Contents

21 Bisimplicial sets

A **bisimplicial set** X is a simplicial object

$$X : \Delta^{op} \to s\mathbf{Set}$$

in simplicial sets, or equivalently a functor

$$X : \Delta^{op} \times \Delta^{op} \to \mathbf{Set}.$$

I write

$$X_{m,n} = X(\mathbf{m}, \mathbf{n})$$

for the set of **bisimplices** in bidgree (m,n) and

$$X_m = X_{m,*}$$

for the **vertical simplicial set** in horiz. degree m.

Morphisms $X \to Y$ of bisimplicial sets are natural transformations.

$s^2\mathbf{Set}$ is the category of bisimplicial sets.

Examples:

1) $\Delta^{p,q}$ is the contravariant representable functor

$$\Delta^{p,q} = \hom(\ ,(\mathbf{p},\mathbf{q}))$$

on $\Delta \times \Delta$.

$$\Delta^{p,q}_m = \bigsqcup_{\mathbf{m} \to \mathbf{p}} \Delta^q.$$

The maps $\Delta^{p,q} \to X$ classify bisimplices in $X_{p,q}$.

The *bisimplex category* $(\Delta \times \Delta)/X$ has the bisimplices of X as objects, with morphisms the incidence relations

$$\begin{array}{ccc} \Delta^{p,q} & & \\ \downarrow & \searrow & \\ & & X \\ & \nearrow & \\ \Delta^{r,s} & & \end{array}$$

2) Suppose K and L are simplicial sets.

The bisimplicial set $K \tilde{\times} L$ has bisimplices

$$(K \tilde{\times} L)_{p,q} = K_p \times L_q.$$

The object $K \tilde{\times} L$ is the *external product* of K and L.

There is a natural isomorphism

$$\Delta^{p,q} \cong \Delta^p \tilde{\times} \Delta^q.$$

3) Suppose I is a small category and that $X : I \to s\mathbf{Set}$ is an I-diagram in simplicial sets.

Recall (Lecture 04) that there is a bisimplicial set $\underrightarrow{\text{holim}}_I X$ ("the" *homotopy colimit*) with vertical simplicial sets

$$\bigsqcup_{i_0 \to \cdots \to i_n} X(i_0)$$

in horizontal degrees n.

The transformation $X \to *$ induces a bisimplicial set map

$$\pi : \bigsqcup_{i_0 \to \cdots \to i_n} X(i_0) \to \bigsqcup_{i_0 \to \cdots \to i_n} * = BI_n,$$

where the set BI_n has been identified with the discrete simplicial set $K(BI_n, 0)$ in each horizontal degree.

Example: Suppose that G is a group, and that X is a simplicial set with a G-action $G \times X \to X$. If G is identified with a one-object groupoid, then the G-action defines a functor $X : G \to s\mathbf{Set}$ which sends the single object of G to X.

The corresponding bisimplicial set has vertical simplicial sets of the form

$$\bigsqcup_{* \xrightarrow{g_1} * \xrightarrow{g_2} * \ldots \xrightarrow{g_n} *} X \cong G^{\times n} \times X,$$

which is a model in bisimplicial sets for the Borel construction $EG \times_G X$.

Applying the diagonal functor (see below) gives the Borel construction in simplicial sets.

Every simplicial set X determines bisimplicial sets which are constant in each vertical degree or each horizontal degree. We write X for the *constant* bisimplicial set determined by X either horizontally or vertically.

From this point of view, the canonical map π is a map of bisimplicial sets

$$\pi : \underrightarrow{\mathrm{holim}}_I X \to BI.$$

The *diagonal* simplicial set $d(X)$ for bisimplicial set X has simplices

$$d(X)_n = X_{n,n}$$

with simplicial structure maps

$$(\theta, \theta)^* : X_{n,n} \to X_{m,m}$$

for ordinal number maps $\theta : \mathbf{m} \to \mathbf{n}$.

