Contents

14 Simplicial groups

A **simplicial group** is a functor $G : \Delta^{op} \to \mathbf{Grp}$.

A morphism of simplicial groups is a natural transformation of such functors.

The category of simplicial groups is denoted by $s\mathbf{Gr}$.

We use the same notation for a simplicial group G and its underlying simplicial set.

Lemma 14.1 (Moore)**.** *Every simplicial group is a Kan complex.*

The proof of Lemma 14.1 involves the classical simplicial identities. Here's the full list:

$$d_i d_j = d_{j-1} d_i \text{ if } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ 1 & \text{if } i = j, j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \text{ if } i \leq j.$$

Proof. Suppose

$$(x_0, \dots, x_{k-1}, x_{\ell-1}, \dots, x_n)$$

($\ell \geq k+2$) is a family of $(n-1)$-simplices of G such that $d_i x_j = d_{j-1} x_i$ for $i < j$.

Suppose there is an n-simplex $y \in G$ such that $d_i(y) = x_i$ for $i \leq k-1$ and $i \geq \ell$.

Then $d_i x_{\ell-1} = d_i d_{\ell-1}(y)$ for $i \leq k-1$ and $i \geq \ell-1$, and

$$d_i(s_{\ell-2}(x_{\ell-1} d_{\ell-1}(y^{-1}))y) = x_i$$

for $i \leq k-1$ and $i \geq \ell-1$.

Alternatively, suppose $S \subset \mathbf{n}$ and $|S| \leq n$.

Write $\Delta^n\langle S\rangle$ for the subcomplex of $\partial\Delta^n$ which is generated by the faces $d_i \iota_n$ for $i \in S$.

Write

$$G_{\langle S\rangle} := \hom(\Delta^n\langle S\rangle, G).$$

Restriction to faces determines a group homomorphism $d : G_n \to G_{\langle S\rangle}$.

We show that d is surjective, by induction on $|S|$.

There is a $j \in S$ such that either $j-1$ or $j+1$ is not a member of S, since $|S| \leq n$.

Pick such a j, and suppose $\theta : \Delta^n\langle S\rangle \to G$ is a simplicial set map such that $\theta_i = \theta(d_i \iota_n) = e$ for $i \neq j$.

Then there is a simplex $y \in G_n$ such that $d_j(y) = \theta$.

For this, set $y = s_j\theta_j$ if $j+1 \notin S$ or $y = s_{j-1}\theta_j$ if $j-1 \notin S$.

Now suppose $\sigma : \Delta^n\langle S\rangle \to G$ is a simplicial set map, and let $\sigma^{(j)}$ denote the composite

$$\Delta^n\langle S - \{j\}\rangle \subset \Delta^n\langle S\rangle \xrightarrow{\sigma} G.$$

Inductively, there is a $y \in G_n$ such that $d(y) = \sigma^{(j)}$, or such that $d_i y = \sigma_i$ for $i \neq j$. Let y_S be the restriction of y to $\Delta^n\langle S\rangle$.

The product $\sigma \cdot y_S^{-1}$ is a map such that $(\sigma \cdot y_S^{-1})_i = e$ for $i \neq j$. Thus, there is a $\theta \in G_n$ such that $d(\theta) = \sigma \cdot y_S^{-1}$.

Then $d(\theta \cdot y) = \sigma$. □

The following result will be useful:

Lemma 14.2. *1) Suppose that $S \subset \mathbf{n}$ such that $|S| \leq n$. Then the inclusion $\Delta^n\langle S\rangle \subset \Delta^n$ is anodyne.*

2) If $T \subset S$, and $T \neq \emptyset$, then $\Delta^n\langle T\rangle \subset \Delta^n\langle S\rangle$ is anodyne.

Proof. For 1), we argue by induction on n.

Suppose that k is the largest element of S. There is a pushout diagram

$$\begin{tikzcd} \Delta^{n-1}\langle S-\{k\}\rangle \arrow[r, "d^{k-1}"] \arrow[d] & \Delta^{n}\langle S-\{k\}\rangle \arrow[d] \\ \Delta^{n-1} \arrow[r, "d^k"'] & \Delta^n\langle S\rangle \end{tikzcd} \tag{1}$$

By adding $(n-1)$-simplices to $\Delta^n\langle S\rangle$, one finds a $k \in \mathbf{n}$ such that the maps in the string

$$\Delta^n\langle S\rangle \subset \Lambda^n_k \subset \Delta^n$$

are anodyne. $\square$

Write

$$N_n(G) = \cap_{i<n}\ker(d_i : G_n \to G_{n-1}).$$

The simplicial identities imply that the face map d_n induces a homomorphism

$$d_n : N_n(G) \to N_{n-1}(G).$$

In effect, if $i < n-1$, then $i < n$ and

$$d_i d_n(x) = d_{n-1} d_i(x) = e$$

for $x \in N_n(G)$.

