J))).$$

$\square$

One can form the Postnikov section $P_{-1}D$ for an unbounded chain complex D just as for chain complexes:

$$P_{-1}(D)_p = \begin{cases} 0 & \text{if } p > 0, \\ B_0 D & \text{if } p = 0, \\ D_p & \text{if } p < 0. \end{cases}$$

and there is a short exact sequence

$$0 \to f_{-1}D \to D \to P_{-1}D \to 0.$$

Note that

$$f_{-1}(D)_p = \begin{cases} D_p & \text{if } p > 0, \\ Z_0 D & \text{if } p = 0 \\ 0 & \text{if } p < 0. \end{cases}$$

Then $f_{-1}D$ is an ordinary chain complex, and the functor $D \mapsto f_{-1}D$ is right adjoint to the inclusion functor $\mathrm{Ch}_+(R) \subset \mathrm{Ch}(R)$. This functor preserves weak equivalences, and there is an isomorphism

$$\pi(C, D) \cong \pi(C, f_{-1}D),$$

relating chain homotopy classes in the unbounded category to chain homotopy classes in ordinary chain complexes.

Corollary 12.9. *Suppose that C is a chain complex and that A is a module. Choose an injective resolution $A[0] \to I$ of A. Then there is a natural isomorphism*

$$[C, A[-n]] = H^n(C, A) \cong \pi(C, I[-n]).$$

Proof. Choose a cofibrant model $\epsilon : P \xrightarrow{\simeq} C$ for C. The injective resolution $A[0] \to I$ induces a weak equivalence

$$A[-n] \to f_{-1}I[-n]$$

and so there is are isomorphisms

$$\pi(P, A[-n]) \xrightarrow{\cong} \pi(P, f_{-1}I[-n]) \cong \pi(P, I[-n]) \xleftarrow{\cong} \pi(C, I[-n])$$

for all n, by the Whitehead Theorem and Proposition 12.8. $\square$

Corollary 12.10. *Suppose that A and B are modules and that $B[0] \to I$ is an injective resolution of B. Then there are natural isomorphisms*

$$\operatorname{Ext}^n(A, B) \cong \pi(A[0], I[-n]) \cong H^n \hom(A, I).$$

13 Eilenberg-Cartan resolutions

The following is a formalization of techniques that we've already seen:

Lemma 13.1. *Suppose that $f : C \to D$ is a map of first (respectively third) quadrant bicomplexes such that the chain complex maps $f : C_{p,*} \to D_{p,*}$ are weak equivalences for all p. Then the chain complex map $f_* : \operatorname{Tot}(C) \to \operatorname{Tot}(D)$ is a weak equivalence.*

Proof. There is a natural filtration F_nC of a bicomplex C with

$$F_nC_{p,q} = \begin{cases} C_{p,q} & \text{if } p \le n, \\ 0 & \text{if } p > n \end{cases}$$

In the short exact sequence of bicomplexes

$$0 \to F_{n-1}C \to F_nC \to F_nC/F_{n-1}C \to 0$$

the quotient $F_nC/F_{n-1}C$ consists of a copy of $C_{n,*}$ in horizontal degree n, and is 0 in other horizontal degrees. It follows that there is an isomorphism

$$\mathrm{Tot}(F_nC/F_{n-1}C) \cong C_{n,*}[-n]$$

It follows from the assumptions of the Lemma that all induced maps

$$f_* : \mathrm{Tot}(F_nC/F_{n-1}C) \to \mathrm{Tot}(F_nD/F_{n-1}D)$$

are weak equivalences.

If C and D are first quadrant bicomplexes, then by increasing induction on n, all maps

$$f_* : \mathrm{Tot}(F_nC) \to \mathrm{Tot}(F_nD)$$

are weak equivalences. There is a functorial isomorphism

$$H_k(\mathrm{Tot}(C)) \cong H_k(\mathrm{Tot}_n(C)) \tag{1}$$

for n sufficiently large, for all k, so that

$$f_* : \mathrm{Tot}(C) \to \mathrm{Tot}(D)$$

is a homology isomorphism.

