Lecture 010 (November 27, 2009)

22 Bisimplicial modules

A *bisimplicial R-module* A is a functor

$$A : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to R - \mathbf{Mod}.$$

Equivalently, A is a simplicial object in simplicial modules. The category $s^2(R - \mathbf{Mod})$ of bisimplicial modules is the obvious thing, namely bisimplicial modules and their natural transformations.

Usually, we write $A_{p,q} = A(\mathbf{p}, \mathbf{q})$. The object A has horizontal and vertical simplicial structure maps, defined by

$$\theta_h^* = (\theta, 1)^* : A_{p,q} \to A_{r,q}$$

and

$$\gamma_v^* = (1, \gamma)^* : A_{p,q} \to A_{p,s}$$

for ordinal number morphisms $\theta : \mathbf{r} \to \mathbf{p}$ and $\gamma : \mathbf{s} \to \mathbf{q}$, respectively.

A bisimplicial module A has an associated *Moore bicomplex*, usually with the same notation. The group of (p, q)-chains of A is just the group $A_{p,q}$.

The horizontal boundary $\partial_h : A_{p,q} \to A_{p-1,q}$ is defined by

$$\partial_h = \sum_{i=0}^{p} (-1)^i (d_i)_h.$$

The vertical boundary $\partial_v : A_{p,q} \to A_{p,q-1}$ is defined by

$$\partial_v = \sum_{i=0}^{q} (-1)^i (d_i)_h.$$

Examples:

1) Suppose that C and D are simplicial modules. Then the *external tensor product* $C \otimes D$ with

$$(C \otimes D)_{p,q} = C_p \otimes D_q$$

is a simplicial module. If X and Y are simplicial sets, then the external product $X \times Y$ with $(X \times Y)_{p,q} = X_p \times Y_q$ is a bisimplicial set, and the associated free bisimplicial R-module $R(X \times Y)$ has a canonical isomorphism

$$R(X \times Y) \cong R(X) \otimes R(Y)$$

with the external tensor product.

2) Every bisimplicial set $X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathbf{Set}$ determines a free bisimplicial module $R(X)$ with

$$R(X)_{p,q} = R(X_{p,q}).$$

Bisimplicial sets are a major source of bisimplicial bisimplicial modules, and hence bicomplexes, via this mechanism.

3) Suppose that I is a small category and that the functor $Y : I \to \mathbf{Set}$ is a functor. The *translation category* $E_I Y$ has for objects all pairs (i, x) with $x \in Y(i)$. The morphisms $\alpha : (i, x) \to (j, y)$ of $E_I Y$ are morphisms $\alpha : i \to j$ such that $\alpha_*(x) = y$. The n-simplices of the nerve $BE_I Y$ are the strings of morphisms

$$(i_0, x_0) \xrightarrow{\alpha_1} (i_1, x_1) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} (i_n, x_n)$$

in the category $E_I Y$ of length n. The elements $x_1, \dots, x_n$ are completely specified by x_0 and the morphism $\alpha_1, \dots, x_n$, so that there is a bijection

$$B(E_I Y)_n \cong \bigsqcup_{i_0 \to \dots \to i_n} Y(i_0).$$

If $X : I \to s\mathbf{Set}$ is an I-diagram of simplicial sets, then the functors $X_n : I \to \mathbf{Set}$ form a simplicial object in set-valued diagrams, and so the simplicial sets $B(E_I X_n)$ form a bisimplicial set $B(E_I X)$, which in "horizontal degree n" has the form

$$\bigsqcup_{i_0 \to \dots \to i_n} X(i_0).$$

If Z is a bisimplicial set, the *diagonal* $d(Z)$ is a simplicial set with $d(Z)_n = Z_{n,n}$ and with simplicial structure maps

$$Z_{n,n} \xrightarrow{(\theta,\theta)^*} Z_{m,m}$$

associated to $\theta : \mathbf{m} \to \mathbf{n}$.

The diagonal $d(B(E_I X))$ is usually called the *homotopy colimit* of the diagram X, because that's what it is.