This construction defines a functor

$$d : s^2\mathbf{Set} \to s\mathbf{Set}.$$

Recall that X_n denotes the vertical simplicial set in horizontal degree n for a bisimplicial set X. The maps

$$\begin{array}{ccc} X_n \times \Delta^m & \xrightarrow{1\times\theta} & X_n \times \Delta^n \\ {\scriptstyle \theta^*\times 1}\downarrow & & \\ X_m \times \Delta^m & & \end{array}$$

associated to the ordinal number maps $\theta : \mathbf{m} \to \mathbf{n}$ determine morphisms

$$\bigsqcup_{\theta:\mathbf{m}\to\mathbf{n}} X_n \times \Delta^m \rightrightarrows \bigsqcup_{n\geq 0} X_n \times \Delta^n. \qquad (1)$$

There are simplicial set maps

$$\gamma_n : X_n \times \Delta^n \to d(X)$$

defined on r-simplices by

$$\gamma_n(x, \tau : \mathbf{r} \to \mathbf{n}) = \tau^*(x) \in X_{r,r}.$$

The maps in (1) above and the morphisms γ_n, $n \geq 0$ together determine a diagram

$$\bigsqcup_{\theta:\mathbf{m}\to\mathbf{n}} X_n \times \Delta^m \rightrightarrows \bigsqcup_{n\geq 0} X_n \times \Delta^n \xrightarrow{\gamma} d(X). \qquad (2)$$

Exercise: Show that the diagram (2) is a coequalizer in simplicial sets.

Example: There are natural isomorphisms

$$d(K \tilde{\times} L) \cong K \times L.$$

In particular, there are isomorphisms

$$d(\Delta^{p,q}) \cong \Delta^p \times \Delta^q.$$

The diagonal simplicial set $d(X)$ has a filtration by subobjects $d(X)^{(n)}$, $n \geq 0$, where

$$d(X)^{(n)} = \text{image of } \bigsqcup_{p \leq n} X_p \times \Delta^p \text{ in } d(X).$$

The (horizontal) degenerate part of the vertical simplicial set X_{n+1} is filtered by subobjects

$$s_{[r]}X_n = \bigcup_{0 \leq i \leq r} s_i(X_n) \subset X_{n+1}$$

where $r \leq n$. There are natural pushout diagrams of cofibrations

$$\begin{array}{ccc} s_{[r]}X_{n-1} & \xrightarrow{s_{r+1}} & s_{[r]}X_n \\ \downarrow & & \downarrow \\ X_n & \xrightarrow[s_{r+1}]{} & s_{[r+1]}X_n \end{array} \tag{3}$$

and

$$\begin{array}{ccc} (s_{[n]}X_n \times \Delta^{n+1}) \cup (X_{n+1} \times \partial\Delta^{n+1}) & \longrightarrow & d(X)^{(n)} \\ \downarrow & & \downarrow \\ X_{n+1} \times \Delta^{n+1} & \longrightarrow & d(X)^{(n+1)} \end{array} \tag{4}$$

in which all vertical maps are cofibrations.

The natural filtration $\{d(X)^{(n)}\}$ of $d(X)$ and the natural pushout diagrams (3) and (4) are used with glueing lemma arguments to show the following:

Lemma 21.1. *Suppose $f : X \to Y$ is a map of bisimplicial sets such that all maps $X_n \to Y_n$, $n \geq 0$, of vertical simplicial sets are weak equivalences.*

Then the induced map $d(X) \to d(Y)$ is a weak equivalence of diagonal simplicial sets.

Example: Suppose that $G \times X \to X$ is an action of a group G on a simplicial set X. The bisimplicial set

$$\bigsqcup_{* \xrightarrow{g_1} * \xrightarrow{g_2} * \dots \xrightarrow{g_n} *} X \cong G^{\times n} \times X$$

has horizontal path components X/G, and the map to path components defines a simplicial set map

$$\pi : EG \times_G X \to X/G,$$

which is natural in G-sets X.

If the action $G \times X \to X$ is free, then the path components the simplicial sets $EG \times_G X_n$ are isomorphic to copies of the contractible space $EG = EG \times_G G$. It follows that the map π is a weak equivalence in this case.

If the action $G \times X \to X$ is free and X is contractible, then we have weak equivalences

$$\begin{array}{ccc} EG \times_G X & \xrightarrow[\simeq]{\pi} & X/G \\ {\scriptstyle p}\downarrow{\scriptstyle \simeq} & & \\ BG & & \end{array}$$

Model structures

There are multiple closed model structures for bisimplicial sets. Here are three of them:

1) The **projective structure**, for which a map $X \to Y$ of bisimplicial sets is a weak equivalence (respectively projective fibration) if all maps $X_n \to Y_n$ are weak equivalences (respectively fibrations) of simplicial sets. The cofibrations for this structure are called the projective cofibrations.

