Lecture 009 (November 25, 2009)

20 Homotopy colimits

Suppose that I is a small category, and let

$$s(R-\mathbf{Mod})^I$$

be the category of I-diagrams in simplicial modules.

The objects of this category are the functors

$$A : I \to s(R-\mathbf{Mod}),$$

and a morphism $f : A \to B$ between I-diagrams is a natural transformation. Explicitly, f consists of homomorphisms $f : A(i) \to B(i)$, one for each object $i \in I$, such that the diagrams

$$\begin{array}{ccc} A(i) & \xrightarrow{f} & B(i) \\ {\scriptstyle\alpha_*}\downarrow & & \downarrow{\scriptstyle\alpha_*} \\ A(j) & \xrightarrow[f]{} & B(j) \end{array}$$

commute for all morphisms $\alpha : i \to j$ of I.

Say that the natural transformation $f : A \to B$ is a *weak equivalence* (respectively *projective fibration*) if all maps $f : A(i) \to B(i)$ are weak equivalences (respectively fibrations) of simplicial R-modules.

The cofibrations of $s(R-\mathbf{Mod})^I$ are defined by a left lifting property: a morphism $i : C \to D$ is a *projective cofibration* if and only if it has the left lifting property with respect to all trivial fibrations.

The *function complex* $\mathbf{hom}(A, B)$ for I-diagrams A and B is a simplicial R-module such that $\hom(A, B)_n$ is the module of natural transformations

$$A \otimes \Delta^n \to B.$$

Here, if K is a simplicial set then

$$(A \otimes K)(i) := A(i) \otimes R(K)$$

defines the I-diagram $A \otimes K$. One could also write

$$(A \otimes K)(i) = A(i) \otimes R(K)(i)$$

(pointwise tensor product) where $R(K)$ is the constant diagram defined by $R(K)(i) = R(K)$, with identity maps giving the structure as a functor.

Theorem 20.1. *Suppose that I is a small category. Then, with these definitions, the category $s(R-\mathbf{Mod})^I$ of I-diagrams in simplicial modules has the structure of a proper closed simplicial model category.*

Remark 20.2. The model structure of Theorem 20.1 is called the *projective model structure* on the category of I-diagrams in $s(R-\mathbf{Mod})$. The general approach to constructing model categories of this type, on categories $\mathcal{M}^I$ of I-diagrams in cofibrantly generated model categories $\mathcal{M}$ first appeared in [1]. The term "projective model structure" is relatively new, and its use originated in motivic homotopy theory.

Proof of Theorem 20.1. The i-sections functor $A \mapsto A(i)$ has a left adjoint

$$L_i : s(R-\mathbf{Mod}) \to s(R-\mathbf{Mod})^I$$

which is defined by

$$L_i M = M \otimes \hom(i,\ ).$$

Observe that L_i takes monomorphisms to monomorphisms.

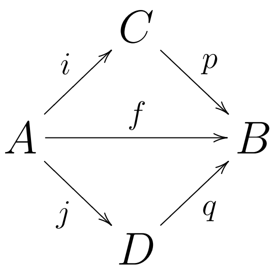
A map $p : A \to B$ is a projective fibration if and only if it has the right lifting property with respect to all maps

$$L_i R(\Lambda^n_k) \to L_i R(\Delta^n).$$

A map $q : C \to D$ is a trivial projective fibration if and only if it has the right lifting property with respect to all maps

$$L_i R(\partial \Delta^n) \to L_i R(\Delta^n).$$

A small object argument therefore implies that every natural transformation $f : A \to B$ has factorizations



such that

1) p is a projective fibration and i is a trivial projective cofibration which has the left lifting property with respect to all projective fibrations, and

2) q is a trivial projective fibration and j is a projective cofibration.

This proves **CM5**, and **CM4** follows by a standard argument.

Properness and the simplicial model axiom are both consequences of the corresponding statements for simplicial modules (Theorem 19.3). $\square$

Corollary 20.3. *The category $Ch_+(R)^I$ of I-diagrams of ordinary chain complexes has a proper closed simplicial model structure for which the weak equivalences (respectively fibrations) are* *defined sectiowise: a natural transformation $C \to D$ is a weak equivalence (respectively projective fibration) if and only if all maps $C(i) \to D(i)$ are weak equivalences (respectively fibrations) of $Ch_+(R)$.*

This is a consequence of Theorem 20.1 and the Dold-Kan correspondence (Theorem 17.2). The basic closed model structure (with properness) can also be derived independently.