Given a bisimplicial module A, there are two functorially associated chain complexes:

1) the *total complex* $\mathrm{Tot}(A)$ of the associated Moore bicomplex A, and
2) the Moore complex $d(A)$ of the diagonal simplicial module.

These two constructions are related by the Generalized Eilenberg-Zilber Theorem, of Dold and Puppe [1]:

Theorem 22.1. *The chain complexes* $d(A)$ *and* $\mathrm{Tot}(A)$ *are naturally chain homotopy equivalent, for all bisimplicial modules* A. *Any two natural maps* $d(A) \to \mathrm{Tot}(A)$ *which induce the same map*

$$H_0(dR(\Delta^{0,0})) \to H_0(\mathrm{Tot}\, R(\Delta^{0,0}))$$

are naturally chain homotopic.

Proof. The external product $\Delta^p \times \Delta^q$ is the (p,q)-cell $\Delta^{p,q} = \hom(\ ,(\mathbf{p},\mathbf{q}))$ in the category of bisimplicial sets, and this identification is natural in p, and q.

There is a bisimplicial module map

$$\phi_{p,q} : A_{p,q} \otimes \Delta^{p,q} \to A.$$

The map

$$\phi_{p,q} : \bigoplus_{(\mathbf{r},\mathbf{s}) \xrightarrow{(\theta,\gamma)} (\mathbf{p},\mathbf{q})} A_{p,q} \to A_{r,s}$$

is the map $(\theta,\gamma)^* : A_{p,q} \to A_{r,s}$ on the summand corresponding to the morphism (θ,γ). Note that $A^{p,q} \otimes \Delta^{p,q}$ is shorthand notation for the object

$$A^{p,q} \otimes R\Delta^{p,q}.$$

All diagrams

$$\begin{tikzcd}
 & A_{p,q} \otimes \Delta^{p,q} \arrow[dr, "\phi_{p,q}"] & \\
A_{p,q} \otimes \Delta^{r,s} \arrow[ur, "1\otimes(\theta,\gamma)"] \arrow[dr, "(\theta,\gamma)^*\otimes 1"'] & & A \\
 & A_{r,s} \otimes \Delta^{r,s} \arrow[ur, "\phi_{r,s}"'] &
\end{tikzcd}$$

commute, and this collection of diagrams in initial among all collections of commutative diagrams

$$\begin{tikzcd}
& A_{p,q} \otimes \Delta^{p,q} \arrow[dr, "f_{p,q}"] & \\
A_{p,q} \otimes \Delta^{r,s} \arrow[ur, "1\otimes(\theta,\gamma)"] \arrow[dr, "(\theta,\gamma)^{*}\otimes 1"'] & & B \\
& A_{r,s} \otimes \Delta^{r,s} \arrow[ur, "f_{r,s}"'] &
\end{tikzcd}$$

It follows that there is a short exact sequence

$$\bigoplus_{(\mathbf{r},\mathbf{s})\xrightarrow{(\theta,\gamma)}(\mathbf{p},\mathbf{q})} A_{p,q}\otimes\Delta^{r,s} \to \bigoplus_{(p,q)} A_{p,q}\otimes\Delta^{p,q} \to A \to 0 \tag{1}$$

which is natural in bisimplicial abelian groups A. The functors $A \mapsto \mathrm{Tot}(A)$ and $A \mapsto d(A)$ are exact and commute with tensoring by abelian groups.