The image of $d_n : N_n(G) \to N_{n-1}(G)$ is normal in G_n, since

$$d_n((s_{n-1}x)y(s_{n-1}x)^{-1}) = x d_n(y) x^{-1}.$$

for $y \in N_{n+1}(G)$ and $x \in G_n$.

Lemma 14.3. *1) There are isomorphisms*

$$\frac{\ker(d_n : N_n(G) \to N_{n-1}(G))}{\operatorname{im}(d_{n+1} : N_{n+1}(G) \to N_n(G)}) \xrightarrow{\cong} \pi_n(G,e)$$

for all $n \geq 0$.

2) The homotopy groups $\pi_n(G,e)$ are abelian for $n \geq 1$.

3) There are isomorphisms

$$\pi_n(G,x) \cong \pi_n(G,e)$$

for any $x \in G_0$.

Proof. The group multiplication on G induces a multiplication on $\pi_n(G,e)$ which has identity represented by $e \in G$ and satisfies an interchange law with the standard multiplication on the simplicial homotopy group $\pi_n(G,e)$.

Thus, the two group structures on $\pi_n(G,e)$ coincide and are abelian for $n \geq 1$.

Multiplication by the vertex x defines a group homomorphism

$$\pi_n(G,e) \to \pi_n(G,x),$$

with inverse defined by multiplication by x^{-1}. $\square$

Corollary 14.4. *A map $f : G \to H$ of simplicial groups is a weak equivalence if and only if it induces isomorphisms*

$$\begin{aligned}
&\pi_0(G) \cong \pi_0(H), \ \ \textit{and} \\
&\pi_n(G,e) \cong \pi_n(H,e), \ \ n \geq 1.
\end{aligned}$$

Lemma 14.5. *Suppose $p : G \to H$ is a simplicial group homomorphism such that $p : G_i \to H_i$ is a surjective group homomorphism for $i \leq n$.*

Then p has the RLP wrt all morphisms $\Lambda^m_k \subset \Delta^m$ for $m \leq n$.

Proof. Suppose given a commutative diagram

$$\begin{tikzcd}
\Lambda^m_k \arrow[r, "\alpha"] \arrow[d] & G \arrow[d, "p"] \\
\Delta^m \arrow[r, "\beta"'] & H
\end{tikzcd}$$

and let K be the kernel of p.

Since $m \leq n$ there is a simplex $\theta : \Delta^m \to G$ such that $p\theta = \beta$. Then $p\theta|_{\Lambda^m_k} = p\alpha$, and there is a simplex $\gamma : \Delta^m \to K$ such that the diagram

$$\begin{tikzcd}
\Lambda^m_k \arrow[r, "\alpha(\theta|_{\Lambda^m_k})^{-1}"] \arrow[d] & K \\
\Delta^m \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

commutes, since K is a Kan complex (Lemma 14.1). Then $(\gamma\theta)|_{\Lambda^m_k} = \alpha$ and $p(\gamma\theta) = \beta$. $\square$

Lemma 14.6. *Suppose $p : G \to H$ is a simplicial group homomorphism such that the induced homomorphisms $N_i(G) \to N_i(H)$ are surjective for $i \leq n$.*

Then p is surjective up to level n.

Proof. Suppose $\beta : \Delta^n \to H$ is an n-simplex, and suppose that p is surjective up to level $n-1$.

p is surjective up to level $n-1$ and is a fibration up to level $n-1$ by Lemma 14.5.

It follows from the proof of Lemma 14.2, ie. the pushouts (1), that p has the RLP wrt to the inclusion $\Delta^{n-1} \subset \Delta^n\langle S\rangle$ defined by the inclusion of the minimal simplex of S.

Thus, there is map $\alpha : \Lambda^n_n \to G$ such that the following commutes

$$\begin{tikzcd} \Lambda^n_n \arrow[r, dotted, "\alpha"] \arrow[d] & G \arrow[d, "p"] \\ \Delta^n \arrow[r, "\beta"'] & H \end{tikzcd}$$

Choose a simplex $\theta : \Delta^n \to G$ which extends α. Then $(\beta p(\theta)^{-1})|_{\Lambda^n_n} = e$ so there is an n-simplex $\gamma \in N_n(G)$ such that $p(\gamma) = \beta p(\theta)^{-1}$.

But then $\beta = p(\gamma\theta)$. $\square$

Lemma 14.7. *The following are equivalent for a simplicial group homomorphism $p : G \to H$:*

1) *The map p is a fibration.*
2) *The induced map $p_* : N_n(G) \to N_n(H)$ is surjective for $n \geq 1$.*

Proof. We will show that 2) implies 1). The other implication is an exercise.

Consider the diagram

$$\begin{tikzcd} G \arrow[r, "p"] \arrow[d] & H \arrow[d] \\ K(\pi_0 G, 0) \arrow[r, "p_*"'] & K(\pi_0 H, 0) \end{tikzcd}$$

where $K(X,0)$ denotes the constant simplicial set on a set X.