All objects are fibrant in the projective model structure on $s(R - \mathbf{Mod})^I$.

Lemma 20.4. *Suppose that $f : A \to B$ is a weak equivalence of cofibrant objects in $s(R - \mathbf{Mod})^I$. Then the induced map*

$$f_* : \varinjlim_I A \to \varinjlim_I B$$

is a weak equivalence.

Proof. Suppose that $p : C \to D$ is a fibration of simplicial R-modules. Then the corresonding map between constant I-diagrams is a projective fibration. It follows by adjointness that if $i : A' \to B'$ is a trivial projective cofibration, then the induced map

$$\varinjlim_I A' \to \varinjlim_I B'$$

is a trivial cofibration of simplicial R-modules.

In general, form the diagram

$$\begin{array}{cccccc} & & A & \xrightarrow{f} & B & \\ & & {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle i_{0*}} & \searrow{\scriptstyle 1} \\ A & \xrightarrow[i_1]{} & A\otimes\Delta^1 & \xrightarrow[f_*]{} & M_f & \\ & & & \searrow{\scriptstyle fs} & & \searrow{\scriptstyle \pi} \\ & & & & & B \end{array} \tag{1}$$

in which the inner square is a pushout. Then the dotted arrow π exists and is uniquely defined. The diagram

$$\begin{array}{ccc} A\oplus A & \xrightarrow{f\oplus 1} & B\oplus A \\ {\scriptstyle (i_0,i_1)}\downarrow & & \downarrow{\scriptstyle (i_{0*},f_*i_1)} \\ A\otimes\Delta^1 & \xrightarrow[f_*]{} & M_f \end{array}$$

is a pushout, so that the map (i_{0*}, f_*i_1) is a cofibration. Then B is cofibrant, so it follows that the map f_*i_1 is a cofibration.

We therefore have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f_*i_1} & M_f \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi} \\ & & B \end{array}$$

in which f_*i_1 is a trivial projective cofibration and the map π has a section by the trivial projective cofibration i_{0*}. Applying the functor $\varinjlim_I$ to the diagram therefore produces a diagram of weak equivalences, by the first paragraph. $\square$

Remark 20.5. The diagram (1) is the mapping cylinder construction for the map $f : A \to B$ in $s(R-\mathbf{Mod})$. Generally, if a functor $F : \mathcal{M} \to \mathcal{N}$ between model categories takes trivial cofibrations to weak equivalences, then F takes weak equivalences between cofibrant objects to weak equivalences, by the same argument.

Write

$$\underrightarrow{\text{holim}}_I A = \varinjlim_I A_c$$

where $A_c \xrightarrow{\simeq} A$ is a cofibrant model of the I-diagram A. The object $\underrightarrow{\text{holim}}_I A$ is called the *homotopy colimit* of the I-diagram A. The choice of cofibrant model A_c can be made functorial, so that

$$A \mapsto \underrightarrow{\text{holim}}_I A$$

defines a functor. Lemma 20.4 implies that any weak equivalence $A \to B$ of I-diagrams induces a weak equivalence

$$\underrightarrow{\text{holim}}_I A \to \underrightarrow{\text{holim}}_I B$$

of simplicial modules. The homotopy colimit functor is the left derived functor of the colimit functor, in exactly the same way that the derived tensor product is a left derived functor of tensor product.

Example 20.6. Suppose that I is the category

$$\begin{array}{ccc} 0 & \rightarrow & 1 \\ \downarrow & & \\ 2 & & \end{array}$$

Then one can show that a diagram

$$\begin{array}{ccc} A_0 & \xrightarrow{i} & A_1 \\ {\scriptstyle j}\downarrow & & \\ A_2 & & \end{array}$$

of simplicial R-modules is projective cofibrant if and only if A_0 is cofibrant and the maps i and j are cofibrations.

Suppose given a pushout diagram

$$\begin{array}{ccc} B_0 & \rightarrow & B_1 \\ {\scriptstyle i}\downarrow & & \downarrow \\ B_2 & \rightarrow & B \end{array}$$

in $s(R-\mathbf{Mod})$ in which i is a cofibration, and suppose that $A \to B$ is a cofibrant model for the I-diagram defined by $B_2 \xleftarrow{i} B_0 \to B_1$. Then by comparing exact sequences we see that the map

$$\underrightarrow{\text{holim}}_I\, A \to \varinjlim_I B = B$$

is a weak equivalence.

