Lecture 008 (November 13, 2009)

17 The Dold-Kan correspondence

Suppose that $A : \mathbf{\Delta}^{op} \to R - \mathbf{Mod}$ is a simplicial R-module. Examples to keep in mind are the free simplicial objects $R(X)$ associated to simplicial sets X and nerves BM of R-modules M.

There are two basic ways to make chain complexes from a simplicial abelian group (or simplicial R-module) A:

1) The *Moore complex*, which is again denoted by A, has n-chains given by A_n, $n \geq 0$, and has boundary maps

$$\partial = \sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1}$$

defined by alternating sums of face homomorphisms.

2) The *normalized chain complex* NA has

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i)$$

and has boundary

$$\partial = (-1)^n d_n : NA_n \to NA_{n-1}.$$

To see that the definition of ∂ for the normalized complex NA makes sense and that $\partial^2 = 0$, observe that

$$d_i d_n(a) = d_{n-1} d_i(a)$$

for $i \leq n-1$.

Suppose that $a = s_j(b)$ in A_n. Then, in the Moore complex

$$\begin{aligned}
\partial(s_j(b)) &= \sum_{i=0}^{n} (-1)^i d_i s_j(b) \\
&= d_0 s_j(b) + \cdots + (-1)^j d_j s_j(b) + (-1)^{j+1} d_{j+1} s_j(b) + \\
&\quad \cdots + (-1)^n d_n s_j(b) \\
&= s_{j-1} d_0(b) + \cdots + (-1)^{j-1} s_{j-1} d_{j-1}(b) + (-1)^{j+2} s_j d_{j+1}(b) + \\
&\quad \cdots + (-1)^n s_j d_{n-1}(b).
\end{aligned}$$

It follows that if we define $DA_n \subset A_n$ to be the subgroup which is generated by all degeneracies $s_j(b)$, $b \in A_{n-1}$, $0 \leq j \leq n-1$, then the boundary in the Moore complex A restricts to a boundary $DA_n \to DA_{n-1}$ and DA acquires the structure of a subcomplex of A.

The definition of the boundary on the normalized complex NA was rigged so that NA is a subcomplex of the Moore chain complex A. Here's a first result:

Theorem 17.1. *Suppose that A is a simplicial R-module. Then we have the following:*

1) *The composite*

$$NA \subset A \to A/DA$$

is an isomorphism of chain complexes.

2) *The inclusion $NA \subset A$ is a chain homotopy equivalence, and in particular a weak equivalence.*

The statements of this result appear as Theorem III.2.1 and Theorem III.2.4, respectively, in [1].

The normalized chains construction defines a functor

$$N : s(R - \mathbf{Mod}) \to Ch_+(R).$$

Suppose that C is an ordinary chain complex, and let $\mathbf{\Delta}_m$ be the category of injective ordinal number morphisms. Then C determines a functor $C : \mathbf{\Delta}_m^{op} \to R - \mathbf{mod}$ with $\mathbf{n} \mapsto C_n$ and which sends a monomorphism $d : \mathbf{m} \to \mathbf{n}$ to

$$d^* = \begin{cases} (-1)^n \partial & \text{if } d = d^n, \\ 1 & \text{if } d = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

To see that the definition actually gives a functor, suppose given ordinal number monomorphisms

$$\mathbf{k} \xrightarrow{d} \mathbf{m} \xrightarrow{d'} \mathbf{n}$$

Then we have the following:

a) If $d'd = d^n$ then $d' = d^n$ and $d = 1$ or, $d' = 1$ and $d = d^n$. If $d'd = 1$ then $d' = 1$ and $d = 1$. In all such cases $(d'd)^* = d^* d'^*$.

b) If the composite $d'd$ is neither 1 nor d^n, then $(d'd)^* = 0$. If both d'^* and d^* are non-zero, then $d' = d^n$ and $d = d^{n-1}$ so that

$$d^* d'^* = (-1)^n \partial (-1)^{n-1} \partial = 0.$$

It follows that $d^* d'^* = (d'd)^*$ for all composable morphisms in $\mathbf{\Delta}_m$, and we have defined a functor $C : \mathbf{\Delta}_m^{op} \to R - \mathbf{Mod}$.