Example: $K(\pi_0 G, 0)$ is the constant simplicial group on the group $\pi_0(G)$.

Every map $K(X,0) \to K(Y,0)$ induced by a function $X \to Y$ is a fibration (exercise), so that the map p_* is a fibration, and the map

$$K(\pi_0 G, 0) \times_{K(\pi_0 H, 0)} H \to H$$

is a fibration.

The functor $G \mapsto N_n(G)$ preserves pullbacks, and the map

$$p' : G \to K(\pi_0 G, 0) \times_{K(\pi_0 H, 0)} H$$

is surjective in degree 0 (exercise).

Then p' induces surjections

$$N_n(G) \to N_n(K(\pi_0 G, 0) \times_{K(\pi_0 H, 0)} H)$$

for $n \geq 0$, and is a fibration by Lemmas 14.5 and 14.6. $\Box$

Here are some definitions:

- A homomorphism $p : G \to H$ of simplicial groups is said to be a **fibration** if the underlying map of simplicial sets is a fibration.
- The homomorphism $f : A \to B$ in $s\mathbf{Gr}$ is a **weak equivalence** if the underlying map of simplicial sets is a weak equivalence.
- A **cofibration** of $s\mathbf{Gr}$ is a map which has the left lifting property with respect to all trivial fibrations.

The forgetful functor $U : s\mathbf{Gr} \to s\mathbf{Set}$ has a left adjoint $X \mapsto G(X)$ which is defined by the free group functor in all degrees.

A map $G \to H$ is a fibration (respectively weak equivalence) of $s\mathbf{Gr}$ iff $U(G) \to U(H)$ is a fibration (resp. weak equivalence) of simplicial sets.

If $i : A \to B$ is a cofibration of simplicial sets, then the map $i_* : G(A) \to G(B)$ of simplicial groups is a cofibration.

Suppose G and H are simplicial groups and that K is a simplicial set.

The simplicial group $G \otimes K$ has

$$(G \otimes K)_n = *_{x \in K_n} G_n$$

(generalized free product, or coproduct in **Gr**).

The **function complex** $\mathbf{hom}(G,H)$ for simplicial groups G,H is defined by

$$\mathbf{hom}(G,H)_n = \{G \otimes \Delta^n \to H\}.$$

There is a natural bijection

$$\hom(G \otimes K, H) \cong \hom(K, \mathbf{hom}(G,H)).$$

There is a simplicial group H^K defined as a simplicial set by

$$H^K = \mathbf{hom}(K,H),$$

with the group structure induced from H. There is an exponential law

$$\hom(G \otimes K, H) \cong \hom(G, H^K).$$

Proposition 14.8. *With the definitions of fibration, weak equivalence and cofibration given above the category s***Gr** *satisfies the axioms for a closed simplicial model category.*

Proof. The proof is exercise. A map $p : G \to H$ is a fibration (respectively trivial fibration) if and only if it has the RLP wrt all maps $G(\Lambda^n_k) \to G(\Delta^n)$ (respectively with respect to all $G(\partial\Delta^n) \to G(\Delta^n)$, so a standard small object argument proves the factorization axiom, subject to proving Lemma 14.9 below.

(We need the Lemma to show that the maps $G(A) \to G(B)$ induced by trivial cofibrations $A \to B$ push out to trivial cofibrations).

The axiom **SM7** reduces to the assertion that if $p : G \to H$ is a fibration and $i : K \to L$ is an inclusion of simplicial sets, then the induced homomorphism

$$G^L \to G^K \times_{H^K} H^L$$

is a fibration which is trivial if either i or p is trivial. For this, one uses the natural isomorphism

$$G(X) \otimes K \cong G(X \times K)$$

and the simplicial model axiom for simplicial sets.

$\square$

Lemma 14.9. *Suppose $i : A \to B$ is a trivial cofibration of simplicial sets. Then the induced map $i_* : G(A) \to G(B)$ is a strong deformation retraction of simplicial groups.*

Proof. All simplicial groups are fibrant, so the lift σ exists in the diagram

$$\begin{tikzcd} G(A) \arrow[r, "1"] \arrow[d, "i_*"'] & G(A) \arrow[d] \\ G(B) \arrow[r] \arrow[ur, dotted, "\sigma"] & e \end{tikzcd}$$

The lift h also exists in the diagram

$$\begin{tikzcd} G(A) \arrow[r, "si_*"] \arrow[d, "i_*"'] & G(B)^{\Delta^1} \arrow[d, "{(p_0,p_1)}"] \\ G(B) \arrow[r, "{(i_*\sigma,1)}"'] \arrow[ur, dotted, "h"] & G(B) \times G(B) \end{tikzcd}$$

and h is the required homotopy. $\square$

Corollary 14.10. *The free group functor $G : s\mathbf{Set} \to s\mathbf{Gr}$ preserves weak equivalences.*