There is an obvious corollary of Lemma 20.4:

Corollary 20.7. *Suppose that $f : C \to D$ is a weak equivalence of cofibrant objects in $Ch_+(R)^I$. Then the induced map*

$$f_* : \varinjlim_I A \to \varinjlim_I B$$

is a weak equivalence.

Here is a first application of this result:

Every diagram $M : I \to R - \mathbf{Mod}$ of R-modules can be identified with a diagram $M[0] : I \to Ch_+(R)$ of chain complexes. This diagram of chain complexes has a cofibrant model $P \xrightarrow{\simeq} M[0]$. The *higher derived functors* $\varinjlim_n M$ *of the colimit* are defined by

$$\varinjlim_n M = H_n(\varinjlim_I P) = H_n(\underrightarrow{\text{holim}}_I\, M[0]).$$

21 Cosimplicial chain compexes

Cosimplicial objects in a category $\mathcal{C}$ are (covariant) functors $\mathbf{\Delta} \to \mathcal{C}$. Write $c\mathcal{C}$ for the corresponding category. Bicosimplicial objects are functors

$$\mathbf{\Delta} \times \mathbf{\Delta} \to \mathcal{C},$$

or equivalently, cosimplicial objects in $s\mathcal{C}$. Write $c^2\mathcal{C}$ for the category of bicosimplicial objects in $\mathcal{C}$.

We shall be investigating the projective model structures for both cosimplicial and bicosimplicial chain complexes (or simplicial modules) in this section.

Cosimplicial spaces are functors $\mathbf{\Delta} \to s\mathbf{Set}$, and are the subject of the Bousfield-Kan Springer Lecture Note [1]. We shall use some of the ideas from that source in this section.

Suppose that $A : \mathbf{\Delta} \to R - \mathbf{Mod}$ is a cosimplicial module. Write $M^{n-1}A$ for the subobject of $(A^{n-1})^n$ consisting of all n-tuples $(a_0, \dots, a_{n-1})$ such that $s^i a_j = s^{j-1} a_i$ for $i < j$. The module $M^{n-1}A$ is traditionally called a *matching object*. There is a canonical map

$$s : A^n \to M^{n-1}A$$

which is defined by

$$s(a) = (s^0 a, s^1 a, \dots, s^{n-1} a).$$

This makes sense, on account of the cosimplicial identities $s^i s^j(a) = s^{j-1} s^i(a)$ for $i < j$.

Lemma 21.1. *Suppose that A is a cosimplicial module. Then the map $s : A^n \to M^{n-1}A$ is surjective.*

Proof. Suppose first of all that one is given an element of the form

$$a = (a_0, \dots, a_j, 0, \dots, 0).$$

Then

$$s^i a_j = s^j a_{i+1} = 0$$

if $i \geq j$ (recall the cosimplicial identity $s^i s^j = s^j s^{i+1}$ if $i \geq j$). It follows that

$$s^i d^j a_j = d^j s^{i-1} a_j = 0$$

if $i > j$ and $s^j d^j a_j = a_j$, so that $a - s(s^j a_j)$ has the form

$$a - s(s^j a_j) = b = (b_0, \dots b_{j-1}, 0, \dots 0).$$

Inductively, if every such element $b = s(c)$, then $s = s(s^j a_j + c)$. □

Corollary 21.2. *1) Suppose that $p : A \to B$ is a map of cosimplicial module which consists of epimorphisms $p : A^n \to B^n$. Then all induced maps*

$$(p, s) : A^n \to B^n \times_{M^{n-1}B} M^{n-1}A$$

are epimorphisms.

2) *Suppose that $p : A \to B$ is a map of cosimplicial chain complexes (respectively simplicial modules) which consists of fibrations $p : A^n \to B^n$. Then all induced maps*

$$(p, s) : A^n \to B^n \times_{M^{n-1}B} M^{n-1}A$$

are fibrations of chain complexes (respectively simplicial modules).

Proof. Suppose that the cosimplicial abelian group K is the kernel of p. Suppose given an element $(b, (a_0, \dots, a_{n-1}))$ such that $s(b) = p(a_0, \dots, a_{n-1})$. There is an element $a \in A^n$ such that $p(a) = b$. Then $s(a) - (a_0, \dots, a_{n-1})$ is an element of $M^{n-1}K$, and there is an element $x \in K^n$ such that

$$s(x) = s(a) - (a_0, \dots, a_{n-1})$$

by Lemma 21.1. But then $s(a+x) = (a_0, \dots, a_{n-1})$ and $p(a + x) = b$.

