Lecture 007 (November 11, 2009)

14 Spectral sequences: filtered chain complexes

The treatment of spectral sequences given here is a nuts and bolts approach (which means that we chase some elements), that essentially follows Mac Lane's *Homology* [1]. There are, of course, many books. The aim here is simply to get the basic calculational machines running.

Suppose that C is an ordinary chain complex. A *filtration* on C is a sequence of subcomplexes

$$F_0C \subset F_1(C) \subset \cdots \subset F_nC \subset \dots$$

of C such that

$$\cup_{n\geq 0} F_nC = \varinjlim_{n\geq 0} F_nC = C.$$

One often says that the structure consisting of a chain complex C together with a filtration F_nC is a *filtered complex*.

Example 14.1. We have already seen some standard examples. Suppose that E is a first quadrant bicomplex, and let F_nE be the sub-bicomplex with

$$F_nE_{p,q} = \begin{cases} E_{p,q} & \text{if } p \leq n, \\ 0 & \text{if } p > n. \end{cases}$$

Then the subobjects

$$F_n(\mathrm{Tot}(E)) = \mathrm{Tot}(F_n E)$$

define a filtration of $\mathrm{Tot}(E)$.

Generally, any filtration of a bicomplex E (suitably defined) determines a filtration of $\mathrm{Tot}(E)$. There are two canonical filtrations of a bicomplex E, namely the *horizontal filtration* displayed above and the *vertical filtration*, with

$$F'_n E_{p,q} = \begin{cases} E_{p,q} & \text{if } q \leq n, \\ 0 & \text{if } q > n. \end{cases}$$

The horizontal and vertical filtrations of a bicomplex are both of fundamental importance in applications.

Every filtration $F_n = F_n C$ on a chain complex C determines short exact sequences of chain complexes

$$0 \longrightarrow F_{n-1} \xrightarrow{i} F_n \xrightarrow{p} F_n/F_{n-1} \longrightarrow 0$$

with corresponding long exact sequences in homology groups

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_{p+q}F_{p-1} \arrow[d, "i_*"] & & \\
& H_{p+q}F_p \arrow[r, "p_*"] & H_{p+q}(F_p/F_{p-1}) \arrow[r, "\partial"] & H_{p+q-1}F_{p-1} \arrow[d, "i_*"] & \\
& & & H_{p+q-1}F_p \arrow[r, "p_*"] & \cdots
\end{tikzcd}$$

It is a key observation that these long exact sequences fit together to form a "ladder diagram"

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_{p+q}F_{p-2} \arrow[d, "i_*"] & & & \\
\cdots \arrow[r, "\partial"] & H_{p+q}F_{p-1} \arrow[r, "p_*"] \arrow[d, "i_*"] & H_{p+q}(F_{p-1}/F_{p-2}) \arrow[r, "\partial"] & H_{p+q-1}F_{p-2} \arrow[d, "i_*"] & \\
& H_{p+q}F_p \arrow[r, "p_*"] & H_{p+q}(F_p/F_{p-1}) \arrow[r, "\partial"] & H_{p+q-1}F_{p-1} \arrow[r, "p_*"] \arrow[d, "i_*"] & \cdots \\
& & & H_{p+q-1}F_p \arrow[r, "p_*"] & \cdots
\end{tikzcd}$$

The following is the piece of the picture on which

I want to focus:

$$
\begin{tikzcd}
 & & & H_{p+q-1}F_{p-3} \arrow[r, "p_*"] \arrow[d, "i_*"] & \cdots \\
\cdots \arrow[r, "\partial"] & H_{p+q}F_{p-1} \arrow[d, "i_*"] & & H_{p+q-1}F_{p-2} \arrow[r, "p_*"] \arrow[d, "i_*"] & \cdots \\
\cdots \arrow[r, "\partial"] & H_{p+q}F_p \arrow[r, "p_*"] \arrow[d, "i_*"] & H_{p+q}(F_p/F_{p-1}) \arrow[r, "\partial"] & H_{p+q-1}F_{p-1} \arrow[r, "p_*"] \arrow[d, "i_*"] & \cdots \\
\cdots \arrow[r, "\partial"] & H_{p+q}F_{p+1} \arrow[d, "i_*"] & & H_{p+q-1}F_p \arrow[r, "\partial"] \arrow[d, "i_*"] & \cdots \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & H_{p+q}C & & H_{p+q-1}C &
\end{tikzcd}
$$