The proof of Corollary 14.10 uses the **mapping cylinder** construction. Let $f : X \to Y$ be a map of simplicial sets, and form the diagram

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "i_0"'] & Y \arrow[d, "i_{0*}"] \\ X \arrow[r, "i_1"'] & X \times \Delta^1 \arrow[r, "f_*"'] & (X \times \Delta^1) \cup_X Y \end{tikzcd}$$

Let $j = f_* i_1$, and observe that this map is a cofibration since X is cofibrant. The map $pr : X \times \Delta^1 \to X$ induces a map $pr_* : (X \times \Delta^1) \cup_X Y \to Y$ such that $pr_* i_{0*} = 1_Y$ and one sees that the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & (X \times \Delta^1) \cup_X Y \arrow[d, "pr_*"] \\ & Y \end{tikzcd} \tag{2}$$

commutes. In other words, any simplicial set map $f : X \to Y$ has a (natural) factorization as above such that j is a cofibration and pr_* has a section which is a trivial cofibration.

Remark: A functor $s\mathbf{Set} \to \mathscr{M}$ taking values in a model category which takes trivial cofibrations to weak equivalences must preserve weak equivalences.

A similar statement holds for functors defined on any category of cofibrant objects and taking values in $\mathscr{M}$.

Remark: The construction of (2) is an abstraction of the classical replacement of the map f by a cofibration. It is dual to the replacement of a map by a fibration in a category of fibrant objects displayed in (1 — see p. 21) of Section 13.

Remark: We have used the forgetful-free group functor adjunction to induce a model structure on $s\mathbf{Gr}$ from that on simplicial sets, in such a way that the functors

$$G : s\mathbf{Set} \leftrightarrows s\mathbf{Gr} : U$$

form a Quillen adjunction.

15 Simplicial modules

$s(R-\mathbf{Mod})$ is the category of simplicial R-modules, where R is some unitary ring.

The forgetful functor $U : s(R-\mathbf{Mod}) \to s\mathbf{Set}$ has a left adjoint

$$R : s\mathbf{Set} \to s(R-\mathbf{Mod}).$$

$R(X)_n$ is the free R-module on the set X_n for $n \geq 0$.

$s(R-\mathbf{Mod})$ has a closed model structure which is induced from simplicial sets by the forgetful-free abelian group functor adjoint pair, in the same way that the category $s\mathbf{Gr}$ of simplicial groups acquires its model structure.

A morphism $f : A \to B$ of simplicial R-modules is a **weak equivalence** (respectively **fibration**) if the underlying morphism of simplicial sets is a weak equivalence (respectively fibration).

A **cofibration** of simplicial R-modules is a map which has the LLP wrt all trivial fibrations.

Examples of cofibrations of $s(R-\mathbf{Mod})$ include all maps $R(A) \to R(B)$ induced by cofibrations of simplicial sets.

Suppose A and B are simplicial groups and that K is a simplicial set. Then there is a simplicial abelian group $A \otimes K$ with

$$(A \otimes K)_n = \bigoplus_{x \in K_n} A_n \cong A_n \otimes R(K)_n.$$

The **function complex** $\mathbf{hom}(A,B)$ for simplicial abelian groups A,B is defined by

$$\mathbf{hom}(A,B)_n = \{A \otimes \Delta^n \to B\}.$$

Then there is a natural bijection

$$\hom(A \otimes K, B) \cong \hom(K, \mathbf{hom}(A,B)).$$

There is a simplicial module B^K defined as a simplicial set by

$$B^K = \mathbf{hom}(K, B),$$

with R-module structure induced from B.

There is an exponential law

$$\hom(A \otimes K, B) \cong \hom(A, B^K).$$

Proposition 15.1. *With the definitions of fibration, weak equivalence and cofibration given above the category $s(R-\mathbf{Mod})$ satisfies the axioms for a closed simplicial model category.*

Proof. The proof is by analogy with the corresponding result for simplicial groups (Prop. 14.8). □

The proof of Proposition 15.1 also uses the following analog of Lemma 14.9, in the same way:

Lemma 15.2. *Suppose $i : A \to B$ is a trivial cofibration of simplicial sets. Then the induced map $i_* : R(A) \to R(B)$ is a strong deformation retraction of simplicial R-modules.*

Corollary 15.3. *The free R-module functor*

$$R : s\mathbf{Set} \to s(R-\mathbf{Mod})$$

preserves weak equivalences.

Once again, the adjoint functors

$$R : s\mathbf{Set} \leftrightarrows s(R - \mathbf{Mod}) : U$$

form a Quillen adjunction.

Example: $R = \mathbb{Z}$: The category $s(\mathbb{Z} - \mathbf{Mod})$ is the category of simplicial abelian groups, also denoted by $s\mathbf{Ab}$.