Theorem 21.9 implies that there is a map of chain complexes

$$f : d(R(\Delta^{p,q})) \to \mathrm{Tot}(R(\Delta^{p,q}))$$

which is natural in (p,q), and induces the identity map

$$R \cong H_0(d(R(\Delta^{0,0})) \to H_0(\mathrm{Tot}(R(\Delta^{0,0})) \cong R.$$

This natural map f determines a comparison diagram

$$\begin{array}{ccc} \oplus_{(\theta,\gamma)}\ A_{p,q}\otimes d(R\Delta^{r,s}) & \longrightarrow & \oplus_{(p,q)}\ A_{p,q}\otimes d(R\Delta^{p,q}) \\ \downarrow & & \downarrow \\ \oplus_{(\theta,\gamma)}\ A_{p,q}\otimes \mathrm{Tot}(R\Delta^{r,s}) & \longrightarrow & \oplus_{(p,q)}\ A_{p,q}\otimes \mathrm{Tot}(R\Delta^{p,q}) \end{array}$$

It follows that there is an induced natural map

$$f_* : d(A) \to \mathrm{Tot}(A).$$

Any two natural maps $f, f' : d(A) \to \mathrm{Tot}(A)$ having the same effect on $H_0(d(R(\Delta^{0,0})))$ restrict to bicosimplicial chain complex maps

$$f, f' : d(R(\Delta^{p,q})) \to \mathrm{Tot}(R(\Delta^{p,q})),$$

which maps are chain homotopic by Theorem 21.9. The chain homotopy construction respects colimits, so the maps $f, f' : d(A) \to \mathrm{Tot}(A)$ are naturally chain homotopic.

The map f has a natural chain homotopy inverse

$$g : \mathrm{Tot}(R\Delta^{p,q}) \to d(R\Delta^{p,q})$$

which induces a natural map

$$g_* : \mathrm{Tot}(A) \to d(A),$$

and g_* is a chain homotopy inverse for f_* □

Remark 22.2. Theorem 22.1 appears as Theorem IV.2.4 in [2]. There is an error in the proof of that result which is repaired in the proof given here.

The following is the classical Eilenberg-Zilber Theorem:

Corollary 22.3 (Eilenberg-Zilber)**.** *Suppose that X and Y are simplicial sets. Then there is a natural chain homotopy equivalence*

$$f : R(X \times Y) \xrightarrow{\simeq} R(X) \otimes R(Y).$$

Any two such natural maps f are naturally chain homotopic

Remark 22.4. There are explicit choices for the natural map f and its natural homotopy inverse g, namely the Alexander-Whitney and shuffle maps respectively. See the ancient texts for descriptions of these maps.

The Eilenberg-Zilber Theorem leads immediately to the definition of the cup product for the cohomology $H^*(X, R)$ of a space X with coefficients in a ring R. Suppose that the chain maps

$$\alpha : R(X) \to R[-p], \;\; \beta : R(X) \to R[-q]$$

represent elements (chain homotopy classes) $[\alpha] \in H^p(X, R)$ and $[\beta] \in H^q(X, R)$, respectively. Then the composite chain map

$$\begin{tikzcd}
R(X) \arrow[r, "\Delta_*"] & R(X \times X) \arrow[r, "f"] & R(X) \otimes R(X) \arrow[d, "\alpha\otimes\beta"] \\
& & R[-p] \otimes R[-q] \cong R[-p-q]
\end{tikzcd}$$

represents an element

$$[\alpha] \cup [\beta] \in H^{p+q}(X, R) = [R(X), R[-p-q]],$$

which is the *cup product* of the classes $[\alpha]$ and $[\beta]$.

There is a natural *twist map*

$$\tau : R(X) \otimes R(Y) \to R(Y) \otimes R(X)$$

which is defined on generators by

$$\tau(x \otimes y) = (-1)^{pq} y \otimes x$$

for $x \in X_p$ and $y \in Y_q$. There is a natural map $t : X \times Y \to Y \times X$ of simplicial sets which is defined by $t(x, y) = (y, x)$. The diagram

$$\begin{tikzcd}
H_0R(\Delta^0 \times \Delta^0) \arrow[r, "f_*"] \arrow[d, "t_*"'] & H_0(R(\Delta^0) \otimes R(\Delta^0)) \arrow[d, "\tau_*"] \\
H_0R(\Delta^0 \times \Delta^0) \arrow[r, "f_*"'] & H_0(R(\Delta^0) \otimes R(\Delta^0))
\end{tikzcd}$$

commutes, and so it follows that the diagram

$$\begin{tikzcd} R(X\times Y) \arrow[r, "f"] \arrow[d, "t_*"'] & R(X)\otimes R(Y) \arrow[d, "\tau"] \\ R(Y\times X) \arrow[r, "f"'] & R(Y)\otimes R(X) \end{tikzcd}$$

is naturally chain homotopy comutative.