We have the following definitions of abelian groups:

$$
\begin{aligned}
Z^r_{p,q} &:= \partial^{-1}(\mathrm{im}(i_*^{r-1})),\\
B^r_{p,q} &:= p_*(\ker(i_*^{r-1})),\\
E^r_{p,q} &= Z^r_{p,q}/B^r_{p,q}.
\end{aligned}
$$

Here, i_*^{r-1} denotes the composite map,

$$
\begin{aligned}
&H_{p+q-1}F_{p-r} \to H_{p+q-1}F_{p-1}, \text{ and}\\
&H_{p+q}F_p \to H_{p+q}F_{p+r-1},
\end{aligned}
$$

respectively.

All of these groups are R-modules if the filtered chain complex is defined in the category of R-

modules. The definitions are natural in morphisms of filtered complexes, suitably defined.

Observe that

$$E^1_{p,q} = H^{p+q}(F_p/F_{p-1})$$

since i^0_* is the identity map, so that

$$Z^1_{p,q} = H_{p+q}(F_p/F_{p-1}) \text{ and } B^1_{p,q} = 0.$$

There is a homomorphism

$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1},$$

called the r-*differential*, which is defined by $[u] \mapsto [p_*(v)]$ where $i^{r-1}_*(v) = \partial(u)$.

Observe that if $u, u' \in Z^r_{p,q}$ with $u = u' + p_*(x)$ for some $x \in H_{p+q}F_p$ then $\partial(u) = \partial(u')$ since $\partial p_* = 0$, so that $d^r([u])$ is independent of the choice of representative u.

If $i^{r-1}_*(v') = i^{r-1}_*(v) = \partial(u)$, then $v - v' \in \ker(i^{r-1}_*)$ so that $p_*(v) = p_*(v') + p_*(v - v')$ and $[p_*(v)] = [p_*(v')]$ in $E^r_{p-r,q+r-1}$. It follows that d^r is well defined for all p, q, r.

Lemma 14.2. *1) The composite*

$$E^r_{p+r,q-r+1} \xrightarrow{d^r} E^r_{p,q} \xrightarrow{d^r} E^r_{p-r,q+r-1}$$

is the 0 *map.*

2) There is an isomorphism

$$\ker(d^r)/\operatorname{im}(d^r) \cong E^{r+1}_{p,q}.$$

Proof. The first claim is an exercise.

There are inclusions

$$B^r_{p,q} \subset B^{r+1}_{p,q} \subset Z^{r+1}_{p,q} \subset Z^r_{p,q}.$$

Let ν be the composite map

$$Z^{r+1}_{p,q} \subset Z^r_{p,q} \to E^r_{p,q}.$$

The sequence

$$Z^{r+1}_{p,q} \xrightarrow{\nu} E^r_{p,q} \xrightarrow{d^r} E^r_{p-r.q+r-1}$$

is exact. To see this, observe that

- If $u \in Z^{r+1}_{p,q}$, then $\partial(u) = i^r_*(v)$ for some $v \in H_{p+q-1}F_{p-r-1}$, and so $d^r([v]) = [p_* i_*(v)] = 0$.
- If $d^r([u]) = [p_*(v)] = 0$, then $p_*(v) = p_*(w)$ where $i^{r-1}_*(w) = 0$, so that $p_*(v-w) = 0$ and $v - w = i_*(v')$. But then

  $$i^r_*(v') = i^{r-1}_*(v-w) = i^{r-1}_*(v) = \partial(u),$$

  and $u \in Z^{r+1}_{p,q}$.