If C and D are third quadrant bicomplexes, then by decreasing induction on n all maps

$$f_* : \mathrm{Tot}(C/F_nC) \to \mathrm{Tot}(D/F_nD)$$

are weak equivalences. There is a functorial isomorphism

$$H_k(\mathrm{Tot}(C)) \cong H_k(\mathrm{Tot}(C/F_nC)) \qquad (2)$$

for n sufficiently small, for all k, so that

$$f_* : \mathrm{Tot}(C) \to \mathrm{Tot}(D)$$

is a homology isomorphism. $\square$

Remark 13.2. The isomorphisms (1) and (2) are *convergence isomorphisms* for the standard filtration, for first and third quadrant bicomplexes respectively.

Lemma 13.3. *Suppose that the sequence of module homomorphisms*

$$0 \to A \to B \to C \to 0 \qquad (3)$$

is exact, and suppose given monomorphisms $i_A : A \to I(A)$ and $i_C : C \to I(C)$ with $I(A)$ and

$I(C)$ injective. Then there is a comparison of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "j"] \arrow[d, "i_A"] & B \arrow[r, "p"] \arrow[d, "i_B"] & C \arrow[r] \arrow[d, "i_C"] & 0 \\ 0 \arrow[r] & I(A) \arrow[r] & I(B) \arrow[r] & I(C) \arrow[r] & 0 \end{tikzcd} \tag{4}$$

in which the module $I(B)$ is injective and i_B is a monomorphism.

Proof. Choose a map $\sigma : B \to I(A)$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i_A"'] & B \arrow[dl, "\sigma"] \\ I(A) \end{tikzcd}$$

commutes. Form the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "j"] \arrow[d, "i_A"] & B \arrow[r, "p"] \arrow[d, "(\sigma, i_C \cdot p)"] & C \arrow[r] \arrow[d, "i_C"] & 0 \\ 0 \arrow[r] & I(A) \arrow[r] & I(A) \oplus I(C) \arrow[r] & I(C) \arrow[r] & 0 \end{tikzcd}$$

and set $I(B) = I(A) \oplus I(C)$. The map $i_B = (\sigma, I_C \cdot p)$ is a monomorphism by the Snake Lemma (Lemma 1.2). $\square$

Corollary 13.4. *Suppose given a short exact sequence*

$$0 \to A \to B \to C \to 0$$

of module homomorphisms and injective resolutions $\eta_A : A \to I(A)$ and $\eta_C : C \to I(C)$ of A and C respectively. Then there is a comparison of short exact sequences of cochain complexes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "j"] \arrow[d, "\eta_A"] & B \arrow[r, "p"] \arrow[d, "\eta"] & C \arrow[r] \arrow[d, "\eta_C"] & 0 \\
0 \arrow[r] & I(A) \arrow[r] & I(B) \arrow[r] & I(C) \arrow[r] & 0
\end{tikzcd}$$

in which the map $\eta : B \to I(B)$ is an injective resolution of B.

Proof. Form the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "j"] \arrow[d, "\eta_A"] & B \arrow[r, "p"] \arrow[d, "\eta"] & C \arrow[r] \arrow[d, "\eta_C"] & 0 \\
0 \arrow[r] & I_0(A) \arrow[r] & I_0(B) \arrow[r] & I_0(C) \arrow[r] & 0
\end{tikzcd}$$

as in Lemma 13.3, and let the exact sequence

$$0 \to Z_{-1}(A) \to B' \to Z_{-1} \to 0$$

be the cokernel of the vertical map. Then by Lemma 13.3 there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I_0(A) \arrow[r] \arrow[d, "p"'] & I_0(B) \arrow[r] \arrow[d, "p"] & I_0(C) \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & Z_{-1}(A) \arrow[r] \arrow[d, "i"'] & B' \arrow[r] \arrow[d, "i"] & Z_{-1}(C) \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & I_{-1}(A) \arrow[r] & I_{-1}(B) \arrow[r] & I_{-1}(C) \arrow[r] & 0
\end{tikzcd}$$

in which all vertical maps p are epimorphisms, all maps i are monomorphisms, and the composites $ip : I_0(A) \to I_{-1}(A)$ and $ip : I_0(C) \to I_{-1}(C)$ coincide with the boundary homomorphisms for the chosen injective resolutions of A and C respectively.