It follows (exercise) that the cup product on $H^*(X,R)$ is *graded commutative* in the sense that the equation

$$[\alpha]\cup[\beta] = (-1)^{pq}[\beta]\cup[\alpha]$$

for $[\alpha]\in H^p(X,R)$ and $[\beta]\in H^q(X,R)$, provided that R is a commutative ring.

It is also an exercise to show that the cup product on $H^*(X,R)$ is associative. Use Proposition 21.10.

23 Bisimplicial sets and spectral sequences

There are some simple consequences of the Generalized Eilenberg-Zilber Theorem (Theorem 22.1) that we can just write down:

Corollary 23.1. *Suppose that A is a bisimplicial module. Then there are spectral sequences*

$$E^2_{p,q} = H^h_p H^v_q A \Rightarrow H_{p+q}d(A) = \pi_{p+q}(d(A),0),$$

and

$$E^2_{p,q} = H^v_p H^h_q A \Rightarrow H_{p+q} d(A).$$

Corollary 23.2. *Suppose that A is an abelian group and that X and Y are spaces (ie. simplicial sets). Then there is a spectral sequence*

$$E^2_{p,q} = H_p(X, H_q(Y, A)) \Rightarrow H_{p+q}(X \times Y, A).$$

The spectral sequence of Corollary 23.2 is the *Künneth spectral sequence*; it has analogs in a variety of contexts, including categories of diagrams of spaces. It makes virtually no sense to use this result for ordinary spaces, where it collapses to give the ordinary Künneth Theorem (Theorem 10.11). The diagram-theoretic variants have much more content.

Here's a more interesting construction. Let $p : X \to Y$ be a morphism of simplicial sets, and consider the pullbacks

$$\begin{tikzcd} p^{-1}(\sigma) \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\sigma"'] & Y \end{tikzcd}$$

over all simplices $\sigma : \Delta^n \to Y$ of the base. Any morphism of simplices

$$\begin{tikzcd} \Delta^m \arrow[d] \arrow[dr, "\tau"] & \\ & Y \\ \Delta^n \arrow[ur, "\sigma"'] & \end{tikzcd} \tag{2}$$

induces a morphism $p^{-1}(\tau) \to p^{-1}(\sigma)$, and the collection of induced maps $p^{-1}(\sigma) \to X$ induces a weak equivalence

$$\underrightarrow{\text{holim}}_{\Delta^n \xrightarrow{\sigma} Y} \, p^{-1}(\sigma) \xrightarrow{\simeq} X. \tag{3}$$

See Lemma IV.5.2 of [2] — the proof of this result is not difficult.

The simplices $\Delta^n \to Y$ and their morphisms (2) define a category $\mathbf{\Delta}/Y$, which is called the *simplex category* of Y. The functor $\sigma \mapsto p^{-1}(\sigma)$ defined above is a functor $\mathbf{\Delta}/X \to s\mathbf{Set}$ which indexes the homotopy colimit in (3).

It follows that there is a spectral sequence, with

$$E^2_{p,q} = \varinjlim{}_p H_q(p^{-1}(*), A) \Rightarrow H_{p+q}(Y, A) \tag{4}$$

This is the *Grothendieck spectral sequence* for the map p.

If the map p happens to be a fibration, this spectral sequence reduces to the standard variants of the Serre spectral sequence. All induced maps

$p^{-1}(\tau) \to p^{-1}(\sigma)$ are weak equivalences, and the functor $\sigma \mapsto H_q(p^{-1}(\sigma))$ are morphism inverting and induce functors $\pi(Y) \to \mathbf{Ab}$ which are defined on the fundamental groupoid $\pi(Y)$ of Y.