The resulting composite map

$$Z^{r+1}_{p,q} \to \ker(d^r) \to \ker(d^r)/\operatorname{im}(d^r)$$

has kernel $B^{r+1}_{p,q}$. In effect, suppose that $u \mapsto 0$ under this map. Then $[u] = [p_*(v')]$ in $E^r_{p,q}$, where $i^{r-1}_*(v') = \partial(w)$ for some w, or equivalently $i^r_*(v') = 0$. But then $u = p_*(v') + p_*(v)$ where $i^{r-1}_*(v) = 0$, so that $u = p_*(v' + v)$ with $i^r_*(v + v') = 0$, so that $u \in B^{r+1}_{p,q}$. $\square$

Here are some observations:

1) The morphism $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ is the composite

   $$H_{p+q}(F_p/F_{p-1}) \xrightarrow{\partial} H_{p+q-1}F_{p-1} \xrightarrow{p_*} H_{p+q-1}F_{p-1}/F_{p-2},$$

   and so the groups $E^2_{p,q}$ are the homology groups of the chain complex

   $$H_{p+q+1}F_{p+1}/F_p \xrightarrow{p_*\partial} H_{p+q}F_p/F_{p-1} \xrightarrow{p_*\partial} H_{p+q-1}F_{p-1}/F_{p-2}$$

2) Tacitly, $F_p = 0$ for $p < 0$ so that $E^1_{p,q} = 0$ for $p < 0$. It follows that $E^r_{p,q} = 0$ for $p < 0$.

   In good examples, such as the standard filtrations of the total complex $\mathrm{Tot}(E)$ of a first quadrant bicomplex E, we also have $E^1_{p,q} = 0$ and hence $E^r_{p,q} = 0$ if $q < 0$.

3) The number $p + q$ is the *total degree* of the group $E^r_{p,q}$. The differential

   $$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1},$$

decreases total degree by 1, for all $r \geq 1$.

The picture that emerges is a sequence of arrays of groups $E^r_{p,q}$ — the array $E^r_{p,q}$ is called the E^r-*term* — with differentials d^r in the E^r-term with isomorphisms $H_{p,q}(E^r) \cong E^{r+1}_{p,q}$. This is a *spectral sequence.*

Spectral sequences are a calculational tool, but to actually make calculations we need a few more definitions.

Define subgroups $Z^\infty_{p,q}$ and $B^\infty_{p,q}$ of $H_{p+q}(F_p/F_{p-1})$ by

$$\begin{aligned} Z^\infty_{p,q} &= \ker(\partial : H_{p+q}F_p/F_{p-1} \to H_{p+q-1}F_{p-1}) \\ B^\infty_{p,q} &= p_*(\ker(H_{p+q}F_p \to H_{p+q}C)) \end{aligned}$$

Then $B^\infty_{p,q} \subset Z^\infty_{p,q}$ (exercise), and we define

$$E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}.$$

Finally, set

$$F_pH_{p+q}C = \operatorname{im}(H_{p+q}F_p \to H_{p+q}C).$$

Then the subgroups

$$\cdots \subset F_pH_{p+q}C \subset F_{p+1}H_{p+q}C \subset \ldots$$

define a filtration of $H_{p+q}C$, and we have the following:

Lemma 14.3. *There are (natural) short exact sequences*

$$0 \to F_{p-1}H_{p+q}C \to F_pH_{p+q}C \xrightarrow{\pi} E^\infty_{p,q} \to 0.$$

The proof of this result is an exercise: the map

$$\pi : F_pH_{p+q}C \to E^\infty_{p,q}$$

is defined by $\pi(x) = [p_*(y)]$, where $y \in H_{p+q}F_p$ maps to x under the map

$$H_{p+q}F_p \to H_{p+q}C.$$

We have inclusions

$$B^r_{p,q} \subset B^\infty_{p,q} \subset Z^\infty_{p,q} \subset Z^r_{p,q}$$

for all $r \geq 1$. We are filtering an ordinary chain complex, so that

$$Z^\infty_{p,q} = Z^r_{p,q}$$

if $r \geq p+1$. We say that the spectral sequence *converges* if

$$B^r_{p,q} = B^\infty_{p,q}$$

for r sufficiently large, for all p, q. This would mean that

$$E^r_{p,q} = E^\infty_{p,q}$$

for r sufficiently large, for all p, q. In this case, we write

$$E^r_{p,q} =? \Rightarrow H_{p+q}C$$

to indicate that we have a spectral sequence for which one can calculate the E^r-term (usually for $r = 1$ or $r = 2$, and there's some fun in the calculation) and which converges to give a calculation of the homology H_*C.