Define a simplicial R-module $\Gamma(C)$ by

$$\Gamma(C)_n = \bigoplus_{\sigma:\mathbf{n} \twoheadrightarrow \mathbf{k}} C_k.$$

where the direct sum is indexed over all ordinal number epimorphisms $\sigma : \mathbf{n} \twoheadrightarrow \mathbf{k}$ which have source $\mathbf{n}$. Suppose that $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number maps, and define θ^* to be the homomorphism such that all diagrams

$$\begin{array}{ccc} C_k & \xrightarrow{in_\sigma} & \oplus_{\sigma:\mathbf{n}\to\mathbf{k}}\ C_k \\ {\scriptstyle d^*}\downarrow & & \downarrow{\scriptstyle \theta^*} \\ C_r & \xrightarrow[in_{\sigma'}]{} & \oplus_{\nu:\mathbf{m}\to\mathbf{r}}\ C_r \end{array}$$

commute, where the diagram

$$\begin{array}{ccc} \mathbf{m} & \xrightarrow{\theta} & \mathbf{n} \\ {\scriptstyle \sigma'}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \mathbf{r} & \xrightarrow[d]{} & \mathbf{k} \end{array}$$

defines an epi-monic factorization of the composite $\sigma\theta$.

There is a natural simplicial R-module homomorphism

$$\Psi : \Gamma(NA) \to A$$

which is defined by the homomorphisms

$$\Psi_n : \bigoplus_{\sigma:\mathbf{n}\to\mathbf{k}} NA_k \to A_n.$$

The homomorphism Ψ_n is defined on the summand NA_k corresponding to the epimorphism $\sigma : \mathbf{n} \to k$ to be the composite

$$NA_k \subset A_k \xrightarrow{\sigma^*} A_n.$$

Then we have the following

Theorem 17.2. *The map Ψ is an isomorphism for all simplicial R-modules A. The functors Ψ and N determine an equivalence of categories*

$$N : s(R - \mathbf{Mod}) \leftrightarrows Ch_+(R) : \Gamma$$

Observe that Theorem 17.1 implies that there is an isomorphism

$$C \xrightarrow{\cong} N\Gamma(C)$$

for all chain complexes C.

The statements of Theorem 17.2 appear as Proposition III.2.2 and Corollary III.2.3 of [1]. This equivalence of categories is called the *Dold-Kan correspondence*.

18 Simplicial homotopy theory

I shall now outline some of the basic homotopy theory of simplicial sets. Proofs of these results can be found in [1], but the outline given here is followed with more clarity in [3].

The category $s\mathbf{Set}$ of simplicial sests has a (proper, simplicial) closed model structure for which the cofibrations are the monomorphisms and the weak equivalences are those maps $X \to Y$ which induce weak equivalences $|X| \to |Y|$ of their realizations in topological spaces.

The realization of a simplicial set X is the topological space which is defined by

$$|X| = \varinjlim_{\Delta^n \to X} |\Delta^n|.$$

In other words, $|X|$ is constructed by glueing together copies of the topological simplex $|\Delta^n|$, one for each simplex $\Delta^n \to X$, along the incidence relations between simplices. The realization functor is left adjoint to the singular functor $Y \mapsto S(Y)$, where

$$S(Y)_n = \hom(|\Delta^n|, Y)$$

for a topological space Y. The realization $|X|$ of a simplicial set is a CW-complex, so a map $f : X \to Y$ is a weak equivalence if and only if the induced map $f_* : |X| \to |Y|$ is a homotopy equivalence.

A model category is *proper* if weak equivalences are stable under pullback along fibrations and under pushout along cofibrations. There is a simplicial set $\mathbf{hom}(X, Y)$, called the *function complex* for all simplicial sets X and Y which enriches the ordinary set of simplicial set maps $X \to Y$: its n-simplices are the simplicial set maps $X \times \Delta^n \to Y$.

The simplicial set category $s\mathbf{Set}$ is *Quillen equivalent* to topological spaces. This means two things:

1) In the adjunction

$$| \ | : s\mathbf{Set} \leftrightarrows \mathbf{CGHaus} : S$$

the realization functor preserves weak equivalences (by definition) and cofibrations, and therefore preserves trivial cofibrations. It follows that the singular functor S preserves fibrations and trivial cofibrations.