If Y is a simply connected space, then its fundamental groupoid is trivial and the spectral sequence (4) has the form

$$E^2_{p,q} = H_p(Y, H_q(F, A)) \Rightarrow H_{p+q}(X, A). \qquad (5)$$

This is the classical homology *Serre spectral sequence* for a fibration p with simply connected base. Here, F is any fibre $p^{-1}(x)$ over a vertex $x : \Delta^0 \to Y$ of Y.

The Künneth spectral sequence of Corollary 23.2 is a very simple example of the Serre spectral sequence.

24 Some calculations

In general (from Section 6), there is a natural short exact sequence of chain complexes

$$0 \to K \to \tilde{C} \to C \to 0,$$

where $\tilde{C}$ is acyclic. The kernel K is isomorphic to $F_0C[1]$, where F_0C is the kernel of the canonical map $C \to P_0C$ to the 0^{th} Postnikov section. In particular, if $C = A[-n]$ for some R-module A where $n \geq 1$ then $F_0A[-n] = A[-n]$ and so $K = A[-n+1]$. It follows in particular (via the Dold-Kan correspondence) that for $n \geq 1$ there are short exact sequences of simplicial modules

$$0 \to K(A, n-1) \to E \to K(A, n) \to 0$$

and hence fibre sequences

$$K(A, n-1) \to E \to K(A, n)$$

such that E is contractible. These fibre sequences are natural in R-modules A. Recall in all that follows that $K(A, 1) = BA$.

There are various ways to show that the circle S^1 (or simplicial circle $\Delta^1/\partial\Delta^1$) is weakly equivalent to $B\mathbb{Z}$. The most common is to observe that the winding function $\mathbb{R} \to S^1$ defined by $t \mapsto e^{2\pi it}$ is a Serre fibration with fibre $\mathbb{Z}$. This construction can be modelled in simplicial sets, and it is an exercise to do so. It is also a standard exercise to show that there are isomorphisms

$$H_n(B\mathbb{Z}, \mathbb{Z}) \cong H_n(S^1, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, 1, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

Example 24.1. In the fibre sequence

$$B\mathbb{Z} \to E \to K(\mathbb{Z}, 2)$$

the space E is contractible and $K(\mathbb{Z}, 2)$ is simply connected. The Serre spectral sequence for this fibre sequence has the form

$$E^2_{p,q} = H_p(K(\mathbb{Z}, 2), H_q(B\mathbb{Z}, \mathbb{Z})) \Rightarrow H_{p+q}(E, \mathbb{Z}).$$

The space E is contractible, so that

$$H_n(E) = H_n(E, \mathbb{Z}) = 0$$

for $n \geq 1$. It follows that $E^\infty_{p,q} = 0$ for $(p, q) \neq (0, 0)$. In effect, the filtration F_pH_n of $H_n = H_n(E)$ has the form

$$\begin{tikzcd} F_0H_n \arrow[r] \arrow[d, "\cong"'] & F_1H_n \arrow[r] \arrow[d] & \cdots & F_{n-1}H_n \arrow[r] & F_nH_n \arrow[r, "="] \arrow[d] & H_n \\ E^\infty_{0,n} & E^\infty_{1,n-1} & & & E^\infty_{n,0} & \end{tikzcd}$$

so that $E^\infty_{p,n-p} = 0$ for $p \leq n$ and for all n, by induction on p.

Now look at the E^2-term:

- $E^2_{1,0} = E^\infty_{1,0} = 0$ since there are no non-trivial differentials in or out. Thus, $H_1(K(\mathbb{Z}, 2)) = 0$.
- The differential $d_2 : E^2_{2,0} \to E^2_{0,1}$ has kernel $E^\infty_{2,0}$ and cokernel $E^\infty_{0,1}$, and is therefore an iso-

morphism. It follows that d_2 induces an isomorphism

$$H_2(K(\mathbb{Z},2)) \cong H_1(B\mathbb{Z}) \cong \mathbb{Z}.$$

Also, $E^2_{2,1} \cong \mathbb{Z}$.