This means that we have a filtration $F_pH_{p+q}C$ of $H_{p+q}C$ with filtration quotients

$$F_pH_{p+q} \to E^\infty_{p,q} \to 0$$

which can be calculated from the corresponding terms $E^r_{p,q}$ by iterating the homology machine which produces $E^{r+1}_{p,q}$ from $E^r_{p,q}$ finitely many times in each bidegree. The references to bidegree and the corresponding total degree give notational cues for keeping track of the calculations.

15 First quadrant bicomplexes

The two spectral sequences associated to a first quadrant bicomplex E are standard and very important examples.

1) Recall that the bicomplex E has a horizontal filtration F_nE with

$$F_nE_{p,q} = \begin{cases} E_{p,q} & \text{if } p \leq n, \\ 0 & \text{if } p > n \end{cases}$$

which induces a filtration

$$F_n \operatorname{Tot}(E) = \operatorname{Tot}(F_n E)$$

of the associated total complex $\operatorname{Tot}(E)$. Recall that there is an isomorphism

$$F_p \operatorname{Tot}(E)/F_{p-1} \operatorname{Tot}(E) \cong E_{p,*}[-p]$$

and so there is an isomorphism

$$E^1_{p,q} = H_{p+q}(E_{p,*}[-p]) \cong H_q E_{p,*} =: H^v_q E_{p,*}.$$

These isomorphisms fit into a commutative diagram

$$\begin{tikzcd}
E^1_{p,q} \arrow[r, "d^1"] \arrow[d, "\cong"'] & E^1_{p-1,q} \arrow[d, "\cong"] \\
H_q E_{p,*} \arrow[r, "\partial^h_*"'] & H_q E_{p-1,*}
\end{tikzcd}$$

so that one could write

$$E^2_{p,q} = H^h_p H^v_q(E)$$

The spectral sequence converges since

$$F_n \operatorname{Tot}(E)_{p+q} = \operatorname{Tot}(E)_{p+q}$$

if $n \geq p+q$, so that we have

$$E^2_{p,q} = H^h_p H^v_q(E) \Rightarrow H_{p+q} \operatorname{Tot}(E). \tag{1}$$

The resulting filtration of $H_n \operatorname{Tot}(E)$ in total degree n has the form

$$\begin{tikzcd} F_0H_n \arrow[r] \arrow[d, "\cong"'] & F_1H_n \arrow[r] \arrow[d] & \cdots & F_{n-1}H_n \arrow[r] & F_nH_n \arrow[r, "\cong"] \arrow[d] & H_n \\ E^\infty_{0,n} & E^\infty_{1,n-1} & & & E^\infty_{n,0} & \end{tikzcd} \tag{2}$$

2) The bicomplex E has a vertical filtration F'_pE with

$$F'_nE_{p,q} = \begin{cases} E_{p,q} & \text{if } q \le n, \\ 0 & \text{if } q > n \end{cases}$$

which induces a filtration

$$F'_n \operatorname{Tot}(E) = \operatorname{Tot}(F'_nE)$$

of the associated total complex $\operatorname{Tot}(E)$. there is an isomorphism

$$F_p \operatorname{Tot}(E)/F_{p-1} \operatorname{Tot}(E) \cong E_{*,p}[-p]$$

and so there is an isomorphism

$$E^1_{p,q} = H_{p+q}(E_{*,p}[-p]) \cong H_qE_{*,p} =: H^h_qE_{*,p}.$$

These isomorphisms fit into a commutative diagram

$$\begin{tikzcd} E^1_{p,q} \arrow[r, "d^1"] \arrow[d, "\cong"'] & E^1_{p-1,q} \arrow[d, "\cong"] \\ H_qE_{*,p} \arrow[r, "\partial^v_*"'] & H_qE_{*,p-1} \end{tikzcd}$$