2) The adjunction maps $\eta : X \to S|X|$ and $\epsilon : |S(Y)| \to Y$ are weak equivalences for simplicial sets X and all spaces Y.

CGHaus is the category of compactly generated Hausdorff spaces.

The fibrations for simplicial sets (this is the interesting bit in the formal setup) are the *Kan fibrations*: these are the maps $X \to Y$ which have the right lifting property with respect to all inclusions $\Lambda^n_k \subset \Delta^n$ of horns in simplices. A *Kan complex* is a fibrant simplicial set.

The k^{th} *horn* Λ^n_k is the subcomplex of Δ^n which is generated by the faces $d_i(\iota_n$, $i \neq k$, of the top simplex. The *boundary* $\partial\Delta^n$ is the subcomplex which is generated by all faces of the top simplex.

Here are some examples:

1) All simplicial groups, and in particular all simplicial abelian groups or simplicial modules, are Kan complexes — see Lemma 14.1 of [3], or Lemma I.3.4 of [1].
2) The singular set $S(Y)$ of a topological space Y is a Kan complex, since $|\Lambda^n_k|$ is a strong deformation retract of $|\Delta^n|$.
3) The nerve BG of a group G is a Kan complex.
4) The simplices $\Delta^n = B\mathbf{n}$ *are not* Kan complexes. In general, the nerve BC of a small category C is a Kan complex if and only if C is a groupoid (exercise).

In general, the set $\pi_0 X$ of *path components* of a simplicial set X is defined by the coequalizer

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \longrightarrow \pi_0 X.$$

One can show that $\pi_0 X \cong \pi_0 |X|$, so the name makes sense.

Classical simplicial homotopy is defined by the cylinder

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\nabla} & X \\ {\scriptstyle (d^1,d^0)}\downarrow & \nearrow_{pr} & \\ X \times \Delta^1 & & \end{array}$$

Here,

$$d^i : X \cong X \times \Delta^0 \xrightarrow{1 \times d^i} X \times \Delta^1$$

is the inclusion along an end point (vertex) of the "interval" Δ^1, and the map (d^1, d^0) can be identified with the cofibration $X \times \partial\Delta^1 \to X \times \Delta^1$ up to isomorphism. The projection pr is a weak equivalence, because its realization is a projection and $|\Delta^1|$ is contractible.

Say that the maps $f, g : X \to Y$ are *simplicially homotopic* if there is a diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle d^1}\downarrow & \searrow^{f} & \\ X \times \Delta^1 & \xrightarrow{h} & Y \\ {\scriptstyle d^0}\uparrow & \nearrow_{g} & \\ X & & \end{array}$$

Here, the map h is the *homotopy*, from f to g.

If X is a Kan complex and $x \in X_0$ is a vertex of X, then one can define combinatorial homotopy groups $\pi_n^s(X, x)$ for $n \geq 1$. The set underlying

$\pi_n^s(X, x)$ is the set of simplicial homotopy classes of maps of pairs

$$\alpha : (\Delta^n, \partial\Delta^n) \to (X, \{x\}),$$

and the group structure is geometrically defined. The group $\pi_n^s(X, x)$ is abelian for $n \geq 1$. One can show that the element α represents the identity element $e \in \pi_n^s(X, x)$ if and only if the lift exists in the diagram

$$\begin{array}{ccc} \partial\Delta^{n+1} & \xrightarrow{(x,\ldots,x,\alpha)} & X \\ \downarrow & \nearrow & \\ \Delta^{n+1} & & \end{array}$$

The Milnor Theorem (Theorem 13.2 of [3], Proposition I.11.2 of [1]) asserts that if X is a Kan complex, then the canonical map $\eta : X \to S|X|$ induces isomorphisms

$\pi_0(X) \xrightarrow{\cong} \pi_0(S|X|) \cong \pi_0|X|$, and
$\pi_n^s(X, x) \xrightarrow{\cong} \pi_n^s(S|X|, x) \cong \pi_n(|X|, x)$ for all $n \geq 1$.