I claim that $H_{2n}(K(\mathbb{Z},2) \cong \mathbb{Z}$ and $H_{2n+1}(K(\mathbb{Z},2) = 0$ for all n.

The argument is an induction on n which repeats the arguments just seen. The differential

$$H_{2n+2}(K(\mathbb{Z},2)) \cong E^2_{2n+2,0} \xrightarrow{d_2} E^2_{2n,1} \cong \mathbb{Z}$$

which has kernel $E^\infty_{2n+2,0}$ and cokernel $E^\infty_{2n,1}$ and therefore must be an isomorphism. There are no non-trivial differentials out of $E^2_{2n+3,0}$ so that

$$H_{2n+3}(K(\mathbb{Z},2)) \cong E^2_{2n+3,0} \cong E^\infty_{2n+3,0} = 0.$$

In general, for a functor $X : I \to \mathbf{Set}$, there is a functor $E_I X \to I$ which is defined by $(i,x) \mapsto i$. Identify a group G with a one-object category, with object $*$.

The group multiplication $G \times G \to G$ defines a functor $G : G \to \mathbf{Set}$ which takes the object $*$ to G and takes a morphism $* \xrightarrow{g} *$ to the function $G \to G$ which is defined by $h \mapsto hg^{-1}$. It is standard to write $EG = B(E_G G)$ for the nerve of the corresponding translation category. The functor $E_G(G) \to G$ induces a simplicial set map $EG \to BG$. Observe that there is a G-action $G \times EG \to EG$ which is defined by the left G-action on the object level, and that the quotient map $EG \to EG/G$ can be identified with the canonical map $EG \to BG$.

The element $(*, e)$ is terminal in $E_G(G)$ so that EG is constractible. The corresponding weak equivalence $\mathbb{Z}(EG) \to \mathbb{Z}$ of chain complexes gives $\mathbb{Z}(EG)$ the structure of a free G-resolution of the trivial G-modules $\mathbb{Z}$. Upon tensoring with any trivial G-modules A over G, we find an isomorphism

$$\mathbb{Z}(EG) \otimes_G A \cong \mathbb{Z}(BG) \otimes_{\mathbb{Z}} A.$$

It follows that there is an isomorphism

$$\operatorname{Tor}_p^G(\mathbb{Z}, A) \cong H_p(BG, A)$$

for all trivial G-modules A, and for all $p \geq 0$.

Example 24.2. We have seen (Section 9) that

$$H_p(B(\mathbb{Z}/n), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0, \\ \mathbb{Z}/n & \text{if } p = 2n+1,\ n \geq 0, \\ 0 & \text{if } p = 2n,\ n > 0. \end{cases}$$

Recall that there is a fibre sequence

$$B(\mathbb{Z}/n) \to E \to K(\mathbb{Z}/n, 2)$$

with E contractible. The corresponding Serre spectral sequence has the form

$$E^2_{p,q} = H_p(K(\mathbb{Z}/n, 2), H_q(B(\mathbb{Z}/n))) \Rightarrow H_{p+q}(E).$$

We have $E^\infty_{p,q} = 0$ for $(p, q) \neq (0, 0)$, as before. It follows that

$$H_1(K(\mathbb{Z}/n, 2)) = E^2_{1,0} = E^\infty_{1,0} = 0.$$

The sequence

$$0 \to E^\infty_{2,0} \to E^2_{2,0} \xrightarrow{d^2} E^2_{0,1} \to E^\infty_{0,1} \to 0$$

is exact, so it also follows that the differential d^2 defines an isomorphism

$$H_2(K(\mathbb{Z}/n, 2)) \cong H_1(B(\mathbb{Z}/n)) \cong \mathbb{Z}/n.$$

The groups $E^2_{1,1}$ and $E^2_{0,2} = H_2(B(\mathbb{Z}/n))$ are trivial, so that

$$H_3(K(\mathbb{Z}/n, 2)) = E^2_{3,0} = E^\infty_{3,0} = 0.$$

Example 24.3. Observe (exercise) that there is a natural isomorphism

$$H_1(BG) \cong G/[G, G]$$

which induced by the canonical map $BG \to \mathbb{Z}(BG)$ — this is the Hurewicz map for BG. In general, the canonical map $X \to \mathbb{Z}(X)$ is *Hurewicz map* for a simplicial set X. It follows that there is a natural isomorphism

$$H_1(BA) \cong A$$

for abelian groups A.