so that one could write

$$E^2_{p,q} = H^v_p H^h_q(E)$$

The spectral sequence converges by the same argument as before, so that we have

$$E^2_{p,q} = H^v_p H^h_q(E) \Rightarrow H_{p+q}\,\mathrm{Tot}(E). \qquad (3)$$

The spectral sequence (3) is effectively the "same" spectral sequence as the spectral sequence (1) which is obtained by filtering in horizontal degree, except that we've reversed the roles of p and q. Reversing the order of calculating vertical and horizontal homology can have strikingly different outcomes, and this frequently gives two very useful and complementary apporoaches to calculating the same thing, which is the homology of the total complex $\mathrm{Tot}(E)$.

Example 15.1. Suppose that C is a chain complex of right R-modules and that A is a left R-module. Suppose that the bicomplex map

$$P(C)_{p,q} \to C_p$$

is an Eilenberg-Cartan resolution of C (as in Section 13). Then $\mathrm{Tot}(P(C)) \to C$ is a weak equivalence and $\mathrm{Tot}(P(C))$ is a complex of projectives.

Recall that the higher torsion product functors $\mathrm{Tor}_*(C, A)$ (Section 10) are defined by

$$\mathrm{Tor}_n(C, A) = H_n(\mathrm{Tot}(P(C)) \otimes_R A)$$

up to isomorphism.

There is an isomorphism

$$\mathrm{Tot}(P(C)) \otimes_R A \cong \mathrm{Tot}(P(C) \otimes_R A).$$

1) Filter the bicomplex $P(C) \otimes A$ in the horizontal direction:

$$F_n(P(C) \otimes A)_{p,q} = \begin{cases} P(C)_{p,q} \otimes A & \text{if } p \leq n, \\ 0 & \text{if } p > n. \end{cases}$$

and

$$F_n \,\mathrm{Tot}(P(C) \otimes A) = \mathrm{Tot}(F_n(P(C) \otimes A).$$

Then

$$F_p \,\mathrm{Tot}(P(C) \otimes A)/F_{p-1} \,\mathrm{Tot}(P(C) \otimes A) \cong P(C)_{p,*} \otimes A[-p]$$

where $P(C)_{p,*} \to C_p$ is a projective resolution, so that

$$E^1_{p,q} \cong \mathrm{Tor}_q(C_p, A).$$

It follows that

$$E^2_{p,q} \cong H_p \,\mathrm{Tor}_q(C, A) \Rightarrow \mathrm{Tor}_{p+q}(C, A). \qquad (4)$$

2) Filter the bicomplex $P(C) \otimes A$ in the vertical direction:

$$F'_n(P(C) \otimes A)_{p,q} = \begin{cases} P(C)_{p,q} \otimes A & \text{if } q \leq n, \\ 0 & \text{if } q > n. \end{cases}$$

and

$$F'_n \operatorname{Tot}(P(C) \otimes A) = \operatorname{Tot}(F'_n(P(C) \otimes A).$$

Then

$$F'_p \operatorname{Tot}(P(C) \otimes A)/F'_{p-1} \operatorname{Tot}(P(C) \otimes A) \cong P(C)_{*,p} \otimes A[-p]$$

and

$$E^1_{p,q} = H_q(P(C)_{*,p} \otimes A).$$

The chain complexes $P(C)_{*,p}$ are split, and so there are isomorphisms

$$H_q(P(C)_{*,p} \otimes A) \cong H_q(P(C)_{*,p}) \otimes A$$

by Lemma 13.7. The chain complex $p \mapsto H_q(P(C))_{*,p}$ is a projective resolution of H_qC since $P(C) \to C$ is an Eilenberg-Cartan resolution, and it follows that there is an isomorphism

$$E^2_{p,q} \cong \operatorname{Tor}_p(H_qC, A).$$

In summary, this spectral sequence has the form

$$E^2_{p,q} = \operatorname{Tor}_p(H_qC, A) \Rightarrow H_{p+q} \operatorname{Tor}_{p+q}(C, A). \quad (5)$$

The spectral sequence (5) is the *universal coefficients spectral sequence*. It is a generalization of the universal coefficients theorem (Theorem 9.17).