Construct the resolution $\eta : B \to I(B)$ inductively. The composite $ip : I_0(B) \to I_{-1}(B)$ is the first differential. $\square$

Lemma 13.5. *Suppose that A is a chain complex (unbounded). Then there is a bicomplex morphism $\eta : A_p \to I(A)_{p,q}$ such that all induced cochain complex maps*

$$\begin{aligned}
&A_p \to I(A)_{p,*}\\
&Z_pA \to Z_pI(A)_{*,0} \to Z_pI(A)_{*,-1} \to \dots\\
&B_pA \to B_pI(A)_{*,0} \to B_pI(A)_{*,-1} \to \dots\\
&H_p(A) \to H_pI(A)_{*,0} \to H_pI(A)_{*,-1} \to \dots
\end{aligned}$$

are injective resolutions.

Proof. Write $B_p = B_p(A)$, $Z_p = Z_p(A)$ and $H_p = H_pA$ for the boundaries, cycles and homology groups of A, respectively.

Choose injective resolutions $\eta_{B_p} : B_p \to I(B_p)$ and $\eta_{H_p} : H_p \to I(H_p)$ for all k.

Use Corollary 13.4 to choose injective resolutions $\eta_{Z_p} : Z_p \to I(Z_p)$ such that the diagrams

$$\begin{tikzcd}
0 \arrow[r] & B_p \arrow[r, "j"] \arrow[d, "\eta_{B_p}"] & Z_p \arrow[r, "\pi"] \arrow[d, "\eta_{Z_p}"] & H_p \arrow[r] \arrow[d, "\eta_{H_p}"] & 0 \\
0 \arrow[r] & I(B_p) \arrow[r, "j_*"'] & I(Z_p) \arrow[r, "\pi_*"'] & I(H_p) \arrow[r] & 0
\end{tikzcd}$$

commute for all k. Use Corollary 13.4 again to choose injective resolutions $\eta_p : A_p \to I(A_p)$ so that the diagrams

$$\begin{tikzcd}
0 \arrow[r] & Z_p \arrow[r, "i"] \arrow[d, "\eta_{Z_p}"] & A_p \arrow[r, "q"] \arrow[d, "\eta_p"] & B_{p-1} \arrow[r] \arrow[d, "\eta_{B_{p-1}}"] & 0 \\
0 \arrow[r] & I(Z_p) \arrow[r, "i_*"'] & I(A_p) \arrow[r, "q_*"'] & I(B_{p-1}) \arrow[r] & 0
\end{tikzcd}$$

commute for all k.

The commutative diagrams

$$\begin{tikzcd}
A_p \arrow[r, "q"] \arrow[d, "\eta_p"'] & B_{p-1} \arrow[r, "j"] \arrow[d, "\eta_{B_{p-1}}"] & Z_{p-1} \arrow[r, "i"] \arrow[d, "\eta_{Z_{p-1}}"] & A_{p-1} \arrow[d, "\eta_{p-1}"] \\
I(A_p) \arrow[r, "q_*"'] & I(B_{p-1}) \arrow[r, "j_*"'] & I(Z_{p-1}) \arrow[r, "i_*"'] & I(A_{p-1})
\end{tikzcd} \tag{5}$$

imply that the lower composites define boundary maps

$$\partial = i_* j_* q_* : I(A_p) \to I(A_{p-1})$$

which give a bicomplex $I(A)$, and that the maps η_p define a bicomplex map $\eta : A \to I(A)$.

By construction, $Z_pI(A) = I(Z_p)$, $B_pI(A) = I(B_p)$ and $H_pI(A) = I(H_pA)$. $\square$

Remark 13.6. In the diagram (5) the chain maps q_* are epimorphisms and the maps i_* and j_* are monomorphisms. The short exact sequences

$$0 \to I(B_p)_q \xrightarrow{j_*} I(Z_p)_q \xrightarrow{\pi_*} I(H_p)_q \to 0$$
$$0 \to I(Z_p)_q \xrightarrow{i_*} I(A_p)_q \xrightarrow{q_*} I(B_{p-1})_q \to 0$$

are short exact sequences of injective modules, and are therefore split.

Say that a chain complex C is *split* if all exact sequences

$$0 \to B_pC \to Z_pC \to H_pC \to 0$$
$$0 \to Z_pC \to C_p \to B_{p-1}C \to 0$$

are split exact. All (vertical) chain complexes $I(A_*)_q$ in the bicomplex $I(A)$ of Lemma 13.5 are split.