The adjunction homomorphism $\eta : X \to U\mathbb{Z}(X)$ for this case is usually written as

$$h : X \to \mathbb{Z}(X)$$

and is called the **Hurewicz homomorphism**. More on this later.

Simplicial R-modules are simplicial groups, so we know a few things:

- For a simplicial R-module A the modules $N_nA = \cap_{i<n} \ker(d_i)$ and the morphisms

  $$N_nA \xrightarrow{(-1)^n d_n} N_{n-1}A$$

  form an ordinary chain complex, called the **normalized chain complex** of A. The assignment $A \mapsto NA$ defines a functor

  $$N : s(R - \mathbf{Mod}) \to Ch_+(R).$$

- There is a natural isomorphism
$$\pi_n(A,0) \cong H_n(NA),$$
and a map $f : A \to B$ is a weak equivalence if and only if the induced chain map $NA \to NB$ is a homology isomorphism (Corollary 14.4).
- A map $p : A \to B$ is a fibration of $s(R - \mathbf{mod})$ if and only if the induced map $p_* : NA \to NB$ is a fibration of $Ch_+(R)$ (Lemma 14.7).

This precise relationship between simplicial modules and chain complexes is not an accident.

The **Moore complex** $M(A)$ for a simplicial module A has n-chains given by $M(A)_n = A_n$ and boundary
$$\partial = \sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1}.$$
The fact that $\partial^2 = 0$ is an exercise involving the simplicial identities $d_i d_j = d_{j-1} d_i$, $i < j$.

The construction is functorial:
$$M : s(R - \mathbf{Mod}) \to Ch_+(R).$$
The Moore chains functor is *not* the normalized chains functor, but the inclusions $N_n A \subset A_n$ determine a natural chain map
$$N(A) \subset M(A).$$

Example: If Y is a space, the n^{th} **singular homology module** $H_n(Y,R)$ with coefficients in R is defined by

$$H_n(Y,R) = H_nM(R(S(Y))).$$

If N is any R-module, then

$$H_n(Y,N) = H_n(M(R(S(Y)) \otimes_R N))$$

defines the n^{th} **singular homology** module of Y with coefficients in N.

The subobject $D(A)_n \subset M(A)_n$ is defined by

$$D(A)_n = \langle s_j(y) \mid 0 \leq j \leq n-1, y \in A_{n-1} \rangle.$$

$D(A)_n$ is the submodule generated by degenerate simplices.

The Moore chains boundary ∂ restricts to a boundary map $\partial : D(A)_n \to DA_{n-1}$ (exercise), and the inclusions $D(A)_n \subset A_n$ form a natural chain map

$$D(A) \subset M(A).$$

Here's what you need to know:

Theorem 15.4. *1) The composite chain map*

$$N(A) \subset M(A) \to M(A)/D(A)$$

is a natural isomorphism.

2) The inclusion $N(A) \subset M(A)$ is a natural chain homotopy equivalence.

Proof. There is a subcomplex $N_j(A) \subset M(A)$ with $N_jA_n = NA_n$ if $n \leq j+1$ and

$$N_jA_n = \cap_{i=0}^{j} \ker(d_j) \text{ if } n \geq j+2.$$

$D_j(A_n) :=$ the submodule of A_n generated by all $s_i(x)$ with $i \leq j$.

1) We show that the composite

$$\phi : N_j(A_n) \to A_n \to A_n/D_j(A_n)$$

is an isomorphism for all $j < n$, by induction on j.

There is a commmutative diagram

$$\begin{tikzcd}
 & N_{j-1}A_{n-1} \arrow[r, "s_j"] \arrow[d, "\phi", "\cong"'] & N_{j-1}A_n \arrow[d, "\phi", "\cong"'] & N_jA_n \arrow[l, "i"'] \arrow[d, "\phi"] & \\
0 \arrow[r] & A_{n-1}/D_{j-1}A_{n-1} \arrow[r, "s_j"'] & A_n/D_{j-1}A_n \arrow[r] & A_n/D_jA_n \arrow[r] & 0
\end{tikzcd}$$

in which the bottom sequence is exact and i is the obvious inclusion.

If $[x] \in A_n/D_jA_n$ for $x \in N_{j-1}A_n$, then $[x - s_jd_jx] = [x]$ and $x - s_jd_jx \in N_jA_n$, so $\phi : N_jA_n \to A_n/d_jA_n$ is surjective.

If $\phi(x) = 0$ for $x \in N_jA_n$ then $x = s_j(y)$ for some $y \in N_{j-1}A_{n-1}$. But $d_jx = 0$ so $0 = d_js_jy = y$.

For 2), we have $N_{j+1}A \subset N_jA$ and

$$NA = \cap_{j \geq 0} N_jA$$

in finitely many stages in each degree.

We show that $i : N_{j+1}A \subset N_jA$ is a chain homotopy equivalence (this is cheating a bit, but is easily fixed — see [2, p.149]).