The second claim follows from the first. One uses Lemma 19.4 together with the fact the fact that the normalized complex functor preserves finite limits up to natural isomorphism. □

Lemma 21.3. *Suppose that K is a cosimplicial object in chain complexes (respectively sim-*

plicial modules) such that all objects K^n are acyclic. Then the map

$$s : K^n \to M^{n-1}K$$

is a trivial fibration.

Proof. Write $M^{n-1}_{\leq k}K$ for the set of $(k+1)$-tuples $(a_0, \dots, a_k) \in K^{n-1}$ such that $s^i a_j = s^{j-1} a_i$ for $i < j \leq k$. Then there are canonical maps $s : K^n \to M^{n-1}_{\leq k}K$ and pullback diagrams

$$\begin{array}{ccc} M^{n-1}_{\leq k}K & \xrightarrow{pr_k} & K^{n-1} \\ {\scriptstyle s_*}\downarrow & & \downarrow{\scriptstyle s} \\ M^{n-1}_{<k}K & \xrightarrow[s^{k-1}]{} & M^{n-2}_{<k}K \end{array}$$

All maps $d : K^n \to M^{n-1}_{\leq k}K$ are fibrations, by a variant of the argument for Lemma 21.1.

Inductively, all maps $s : K^{n-1} \to M^{n-2}_{\leq i}K$ are trivial fibrations, so that the map s_* in the diagram is a trivial fibration. Inductively again, the map $s : K^n \to M^{n-1}_{<k}K$ is a weak equivalence, so that the map $s : K^n \to M^{n-1}_{\leq k}K$ is a trivial fibration. □

Corollary 21.4. *Suppose that $p : A \to B$ is a map of cosimplicial chain complexes (respectively simplicial modules) which consists of trivial fibrations $p : A^n \to B^n$. Then all induced* *maps*

$$(p,s) : A^n \to B^n \times_{M^{n-1}B} M^{n-1}A$$

are trivial fibrations of chain complexes (respectively simplicial modules).

Proof. The map (p,s) is surjective in all simplicial degrees since p is surjective for all n and in all simplicial degrees. Suppose that the cosimplicial object K is the kernel of p. The kernel of (p,s) is isomorphic to the kernel of the map $s : K^n \to M^{n-1}K$, which kernel is acyclic by Lemma 21.3. □

Corollary 21.5. *1) Suppose that $p : A \to B$ is a projective fibration of bicosimplicial chain complexes (or simplicial modules), and identify this map with a morphism of cosimplicial objects in cosimplicial objects. Then all induced maps*

$$(p,s) : A^n \to B^n \times_{M^{n-1}B} M^{n-1}A$$

are projective fibrations of cosimplicial objects.

2) Suppose that $p : A \to B$ is a trivial projective fibration of bicosimplicial chain com-

plexes (or simplicial modules). Then all induced maps

$$(p,s) : A^n \to B^n \times_{M^{n-1}B} M^{n-1}A$$

are trivial projective fibrations of cosimplicial objects.

Suppose again that $A : \mathbf{\Delta} \to R - \mathbf{Mod}$ is a cosimplicial module. The module L^nA is defined by the coequalizer

$$\bigoplus_{i<j} A^{n-2} \rightrightarrows \bigoplus_{0 \leq i \leq n} A^{n-1} \to L^nA$$

which is defined by the cosimplicial identities $d^jd^i = d^id^{j-1}$ for $i < j$. In effect, the two composites

$$A^{n-2} \xrightarrow{in_{i<j}} \bigoplus_{i<j} A^{n-2} \rightrightarrows \bigoplus_{0 \leq i \leq n} A^{n-1}$$

are the composites

$$A^{n-2} \xrightarrow{d^{j-1}} A^{n-1} \xrightarrow{in_i} \bigoplus_{0 \leq i \leq n} A^{n-1}$$

and

$$A^{n-2} \xrightarrow{d^i} A^{n-1} \xrightarrow{in_j} \bigoplus_{0 \leq i \leq n} A^{n-1}.$$

There is a canonical map $d : L^{n-1}A \to A^n$ which is defined by the coface d^i on the i^{th} summand.