It follows that a map $f : X \to Y$ of Kan complexes is a weak equivalence if and only if it induces isomorphisms in all possible combinatorial homotopy groups.

If G is a simplicial group, then the set of all maps

$$\alpha : (\Delta^n, \partial\Delta^n) \to (G, \{e\}),$$

has a group structure which it inherits from the group structure $G \times G \to G$ on G, and this group structure induces the group structure on $\pi_n(G, e)$ by a standard interchange law trick. That same trick implies that $\pi_1(G, e)$ is abelian.

Write

$$ZG_n = \cap_{i=0}^{n} \ker(d_i)$$

and

$$NG_n = \cap_{i=0}^{n-1} \ker(d_i).$$

We have "proved"

Lemma 18.1. *Suppose that G is a simplicial group. Then there is a short exact sequence*

$$NG_{n+1} \xrightarrow{d_{n+1}} ZG_n \to \pi_n^s(G, e) \to e.$$

The other thing to know about simplicial groups is the following:

Lemma 18.2. *A homomorphism $f : G \to H$ of simplicial groups is a weak equivalence if and only if it induces isomorphisms*

$$\pi_0 G \xrightarrow{\cong} \pi_0 H, \text{ and}$$
$$\pi_n^s(G, e) \xrightarrow{\cong} \pi_n^s(H, e) \text{ for all } n \geq 1.$$

Proof. Multiplication by $x \in G_0$ induces an isomorphism

$$x_* : \pi_n^s(G, e) \xrightarrow{\cong} \pi_n^s(G, x).$$

$\square$

Corollary 18.3. *1) There is a natural R-module isomorphism*

$$\pi_n^s(A, 0) \cong H_n(NA)$$

for all simplicial R-modules A.

2) A map $f : A \to B$ of simplicial R-modules is a weak equivalence of simplicial sets if and only if the map $f : NA \to NB$ is a homology isomorphism.

19 Model structures for simplicial modules

Lemma 19.1. *The free module functor*

$$R : s\mathbf{Set} \to s(R - \mathbf{Mod})$$

preserves weak equivalences.

This result is a special case of Corollary 14.9 of [3].

Proof. It is enough to show that the functor R takes trivial cofibrations to weak equivalences. In effect, every weak equivalence $f : X \to Y$ of simplicial sets has a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

where i is a trivial cofibration and p is a trivial fibration. Every simplicial set is cofibrant, so that the trivial fibration p has a section $\sigma : Y \to Z$, which section is a trivial cofibration.

Suppose that $i : A \to B$ is a trivial cofibration of simplicial sets. Then $R(A)$ is a fibrant simplicial set, so the lifts σ and σ_* exist in the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\eta} & R(A) & \xrightarrow{1} & R(A) \\ {\scriptstyle i}\downarrow & & {\scriptstyle i_*}\downarrow & \overset{\sigma}{\nearrow} \; \overset{\sigma_*}{\nearrow} & \downarrow \\ B & \xrightarrow[\eta]{} & R(B) & \longrightarrow & e \end{array}$$

The morphism σ_* is a simplicial R-module homomorphism. The lifts h and h_* also exist in the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\eta} & R(A) & \xrightarrow{si_*} & R(B)^{\Delta^1} \\ {\scriptstyle i}\downarrow & & {\scriptstyle i_*}\downarrow & \overset{h}{\nearrow} \; \overset{h_*}{\nearrow} & \downarrow{\scriptstyle (p_0,p_1)} \\ B & \xrightarrow[\eta]{} & R(B) & \xrightarrow[(i_*\sigma_*,1)]{} & R(B) \times R(B) \end{array}$$

where $R(B)^{\Delta^1}$ is the simplicial R-module defined by the function complex $\mathbf{hom}(\Delta^1, R(B))$, and the diagram of simplicial R-module homomorphisms

$$\begin{array}{ccc} & & \mathbf{hom}(\Delta^1, R(B)) \\ & \overset{s}{\nearrow} & \downarrow{\scriptstyle (p_0,p_1)} \\ \mathbf{hom}(\Delta^0, R(B)) & \xrightarrow[\Delta]{} & \mathbf{hom}(\partial\Delta^1, R(B)) \end{array}$$

is the (standard) path object for the Kan complex $R(B)$. $\square$

Observe that we end up showing in the proof of Lemma 19.1 that the free R-module functor takes trivial cofibrations of simplicial sets to strong deformation retractions of simplicial R-modules.