There are chain maps $f : N_jA \to N_{j+1}A$ defined by

$$f(x) = \begin{cases} x - s_{j+1}d_{j+1}(x) & \text{if } n \geq j+2, \\ x & \text{if } n \leq j+1. \end{cases}$$

Write $t = (-1)^j s_{j+1} : N_jA_n \to N_jA_{n+1}$ if $n \geq j+1$ and set $t = 0$ otherwise. Then $f(i(x)) = x$ and

$$1 - i \cdot f = \partial t + t\partial.$$

□

Suppose A is a simplicial R-module. Every monomorphism $d : \mathbf{m} \to \mathbf{n}$ induces a homomorphism $d^* : NA_n \to NA_m$, and $d^* = 0$ unless $d = d^n$.

Suppose C is a chain complex. Associate the module C_n to the ordinal number $\mathbf{n}$, and associate to each ordinal number monomorphism d the morphism $d^* : C_n \to C_m$, where

$$d^* = \begin{cases} 0 & \text{if } d \neq d^n, \\ (-1)^n\partial : C_n \to C_{n-1} & \text{if } d = d^n. \end{cases}$$

Define

$$\Gamma(C)_n = \bigoplus_{s:\mathbf{n}\twoheadrightarrow\mathbf{k}} C_k.$$

The ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ induces an R-module homomorphism

$$\theta^* : \Gamma(C)_n \to \Gamma(C)_m$$

which is defined on the summand corresponding to the epi $s : \mathbf{n} \twoheadrightarrow \mathbf{k}$ by the composite

$$C_k \xrightarrow{d^*} C_r \xrightarrow{in_t} \bigoplus_{\mathbf{m}\twoheadrightarrow\mathbf{r}} C_r,$$

where the ordinal number maps

$$\mathbf{m} \overset{t}{\twoheadrightarrow} \mathbf{r} \overset{d}{\rightarrowtail} \mathbf{k}$$

give the epi-monic factorization of the composite

$$\mathbf{m} \xrightarrow{\theta} \mathbf{n} \overset{s}{\twoheadrightarrow} \mathbf{k}.$$

and d^* is induced by d according to the prescription above.

The assignment $C \mapsto \Gamma(C)$ is defines a functor

$$\Gamma : Ch_+(R) \to s(R - \mathbf{Mod}).$$

Theorem 15.5 (Dold-Kan)**.** *The functor Γ is an inverse up to natural isomorphism for the normalized chains functor N.*

The equivalence of categories defined by the functors N and Γ is the **Dold-Kan correspondence**.

Proof. One can show that

$$D(\Gamma(C))_n = \bigoplus_{s:\mathbf{n}\twoheadrightarrow\mathbf{k},k\leq n-1} C_k,$$

so there is a natural isomorphism

$$C \cong M(\Gamma(C))/D(\Gamma(C)) \cong N(\Gamma(C))$$

There is a natural homomorphism of simplicial modules

$$\Psi : \Gamma(NA) \to A,$$

which in degree n is the homomorphism

$$\bigoplus_{s:\mathbf{n}\twoheadrightarrow\mathbf{k}} NA_k \to A_n$$

defined on the summand corresponding to $s : \mathbf{n} \twoheadrightarrow \mathbf{k}$ by the composite

$$NA_k \subset A_k \xrightarrow{s^*} A_n.$$

Collapsing Ψ by degeneracies gives the canonical isomorphism $NA \cong A/D(A)$, so the map

$$N(\Psi) : N(\Gamma(NA)) \to NA$$

is an isomorphism of chain complexes.

It follows from Lemma 14.6 that the natural map Ψ is surjective in all degrees.

The functor $A \mapsto NA$ is exact: it is left exact from the definition, and it preserves epimorphisms by Lemma 14.7.

It follows that the normalized chains functor reflects isomorphisms.

To see this, suppose $f : A \to B$ is a simplicial module map and that the sequence

$$0 \to K \to A \xrightarrow{f} B \to C \to 0$$

is exact. Suppose also that Nf is an isomorphism. Then the sequence of chain complex maps

$$0 \to NK \to NA \xrightarrow{Nf} NB \to NC \to 0$$

is exact, so that $NK = NC = 0$. But then $K = C = 0$ since Ψ is a natural epimorphism, so that f is an isomorphism.

Finally, $N\Psi$ is an isomorphism, so that Ψ is an isomorphism. $\square$

16 Eilenberg-Mac Lane spaces

Under the Dold-Kan correspondence

$$\Gamma : Ch_+(R) \leftrightarrows s(R-\mathbf{Mod}) : N$$

a map $f : A \to B$ of simplicial modules is a weak equivalence (respectively fibration, cofibration) if and only if the induced map $f_* : NA \to NB$ is a weak equivalence (resp. fibration,cofibration) of $Ch_+(R)$.