Suppose that A and B are cosimplicial objects (in modules, chain complexes, or simplicial modules), and that $f^k : A^k \to B^k$, $k < n$, are homomorphisms which define a map $f : A^{<n} \to B^{<n}$ of truncated cosimplicial objects. Then f extends to a map $f_* : A^{\leq n} \to B^{\leq n}$ if and only if there is a map $f_* : A^n \to B^n$ such that the diagram

$$\begin{array}{ccccc} L^{n-1}A & \xrightarrow{d} & A^n & \xrightarrow{s} & M^{n-1}A \\ {\scriptstyle f_*}\downarrow & & \vdots\,{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ L^{n-1}B & \xrightarrow[d]{} & B^n & \xrightarrow[s]{} & M^{n-1}B \end{array}$$

commutes (exercise).

Lemma 21.6. *Suppose that $i : A \to B$ is a map of cosimplicial chain complexes (or simplicial modules) such that the map $i : A^0 \to B^0$ is a cofibration and all maps*

$$(d, i) : L^{n-1}B \cup_{L^{n-1}A} A^n \to B^n$$

are cofibrations. Then $i : A \to B$ is a cofibration for the projective model structure on cosimplicial chain complexes (respectively simplicial modules).

Proof. Suppose given a lifting problem

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

in cosimplicial objects, where the map i satisfies the conditions of the Lemma and p is a trivial projective fibration. Then p^0 is a trivial fibration so that the lifting θ^0 exists in the diagram

$$\begin{array}{ccc} A^0 & \longrightarrow & X^0 \\ {\scriptstyle i^0}\downarrow & \overset{\theta^0}{\nearrow} & \downarrow{\scriptstyle p^0} \\ B^0 & \longrightarrow & Y^0 \end{array}$$

in cosimplicial degree 0. Then the maps θ^n making up the lifting θ are inductively found by solving the lifting problems

$$\begin{array}{ccc} L^{n-1}B \cup_{L^{n-1}A} A^n & \longrightarrow & X^n \\ {\scriptstyle (d,i)}\downarrow & \overset{\theta^n}{\nearrow} & \downarrow{\scriptstyle (p,s)} \\ B^n & \longrightarrow & Y^n \times_{M^{n-1}Y} M^{n-1}X \end{array}$$

This can be done, since (d, i) is a cofibration by assumption and (p, s) is a trivial fibration by Corollary 21.4. □

Lemma 21.7. *Suppose that* $i : A \to B$ *is a map of bicosimplicial chain complexes (or simplicial modules) such that the map* $i : A^0 \to B^0$ *and all maps*

$$(d, i) : L^{n-1}B \cup_{L^{n-1}A} A^n \to B^n$$

are projective cofibrations of cosimplicial chain complexes (respectively simplicial modules). Then

$i : A \to B$ is a cofibration for the projective model structure on bicosimplicial chain complexes (respectively simplicial modules).

The proof is the same as that of Lemma 21.6.

Corollary 21.8. *1) Suppose that A is a cosimplicial chain complex (or simplicial module) such that A^0 is cofibrant and all maps*

$$d : L^{n-1}A \to A^n$$

are cofibrations. Then A is a projective cofibrant cosimplicial object.

2) Suppose that A is a bicosimplicial chain complex (or simplicial module) such that the cosimplicial object A^0 is projective cofibrant and all maps

$$d : L^{n-1}A \to A^n$$

are projective cofibrations. Then A is a projective cofibrant bicosimplicial object.

Examples:

1) The diagram

$$(\mathbf{m}, \mathbf{n}) \mapsto R(\Delta^m \times \Delta^n)$$

of Moore complexes is a bicosimplicial chain complex. Then

a) The cosimplicial chain complex

$$\mathbf{n} \mapsto R(\Delta^0 \times \Delta^n) \cong R(\Delta^n)$$

is projective cofibrant. In effect, $R(\Delta^0)$ is a cofibrant chain complex and the map

$$d : L^{n-1}R(\Delta^*) \to R(\Delta^n)$$

can be identified up to isomorphism with the inclusion $i_* : R(\partial\Delta^n) \to R(\Delta^n)$ which is induced by the inclusion $i : \partial\Delta^n \subset \Delta^n$. Then the chain complex $R(\Delta^n)$ is obtained from $R(\partial\Delta^n)$ by freely adjoining generators in all degrees, so that i_* is a cofibration.