The spectral sequences (4) and (5) both converge to the same thing, namely the higher torsion groups $\mathrm{Tor}_*(C, A)$. The spectral sequence (4) is more useful in the presence of "coarse" information about the complex C on the chain level, while (5) is used when was one has "fine" information about its homology.

Example 15.2. Choose cofibrant (projective) models $P \xrightarrow{\simeq} C$ and $Q \xrightarrow{\simeq} D$ for C and D respectively, in the respective categories of ordinary chain complexes. Form the bicomplex $P \otimes_R Q$ with

$$(P \otimes_R Q)_{p,q} = P_p \otimes_R Q_q$$

and take the horizontal filtration $F_p(P \otimes_R Q)$. Then in the corresponding spectral sequence

$$E^1_{p,q} = H_q(P_p \otimes_R Q) \cong H_q(P_p \otimes D) \cong P_p \otimes H_q(D).$$

It follows that

$$E^2_{p,q} = \mathrm{Tor}_p(C, H_q D) \Rightarrow \mathrm{Tor}_{p+q}(C, D).$$

Filtering in the vertical direction gives the spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p(H_q C, D) \Rightarrow \mathrm{Tor}_{p+q}(C, D).$$

For either of these spectral sequences, there is more work to do: more spectral sequence calculations as in Example 15.1 may be required to calculate the respective E^2-terms.

16 Filtered cochain complexes and third quadrant bicomplexes

Suppose that C is an unbounded complex with a filtration

$$\cdots \subset F_n C \subset F_{n+1} C \subset \dots C$$

indexed by $n \in \mathbb{Z}$. We still require that

$$\cup_{n\in\mathbb{Z}} F_n C = \varinjlim_{n\in\mathbb{Z}} F_n C = C.$$

The basic spectral sequence machinery of Section 14 applies in this more general case. We still have the picture of interlocking long exact sequences

$$\begin{tikzcd}
 & & & H_{p+q-1}F_{p-3} \arrow[r, "p_*"] \arrow[d, "i_*"] & \cdots \\
\cdots \arrow[r, "\partial"] & H_{p+q}F_{p-1} \arrow[d, "i_*"] & & H_{p+q-1}F_{p-2} \arrow[r, "p_*"] \arrow[d, "i_*"] & \cdots \\
\cdots \arrow[r, "\partial"] & H_{p+q}F_p \arrow[r, "p_*"] \arrow[d, "i_*"] & H_{p+q}(F_p/F_{p-1}) \arrow[r, "\partial"] & H_{p+q-1}F_{p-1} \arrow[r, "p_*"] \arrow[d, "i_*"] & \cdots \\
\cdots \arrow[r, "\partial"] & H_{p+q}F_{p+1} \arrow[d, "i_*"] & & H_{p+q-1}F_p \arrow[r, "\partial"] \arrow[d, "i_*"] & \cdots \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & H_{p+q}C & & H_{p+q-1}C &
\end{tikzcd}$$

but with $p, q \in \mathbb{Z}$. We set

$$\begin{aligned}
Z^r_{p,q} &:= \partial^{-1}(\mathrm{im}(i_*^{r-1})), \\
B^r_{p,q} &:= p_*(\ker(i_*^{r-1})), \\
E^r_{p,q} &= Z^r_{p,q}/B^r_{p,q}.
\end{aligned}$$

and then again a differential

$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

is defined such that the analog of Lemma 14.2 holds (with the same proof): the composite

$$E^r_{p+r,q-r+1} \xrightarrow{d^r} E^r_{p,q} \xrightarrow{d^r} E^r_{p-r,q+r-1}$$

is the 0 map, and there is an isomorphism

$$\ker(d^r)/\operatorname{im}(d^r) \cong E^{r+1}_{p,q}.$$

Subgroups $Z^\infty_{p,q}$ and $B^\infty_{p,q}$ of $H_{p+q}(F_p/F_{p-1})$ are defined by

$$\begin{aligned} Z^\infty_{p,q} &= \ker(\partial : H_{p+q}F_p/F_{p-1} \to H_{p+q-1}F_{p-1}) \\ B^\infty_{p,q} &= p_*(\ker(H_{p+q}F_p \to H_{p+q}C)) \end{aligned}$$