There are natural isomorphisms

$$\pi_n(|A|,0) \cong \pi_n^s(A,0) \cong H_n(N(A)) \cong H_n(M(A)).$$

for simplicial modules A.

Suppose that C is a chain complex.

Take $n \geq 0$. Write $C[-n]$ for the **shifted** chain complex with

$$C[-n]_k = \begin{cases} C_{k-n} & k \geq n, \\ 0 & k < n. \end{cases}$$

There is a natural short exact sequence of chain complexes

$$0 \to C \to \widetilde{C[-1]} \to C[-1] \to 0.$$

In general (see Section 6), $\tilde{D}$ is the acyclic complex with $\tilde{D}_n = D_n \oplus D_{n+1}$ for $n > 0$,

$$\tilde{D}_0 = \{(x,z) \in D_0 \oplus D_1 \mid x + \partial(z) = 0\},$$

and with boundary map defined by

$$\partial(x,z) = (\partial(x), (-1)^n x + \partial(z))$$

for $(x,z) \in \tilde{D}_n$.

For a simplicial module A, the objects $\Gamma(NA[-1])$ and $\Gamma(\widetilde{NA[-1]})$ have special names, due to Eilenberg and Mac Lane:

$$\overline{W}(A) := \Gamma(NA[-1]),$$

and

$$W(A) := \Gamma(\widetilde{NA[-1]}).$$

There is a natural short exact (hence fibre) sequence of simplicial modules

$$0 \to A \to W(A) \to \overline{W}(A) \to 0,$$

(exercise) and there are isomorphisms

$$\pi_n(A) \cong \pi_{n+1}(\overline{W}(A)).$$

The object $\overline{W}(A)$ is a natural delooping of the simplicial module A, usually thought of as either a *suspension* or a *classifying space* for A.

Suppose B is an R-module, and write $B(0)$ for the chain complex concentrated in degree 0, which consists of B in that degree and 0 elsewhere.

Then $B(n) = B(0)[-n]$ is the chain complex with B in degree n. Write

$$K(B,n) = \Gamma(B(n)).$$

There are natural isomorphisms

$$\pi_j K(B,n) \cong H_j(B(n)) \cong \begin{cases} B & j=n \\ 0 & j \neq n. \end{cases}$$

The object $K(B,n)$ (or $|K(B,n)|$) is an **Eilenberg-Mac Lane space of type** (B,n).

This is a standard method of constructing these spaces, together with the natural fibre sequences

$$K(B,n) \to W(K(B,n)) \to K(B,n+1)$$

for modules (or abelian groups) B. These fibre sequences are short exact sequences of simplicial modules.

Non-abelian groups

The non-abelian world is different. Here's an exercise:

Exercise: Show that a functor $f : G \to H$ between groupoids induces a fibration $BG \to BH$ if and only if f has the **path lifting property** in the sense that all lifting problems

$$\begin{tikzcd} * \arrow[r] \arrow[d, "0"'] & G \arrow[d, "f"] \\ \mathbf{1} \arrow[r] \arrow[ur, dotted] & H \end{tikzcd}$$

can be solved.

Suppose G is a group, identified with a groupoid with one object $*$, and recall that the slice category $*/G$ has as objects all group elements (morphisms) $* \xrightarrow{g} *$, and as morphisms all commutative diagrams

$$\begin{tikzcd} & * \arrow[d, "k"] \\ * \arrow[ur, "g"] \arrow[r, "h"'] & * \end{tikzcd}$$

The canonical functor $\pi : */G \to G$ sends the morphism above to the morphism k of G.

The functor π has the path lifting property, and the fibre over the vertex $*$ of the fibration $\pi : B(*/G) \to BG$ is a copy of $K(G,0)$.

One usually writes

$$EG = B(*/G).$$

This is a contractible space, since it has an initial object e and the unique maps $\gamma_g : e \to g$ define a contracting homotopy $*/G \times \mathbf{1} \to */G$.

The Kan complex BG is connected, since it has only one vertex. The long exact sequence in homotopy groups associated to the fibre sequence

$$K(G,0) \to EG \xrightarrow{\pi} BG$$

can be used to show that $\pi_n^s(BG)$ is trivial for $n \neq 1$, and that the boundary map

$$\pi_1^s(BG) \xrightarrow{\partial} G = \pi_0(K(G,0))$$

is a bijection.

For this, there is a surjective homomorphism

$$G \to \pi_1^s(BG),$$

defined by taking g to the homotopy group element $[g]$ represented by the simplex $* \xrightarrow{g} *$. One shows that the composite

$$G \to \pi_1^s(BG) \xrightarrow{\partial} G \tag{3}$$

is the identity on G, so that the homomorphism $G \to \pi_1^s(BG)$ is a bijection.

To see that the composite (3) is the identity, observe that there is a commutative diagram

$$\begin{tikzcd} \Lambda^1_0 \arrow[r, "e"] \arrow[d] & EG \arrow[d, "\pi"] \\ \Delta^1 \arrow[r, "g"'] \arrow[ur, "\gamma_g"] & BG \end{tikzcd}$$

Then $\partial([g]) = d_0(\gamma_g) = g$.