b) The map

$$d : L^{n-1}R(\Delta^* \times \Delta^*) \to R(\Delta^n \times \Delta^*)$$

can be identified with the cosimplicial chain complex map

$$R(\partial\Delta^n \times \Delta^*) \to R(\Delta^n \times \Delta^*)$$

which is induced by the cosimplicial space map

$$\partial\Delta^n \times \Delta^* \to \Delta^n \times \Delta^*.$$

The cochain complex map

$$R(\partial\Delta^n \times \Delta^0) \to R(\Delta^n \times \Delta^0)$$

is a cofibration. The map

$$(d, i) : L^{m-1}R(\Delta^n \times \Delta^*) \cup R(\partial\Delta^n \times \Delta^m) \to R(\Delta^m \times \Delta^n)$$

can be identified up to isomorphism with the cofibration

$$R((\Delta^n \times \partial\Delta^m) \cup (\partial\Delta^n \times \Delta^m)) \to R(\Delta^n \times \Delta^m).$$

It follows that the bicosimplicial chain complex $R(\Delta^* \times \Delta^*)$ is projective cofibrant.

2) A similar argument shows that the bicosimplicial chain complex

$$(\mathbf{m}, \mathbf{n}) \mapsto R(\Delta^m) \otimes R(\Delta^n)$$

is also projective cofibrant.

Theorem 21.9. *There is a map of bicosimplicial chain complexes*

$$f : R(\Delta^m \times \Delta^n) \to R(\Delta^m) \otimes R(\Delta^n)$$

which is a lifting of the identity on H_0, and any two such maps are chain homotopic. The map f is a chain homotopy equivalence.

Proof. There are isomorphisms of bicomsimplicial abelian groups

$$H_0R(\Delta^m \times \Delta^n) \xrightarrow{\cong} H_0R(\Delta^0 \times \Delta^0) \cong R,$$

and

$$H_0(R(\Delta^m) \otimes R(\Delta^n)) \xrightarrow{\cong} H_0(R(\Delta^0) \otimes R(\Delta^0)) \cong R.$$

It follows that there are trivial projective fibrations

$$R(\Delta^m \times \Delta^m) \to R[0]$$

and

$$R(\Delta^n) \otimes R(\Delta^m) \to R[0]$$

of bicosimplicial chain complexes, where $R[0]$ is a constant bicosimplicial object.

The bicosimplicial chain complexes $R(\Delta^m \times \Delta^n)$ and $R(\Delta^m) \otimes R(\Delta^n)$ are therefore projective cofibrant models for the same thing so the map f exists by solving a lifting problem

$$\begin{array}{ccc} & & R(\Delta^*) \otimes R(\Delta^*) \\ & \overset{f}{\nearrow} & \downarrow \\ R(\Delta^* \times \Delta^*) & \longrightarrow & R[0] \end{array}$$

Any two such lifts are chain homotopic, because the chain homotopy construction for chain complexes defines a path object for the projective model structure.

The map $g : R(\Delta^*) \otimes R(\Delta^*) \to R(\Delta^* \times \Delta^*)$ exists as a covering of the identity on the constant object $R[0]$, and is a chain homotopy inverse of f, by the same argument. □

We will see, in the next section, that Theorem 21.9 is a form of the Eilenberg-Zilber Theorem.

Observe as well that a similar argument which is based on the projective model structure for tricosimplicial chain complexes gives the following:

Proposition 21.10. *There is a chain homotopy equivalence of tricosimplicial chain complexes*

$f : R(\Delta^m \times \Delta^n \times \Delta^k) \xrightarrow{\simeq} R(\Delta^m) \otimes R(\Delta^n) \otimes R(\Delta^k).$

which lifts the isomorphism

$H_0 R(\Delta^0 \times \Delta^0 \times \Delta^0) \cong H_0(R(\Delta^0) \otimes R(\Delta^0) \otimes R(\Delta^0)).$

Any two such maps are chain homotopic.

The proof amounts to showing that the tricosimplicial objects

$$R(\Delta^* \times \Delta^* \times \Delta^*)$$

and

$$R(\Delta^*) \otimes R(\Delta^*) \otimes R(\Delta^*)$$

are cofibrant for the projective model structure on tricosimplicial chain complexes.

Higher dimensional analogues, for n-cosimplicial chain complexes, of Theorem 21.9 and Proposition 21.10 can also be proved.

References

[1] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations.* Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 304.