Then $B^\infty_{p,q} \subset Z^\infty_{p,q}$, and we set

$$E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}.$$

Finally, the analog of Lemma 14.3 holds (with the same proof): there are natural short exact sequences

$$0 \to F_{p-1}H_{p+q}C \to F_pH_{p+q}C \xrightarrow{\pi} E^\infty_{p,q} \to 0, \quad (6)$$

where $F_pH_{p+q}C$ is the image of the map

$$H_{p+q}(F_pC) \to H_{p+q}C.$$

Remark 16.1. The definition of $Z^\infty_{p,q}$ which appears above is *not* standard. This group is normally required to be the preimage under the map ∂ of the canonical map

$$\varprojlim_k \; H_{p+q-1}F_k \to H_{p+q-1}F_{p-1}.$$

The two descriptions of $Z^\infty_{p,q}$ coincide for the filtrations of ordinary chain complexes which were discussed in Section 15, but not in general, and the question of when the two corresponding descriptions of $E^{\infty}_{p,q}$ coincide can become a rather delicate issue.

In general, there are inclusions

$$B^{r-1}_{p,q} \subset B^{r}_{p,q} \subset B^{\infty}_{p,q} \subset Z^{\infty}_{p,q} \subset Z^{r}_{p,q} \subset Z^{r-1}_{p,q}$$

We say that the spectral sequence *converges* if for all p, q there is an r such that

1) $Z^{\infty}_{p,q} = Z^{r}_{p,q}$, and
2) $B^{r}_{p,q} = B^{\infty}_{p,q}$.

Generally, even if such a spectral sequence converges, to make effective calcuations you also need to know that

3) $F_n H_{p+q} C = 0$ for some n, for all p, q.

Suppose that E is a third quadrant bicomplex.

1) Filter the bicomplex E in horizontal degrees by setting

$$F_n E_{p,q} = \begin{cases} E_{p,q} & \text{if } p \leq n, \\ 0 & \text{if } p > n. \end{cases}$$

Then the subobjects

$$F_n(\mathrm{Tot}(E)) = \mathrm{Tot}(F_n E)$$

define a filtration of $\mathrm{Tot}(E)$ with $F_0 \mathrm{Tot}(E) = \mathrm{Tot}(E)$, so the filtration on $\mathrm{Tot}(E)$ is *descending.*

Note that condition 2) above follows automatically.

Also, $F_n E_{p,q} = 0$ if $q > n$ so that $F_n \operatorname{Tot}(E)_{p+q} = 0$ if $p + q > n$. It follows that $H_{p-1} F_k = 0$ if $k < p - 1$. Conditions 1) and 3) follow, and so the spectral sequence for the filtered complex converges in the strong sense that it satisfies conditions 1)–3).

Suppose that p and q are positive numbers. There are isomorphisms

$$E^1_{-p,-q} = H_{-p-q}(F_{-p}/F_{-p-1}) \cong H_{-p-q}(E_{-p,*}[p]) \cong H_{-q} E_{-p,*}$$

and it follows that the spectral sequence has the form

$$E^2_{-p,-q} \cong H^h_{-p} H^v_{-q}(E) \Rightarrow H_{-p-q} \operatorname{Tot}(E).$$

The filtration of $H_{-n} \operatorname{Tot}(E)$ has the form

$$\begin{tikzcd} F_{-n}H_{-n} \arrow[r] \arrow[d, "\cong"'] & F_{-(n-1)}H_{-n} \arrow[r] \arrow[d] & \cdots & F_{-1}H_{-n} \arrow[r] & F_0 H_{-n} \arrow[r, "\cong"] \arrow[d] & H_{-n} \\ E^\infty_{-n,0} & E^\infty_{-(n-1),-1} & & & E^\infty_{0,-n} & \end{tikzcd}$$