The classifying space BG for a group G is an Eilenberg-Mac lane space $K(G,1)$. This is a standard model.

Some facts about groupoids

Suppose that H is a connected groupoid. This means that, for any two objects $x, y \in H$ there is a morphism (isomorphism) $\omega : x \to y$.

Fix an object x of H and chose isomorphisms $\gamma_y : y \to x$ for all objects of H, such that $\gamma_x = 1_x$. There is an inclusion functor

$$i : H_x = H(x,x) \subset H.$$

We define a functor $r : H \to H_x$ by conjugation with the maps γ_y: if $\alpha : y \to z$ is a morphism of H, then $r(\alpha) = \gamma_z^{-1}\alpha\gamma_x$, so that the diagrams

$$\begin{tikzcd} x \arrow[r, "\gamma_y"] \arrow[d, "r(\alpha)"'] & y \arrow[d, "\alpha"] \\ x \arrow[r, "\gamma_z"'] & z \end{tikzcd}$$

commute.

The functor r is uniquely determined by the isomorphisms γ_y, and the composite

$$H_x \overset{i}{\subset} H \xrightarrow{r} H_x$$

is the identity.

The maps γ_y define a natural transformation

$$\gamma : i \cdot r \to 1_H.$$

We have shown that the inclusion $BH_x \to BH$ is a homotopy equivalence, even a strong deformation retraction.

It follows that, for arbitrary small groupoids H, there is a homotopy equivalence

$$BH \simeq \bigsqcup_{[x]\in\pi_0(H)} BH(x,x). \tag{4}$$

Thus, a groupoid H has no higher homotopy groups in the sense that $\pi_k(BH,x) = 0$ for $k \geq 2$ and all objects x, since the same is true of classifying spaces of groups.

Example: Group actions

Suppose that $G \times F \to F$ is the action of a group G on a set F.

Recall that the corresponding translation groupoid $E_G F$ has objects $x \in F$ and morphisms $x \to g \cdot x$.

The space $B(E_G F) = EG \times_G F$ is the Borel construction for the action of G on F.

The group of automorphisms $x \to x$ in $E_G F$ can be identified with the subgroup $G_x \subset G$ that stabilizes x. If $\alpha : x \to y$ is a morphism of $E_G F$, then G_x is conjugate to G_y as subgroups of G (exercise).

There is a bijection

$$\pi_0(EG \times_G F) \cong F/G,$$

and the identification (4) translates to a homotopy equivalence

$$EG \times_G F \simeq \coprod_{[x] \in F/G} BG_x. \qquad (5)$$

Then $EG \times_G F$ is contractible if and only if

1) G acts transitively on F, ie. $F/G \cong *$, and
2) the stabilizer subgroups G_x (fundamental groups) are trivial for all $x \in F$.

One usually summarizes conditions 1) and 2) by saying that G acts simply transitively on F, or that G acts principally on F.

In ordinary set theory, this means precisely that there is a G-equivariant isomorphism $G \xrightarrow{\cong} F$.

In the topos world, where $G \times F \to F$ is the action of a sheaf of groups G on a sheaf F, the assertion that the Borel construction $EG \times_G F$ is (locally) contractible is equivalent to the assertion that F is a G-torsor.

The canonical groupoid morphism $E_G F \to G$ has the path lifting property, and hence induces a Kan fibration

$$\pi : EG \times_G F \to BG$$

with fibre F.

The use of this fibration π, in number theory, geometry and topology, is to derive calculations of homology invariants of BG from calculations of the corresponding invariants of the spaces BG_x associated to stabilizers, usually via spectral sequence calcluations.

The Borel construction made its first appearance in the Borel seminar on transformation groups at IAS in 1958-59 [1].

If the action $G \times F \to F$ is simple in the sense that all stabilizer groups G_x are trivial, then all orbits are copies of G up to equivariant isomorphism, and the canonical map

$$EG \times_G F \to F/G$$

is a weak equivalence.

It is a consequence of Quillen's Theorem 23.4 below that if $G \times X \to G$ is an action of G on a simplicial set X, then X is the homotopy fibre of the canonical map $EG \times_G X \to BG$.

It follows that, if the action $G \times X \to X$ is simple in all degrees and the simplicial set X is contractible, then the maps

$$\begin{tikzcd} EG \times_G X \arrow[r, "\simeq"] \arrow[d, "\pi"', "\simeq"] & X/G \\ BG \end{tikzcd}$$

are weak equivalences, so that BG is weakly equivalent to X/G. This is a well known classical result.

References

[1] Armand Borel. *Seminar on transformation groups*. With contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais. Annals of Mathematics Studies, No. 46. Princeton University Press, Princeton, N.J., 1960.

[2] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.