Say that a map $f : A \to B$ of simplicial R-modules is a *weak equivalence* (respectively *fibration*) if the underlying map of simplicial sets is a weak equivalence (respectively fibration). The cofibrations of simplicial R-modules are those maps which have the left lifting property with respect to trivial fibrations.

It is a consequence of Corollary 18.3 that a map $f : A \to B$ is a weak equivalence of simplicial R-modules if and only if the associated chain complex morphism $f_* : NA \to NB$ is a weak equivalence (aka. homology isomorphism).

For simplicial modules A and B there is a simplicial module $\mathbf{hom}(A, B)$ with

$$\mathbf{hom}(A, B)_n = \hom(A \otimes R(\Delta^n), B).$$

There are natural isomorphism

$$\begin{aligned} \hom(K, \mathbf{hom}(A,B)) &\cong \hom(A \otimes R(K), B) \\ &\cong \hom(A, \mathbf{hom}(K,B)) \end{aligned}$$

for all simplicial sets K. Here, $\mathbf{hom}(K,B)$ is the function complex of simplicial sets, endowed with the simplicial R-module structure that it inherits from B.

Remark 19.2. Here, we're using the *tensor product* $A \otimes C$ of simplicial modules, which is defined by

$$(A \otimes C)_n = A_n \otimes C_n.$$

The chain complex $N(A \otimes C)$ *does not* coincide with $NA \otimes NC$ up to isomorphism in the chain complex category. There is, however, a natural weak equivalence between the two objects (this is usually called the Eilenberg-Zilber Theorem), and more will be said about this later.

Theorem 19.3. *With these definitions, the category $s(R - \mathbf{Mod})$ satisfies the axioms for a proper closed simplicial model category.*

Proof. A map $p : A \to B$ is a fibration (respectively trivial fibration) if and only if it has the right lifting property with respect to all maps $R(\Lambda^n_k) \to$

$R(\Delta^n)$ (respectively with respect to all $R(\partial\Delta^n) \to R(\Delta^n)$).

If one is given a pushout diagram

$$\begin{array}{ccc} R(\Lambda^n_k) & \longrightarrow & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_*} \\ R(\Delta^n) & \longrightarrow & A' \end{array}$$

then the map i_* is a weak equivalence. In effect, the diagram of normalized chain complexes

$$\begin{array}{ccc} NR(\Lambda^n_k) & \longrightarrow & NA \\ {\scriptstyle Ni}\downarrow & & \downarrow{\scriptstyle Ni_*} \\ NR(\Delta^n) & \longrightarrow & NA' \end{array}$$

is a pushout, and the map Ni is a weak equivalence by Lemma 19.1. It therefore follows from a small object argument that any morphism $f : A \to B$ has a factorization

$$\begin{array}{ccc} A & \xrightarrow{i} & C \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & B \end{array}$$

where p is a fibration and i is a trivial cofibration which has the left lifting property with respect to all fibrations.

The proof of the other factorization statement is similar: one uses a small object argument to show that every $f : A \to B$ has as factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & D \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle q} \\ & & B \end{array}$$

such that q is a trivial fibration and j is a cofibration. It is an artifact of that proof that j is also a monomorphism. We have therefore proved the factorization axiom **CM5**.

It follows by a standard argument that every trivial cofibration has the left lifting property with respect to all fibrations, giving **CM4**.

If $p : A \to B$ is a fibration of simplicial R-modules and $i : K \to L$ is a cofibration of simplicial sets, then the induced map

$$\mathbf{hom}(L, A) \xrightarrow{(i^*, p_*)} \mathbf{hom}(K, B) \times_{\mathbf{hom}(B,Y)} \mathbf{hom}(L, B)$$

is a fibration, which is trivial if either i or p is trivial. This gives the simplicial model structure.