2) Reversing the roles of p and q (or by using the vertical filtration) we find a spectral sequence with

$$E^2_{-p,-q} = H^v_{-p} H^h_{-q}(E) \Rightarrow H_{-p-q} \operatorname{Tot}(E).$$

In both spectral sequences, the terms $E^r_{-p,-q}$ are only non-zero for $p, q \geq 0$. It is a common (meaning, universal) practice to rewrite

$$E_r^{p,q} = E^r_{-p,-q} \text{ and } E_\infty^{p,q} = E^\infty_{-p,-q}$$

and

$$H^n \operatorname{Tot}(E) = H_{-n} \operatorname{Tot}(E).$$

Then the two spectral sequences are written

$$\begin{aligned} E_2^{p,q} &= H_h^p H_v^q(E) \Rightarrow H^{p+q} \operatorname{Tot}(E), \text{ and} \\ E_2^{p,q} &= H_v^p H_h^q(E) \Rightarrow H^{p+q} \operatorname{Tot}(E). \end{aligned}$$

The differentials

$$d^r : E^r_{-p,-q} \longrightarrow E^r_{-p-r,-q+r-1}$$

are rewritten as

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1},$$

so that they raise total degree by 1 in the new notation.

One also writes

$$F^n H^n \operatorname{Tot}(E) = F_{-n} H_{-n} \operatorname{Tot}(E)$$

so that the filtration of $H^n \operatorname{Tot}(E)$ (for both spectral sequences) takes the form

$$
\begin{tikzcd}
F^n H^n \arrow[r] \arrow[d, "\cong"'] & F^{n-1} H^n \arrow[r] \arrow[d] & \cdots & F^1 H^n \arrow[r] & F^0 H^n \arrow[r, "\cong"] \arrow[d] & H^n \\
E_\infty^{n,0} & E_\infty^{n-1,1} & & & E_\infty^{0,n} &
\end{tikzcd}
\tag{7}
$$

Example 16.2. Suppose that C is a chain complex and that A is a module. Suppose that $A \to I$ is an injective resolution of A. Form the third quadrant bicomplex

$$\hom(C, I)_{-p,-q} = \hom(C_p, I_{-q}).$$

Recall from Lemma 12.7 that there is an isomorphism

$$H^n \operatorname{Tot}(\hom(C, I)) \cong \pi(C, I[-n]).$$

Filter the bicomplex in vertical degrees, so that there are isomorphisms

$$E_1^{p,q} \cong \hom(H_p C, I^q).$$

It follows that the resulting spectral sequence has

$$E_2^{p,q} = \operatorname{Ext}^q(H_p C, A) \Rightarrow \pi(C, I[-p-q]) = [C, A[-p-q]]. \tag{8}$$

The last identification is a consequence of Corollary 12.9.

The spectral sequence (8) is the *universal coefficients spectral sequence in cohomology.*

The standard universal coefficients theorem for cohomology says that, if X is a simplicial set and A is an abelian group, then there is short exact sequence

$$0 \to \mathrm{Ext}(H_{p-1}(X), A) \to H^p(X, A) \to \hom(H_p(X), A) \to 0$$

where the homology groups $H_*(X) = H_*(X, \mathbb{Z})$ have integral coefficients.

This result is usually proved by applying $\hom(\ , A)$ to the exact sequence of chain complexes

$$0 \to Z(X) \to \mathbb{Z}(X) \to B(X)[-1] \to 0$$

where $Z(X)_n = Z_n\mathbb{Z}(X)$, $B(X)_n = B_n\mathbb{Z}(X)$ are defined by cycles and boundaries, respectively, and the differentials are trivial for both $Z(X)$ and $B(X)$.

References

[1] Saunders Mac Lane. *Homology.* Springer-Verlag, Berlin, first edition, 1967. Die Grundlehren der mathematischen Wissenschaften, Band 114.