Lemma 13.7. *Suppose that C is a split chain complex and that A is a module. Then*

1) the complex $C \otimes A$ is split and there is an isomorphism

$$H_n(C \otimes A) \cong H_n(C) \otimes A.$$

2) the cochain complex $\hom(C, A)$ *is split and there is an isomorphism*

$$H_{-n} \hom(C, A) \cong \hom(H_n C, A).$$

The proof is an exercise.

Remark 13.8. The bicomplex $I(A)$ of Lemma 13.5, is called an *Eilenberg-Cartan resolution* of the cochain complex A.

There is a subtle point at work here: we construct the map of bicomplexes $A \to I(A)$ for unbounded complexes A, but in that case the homotopy type of $\mathrm{Tot}(I(A))$ (which is defined on the chain level by infinite direct sums) is hard to analyse. The Eilenberg-Cartan construction is compatible with the Postnikov tower construction, which construction produces shifted cochain complexes, and there is a comparison of towers of fibrations

$$P_n A \to \mathrm{Tot}(I(P_n A))$$

which consists of weak equivalences. A homotopy inverse limit argument (eg. [1, VI.1]) shows that the induced map

$$A \cong \varprojlim_n P_n A \to \varprojlim_n \mathrm{Tot}(I(P_n A))$$

is a weak equivalence of unbounded complexes. The thing on the right is the "correct" Eilenberg-Cartan resolution for the unbounded complex A — it is defined on the chain level by infinite direct products.

Exercise 13.9. Show the projective resolution analogs of Lemma 13.3, Corollary 13.4 and Lemma 13.5:

1) Show that every short exact sequence has a projective cover, suitably defined.

2) Show that every short exact sequences of ordinary chain complexes has a projective resolution.

3) Show that for every chain complex A there is a bicomplex map $P(A) \to A$ with

$$P(A)_{p,q} = P(A_p)_q \to A_p$$

such that the maps

$$\begin{aligned}
&A_p \leftarrow P(A)_{p,*} \\
&Z_pA \leftarrow Z_pP(A)_{*,0} \leftarrow Z_pP(A)_{*,1} \leftarrow \dots \\
&B_pA \leftarrow B_pP(A)_{*,0} \leftarrow B_pP(A)_{*,1} \leftarrow \dots \\
&H_pA \leftarrow H_pP(A)_{*,0} \leftarrow H_pP(A)_{*,1} \leftarrow \dots
\end{aligned}$$

are projective resolutions.

Show that all chain complexes $P(A)_{*,q}$ are split in the sense described in Remark 13.6.

Corollary 13.10. *1) Eilenberg-Cartan resolutions $A \to I_*(A)$ and $P_*(C) \to C$ for cochain complexes A and chain complexes C induce weak equivalences*

$$A \to \mathrm{Tot}(I_*(A)) \text{ and } \mathrm{Tot}(P_*(C)) \to C,$$

respectively.

2) For every cochain complex A there is a weak equivalence $A \to J$ of cochain complexes such that all J_n are injective modules.

3) For every chain complex C there is a weak equivalence $P \to A$ such that all P_n are projective modules.

Of course, we've already seen a better proof of statement 3) — see Remark 4.11.

Corollary 13.11. *Suppose that C is an ordinary chain complex and A is a cochain complex. Choose a weak equivalence $A \to J$ such that J is a cochain complex of injective modules. Then there are isomorphisms*

$$[C, f_{-1}(A[-n])] \cong [C, A[-n]] \cong \pi(C, J[-n]).$$

relating morphisms in (respectively) the derived category of ordinary chain complexes, morphisms in the full derived category, and chain homotopy classes of unbounded chain complexes.

We can make the choice of weak equivalence $A \to J$ with all J^n injective by Corollary 13.11.

Proof. Choose a projective resolution $P \to C$. Then P is a cofibrant unbounded chain complex, by an adjunction argument (extending P to an unbounded complex by putting in zeros is left adjoint to the truncation $D \mapsto f_{-1}D$). There are isomorphisms

$$\begin{aligned}\pi(P, f_{-1}(A[-n])) &\cong \pi(P, f_{-1}(J[-n])) \\ &\cong \pi(P, J[-n]) \\ &\cong \pi(C, J[-n])\end{aligned}$$

since the truncation functor f_{-1} preserves weak equivalences, and by Proposition 12.8. □

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