Weak equivalences pull back along fibrations by the corresponding property for simplicial sets, and weak equivalences push out along cofibrations by a comparison of long exact sequences in homology. Note that every cofibration is a retract of a monomorphism, and is therefore a monomorphism. We have therefore established properness. $\square$

The following is a special case of a result of Quillen [4] (see also Lemma 14.6 of [3]):

Lemma 19.4. *A simplicial R-module homomorphism $p : A \to B$ is a fibration if and only if the induced map $p_* : NA \to NB$ is a fibration of chain complexes.*

Proof. We will show that p is a fibration if p_* is a fibration of chain complexes. The other implication is an exercise.

Generally, if a simplicial group homomorphism $p : G \to H$ is surjective in all degrees, then p is a fibration. To see this, let K be the kernel of p and consider a lifting problem

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\alpha} & G \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow[\beta]{} & H \end{array}$$

There is a simplex $\gamma : \Delta^n \to G$ such that $p(\gamma) = \beta$ since p is surjective in all degrees. Then $\alpha \cdot (\gamma|_{\Lambda^n_k})^{-1}$ is a morphism $\Lambda^n_k \to K$ and there is an extension

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\alpha(\gamma|_{\Lambda^n_k})^{-1}} & K \\ \downarrow & \nearrow_{\theta} & \\ \Delta^n & & \end{array}$$

since K is a Kan complex. Then $\theta\gamma$ is the desired lifting.

It therefore follows from the Dold-Kan correspondence (Theorem 17.2: the fact that Ψ is an isomorphism) that $p : A \to B$ is a fibration if the induced map $p_* : NA \to NB$ is surjective in all degrees.

Form the diagram

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ \downarrow & & \downarrow \\ K(\pi_0 A, 0) & \xrightarrow[p_*]{} & K(\pi_0 B, 0) \end{array}$$

where $K(M, 0)$ denotes the constant simplicial module on the module M. Any module homomorphism $M \to N$ induces a fibration $K(M, 0) \to K(N, 0)$ (exercise), so the induced map

$$K(\pi_0 A, 0) \times_{K(\pi_0 B, 0)} B \to B$$

is a fibration.

The normalized chains functor N preserves pullbacks, and $NK(\pi_0 M, 0) \cong M[0]$ for all modules M. The induced map

$$NA_n \to (\pi_0 A[0] \times_{\pi_0 B[0]} NB)_n$$

is isomorphic to $NA_n \to NB_n$ for $n \geq 1$. It therefore suffices to show that the map

$$A_0 \to \pi_0 A[0] \times_{\pi_0 B[0]} B_0 \qquad (1)$$

is surjective.

An element $([b], a)$ of the pullback $\pi_0 A[0] \times_{\pi_0 B[0]} B_0$ determines a commutative diagram of simplicial module morphisms

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{b} & A \\ {\scriptstyle d^0}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^1 & \xrightarrow[\omega]{} & B \end{array}$$

where $d_1 \omega = a$. Solving the indicated lifting problem produces an element $d_1 \theta \in A_0$ such that $d_1 \theta \mapsto ([b], a)$ under the map (1).

The simplicial module A is a Kan complex, so there is a simplex $\zeta : \Delta^1 \to A$ such that $d_0(\zeta) = b$. Then $d_0(\omega - p(\zeta)) = 0$, and $p_* : NA_1 \to NB_1$ is surjective, so there is an element $v \in NA_1$ such that $p(v) = \omega - p(\zeta)$. Then $d_0(\zeta + v) = d_0(\zeta) = b$ and $p(\zeta + v) = \omega$. Thus, setting $\theta = \zeta + v$ solves the lifting problem. □

Remark 19.5. The model structure on the category $s(R - \mathbf{Mod})$ of simplicial R-modules which is given by Theorem 19.3 is equivalent to the standard model structure on the category $Ch_+(R)$ of ordinary chain complexes of Theorem 5.1 under the Dold-Kan correspondence (Theorem 17.2). Why bother? The answer is that the proof of Theorem 19.3 holds in far greater generality (eg. sheaves of modules) than does the proof of Theorem 5.1. See [2] for more detail.

References

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