The spectral sequence for the fibration

$$BA \to E \to K(A,2)$$

can be used to show that $H_1(K(A,2)) = 0$ and that the differential

$$H_2(K(A,2)) = E^2_{2,0} \xrightarrow{d^2} E^1_{0,1} = H_1(BA) \cong A$$

is an isomorphism, just as before.

We know from the previous examples that

$$H_3(K(B,2)) = 0$$

if B is $\mathbb{Z}$ or $\mathbb{Z}/n$. If B_1 and B_2 are abelian groups which satisfy

$$H_3(K(B_i,2)) = 0$$

then

$$H_3(K(B_1 \oplus B_2, 2)) = 0$$

by a Künneth Theorem argument (exercise). It follows that all finitely generated abelian groups B satisfy $H_3(K(B,2)) = 0$. Finally, every abelian group A is a filtered colimit of its finitely generated abelian subgroups, and so every abelian group A satisfies $H_3(K(A,2)) = 0$.

Example 24.4. Suppose that A is an abelian group. I claim that

$$H_{n+1}(K(A,n)) = 0 = H_p(K(A,n))$$

for all $n \geq 2$ and for $0 < p < n$, and that there is a natural isomorphism

$$H_n(K(A,n)) \cong A.$$

There is a fibre sequence

$$K(A,n) \to E \to K(A,n+1)$$

with E contractible. The Serre spectral sequence

$$E^2_{p,q} = H_p(K(A,n{+}1), H_q(K(A,n))) \Rightarrow H_{p+q}(E)$$

converges to the homology of a contractible space so that $E^\infty_{p,q} = 0$ for $(p,q) \neq (0,0)$. It follows that

$$H_p(K(A,n+1)) \cong E^2_{p,0} = E^\infty_{p,0} = 0$$

for $0 < p < n{+}1$, because $E^2_{p,q} = 0$ for $0 < q < n$. The differential

$$E^2_{n+1,0} = E^{n+1}_{n+1,0} \xrightarrow{d^{n+1}} E^{n+1}_{0,n} = E^2_{0,n}$$

is an isomorphism, since the kernel and cokernel are E^∞ terms. Thus, the diffential d^{n+1} defines a natural isomorphism

$$H_{n+1}(K(A, n+1)) \cong H_n(K(A, n)) \cong A.$$

The groups $E^2_{n+1-p,p}$ are 0 for $0 < p \leq n+1$; in particular,

$$E^2_{0,n+1} = H_{n+1}(K(A, n)) = 0$$

by the inductive assumption. But then

$$H_{n+2}(K(A, n+1)) = E^2_{n+2,0} = E^\infty_{n+2,0} = 0.$$

The results given in Example 24.4 are the calculational results which are required for the proof of the Hurewicz Theorem (Theorem III.3.7 of [2]), which says that if a pointed space X is n-connected and $n \geq 1$, then the Hurewicz map

$$X \to \mathbb{Z}(X) \to \tilde{\mathbb{Z}}(X) = \mathbb{Z}(X)/\mathbb{Z}(*)$$

induces morphisms

$$h : \pi_p(X) \to \tilde{H}_p(X, \mathbb{Z})$$

which are isomorphisms for $p \leq n+1$ and an epimorphism if $p = n+2$.

References

[1] Albrecht Dold and Dieter Puppe. Homologie nicht-additiver Funktoren. Anwendungen. *Ann. Inst. Fourier Grenoble*, 11:201–312, 1961.

[2] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[3] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.