2) The **injective structure**, for which $X \to Y$ is a weak equivalence (respectively cofibration) if all maps $X_n \to Y_n$ are weak equivalences (respectively cofibrations) of simplicial sets. The fibrations for this theory are called the injective fibrations.

3) There is a **diagonal model structure** on $s^2\mathbf{Set}$ for which a map $X \to Y$ is a weak equivalence if it is a *diagonal weak equivalence* ie. that the map $d(X) \to d(Y)$ of simplicial sets is a weak equivalence, and the cofibrations are the monomorphisms of bisimplicial sets as in 2).

The existence of the diagonal structure is originally due to Joyal and Tierney, but they did not publish the result. A proof appears in [3].

The projective structure is a special case of the projective structure for I-diagrams of simplicial sets of Lemma 19.1 (Lecture 07) — it is called the Bousfield-Kan structure in [2, IV.3.1].

The injective structure is similarly a special case of the injective structure for I-diagrams, of Theorem 19.3.

The injective structure is also an instance of the Reedy structure for simplicial objects in a model category [2, IV.3.2,VII.2].

The weak equivalences for both the projective and injective structures are called *level equivalences*.

Lemma 21.1 says that every level equivalence is a diagonal equivalence.

The diagonal functor $X \mapsto d(X)$ is left adjoint to a "singular functor" $X \mapsto d_*(X)$, where

$$d_*(X)_{p,q} = \hom(\Delta^p \times \Delta^q, X).$$

One can show, by verifying a (countable) bounded cofibration condition, that a bisimplicial set map $p : X \to Y$ is a fibration for the diagonal model structure if and only if it has the right lifting property with respect to all trivial cofibrations $A \to B$ which are countable in the sense that all sets of bisimplices $B_{p,q}$ are countable.

The bounded cofibration condition is a somewhat tough exercise to prove — one uses the fact that the diagonal functor has a left adjoint as well as a right adjoint.

22 Homotopy colimits and limits (revisited)

Suppose $X : I \to s\mathbf{Set}$ is an I-diagram which takes values in Kan complexes.

Following [1], one writes

$$\underleftarrow{\mathrm{holim}}_I X = \mathbf{hom}(B(I/?), X),$$

where the function complex is standard, and $B(I/?)$ is the functor $i \mapsto B(I/i)$.

Suppose Y is a simplicial set, and X is still our prototypical I-diagram.

Homotopy colimits

The assignment $i \mapsto \mathbf{hom}(X(i), Y)$ defines an I^{op}-diagram

$$\mathbf{hom}(X,Y) : I^{op} \to s\mathbf{Set}.$$

There is a natural isomorphism of function spaces

$$\mathbf{hom}(\underrightarrow{\mathrm{holim}}_I X, Y) \cong \underleftarrow{\mathrm{holim}}_{I^{op}} \mathbf{hom}(X,Y),$$

where $\underrightarrow{\text{holim}}_I X$ is defined by the coequalizer

$$\bigsqcup_{\alpha:i\to j \text{ in } I} B(j/I)\times X(i) \rightrightarrows \bigsqcup_{i\in \mathrm{Ob}(I)} B(i/I)\times X(i) \to \underrightarrow{\text{holim}}_I X.$$

By looking at maps

$$\underrightarrow{\text{holim}}_I X \to Y,$$

one shows (exercise) that $\underrightarrow{\text{holim}}_I X$ is the diagonal of the bisimplicial set, with vertical n-simplices

$$\bigsqcup_{i_0\to\cdots\to i_n} X(i_0),$$

up to isomorphism.

This is the (standard) description of the homotopy colimit of X that was introduced in Section 9.

This definition of homotopy colimit coincides up to equivalence with the "colimit of projective cofibrant model" description of Section 20.

Here is the key to comparing the two:

Lemma 22.1. *Suppose $X : I \to s\mathbf{Set}$ is a projective cofibrant I-diagram. Then the canonical map*

$$\underrightarrow{\text{holim}}_I X \to \varinjlim_I X$$

is a weak equivalence.

Proof. $\varinjlim_I X_m$ is the set of path components of the simplicial set

$$\bigsqcup_{i_0 \to \cdots \to i_n} X(i_0)_m,$$

so $\varinjlim_I X$ can be identified with the simplicial set of horizontal path components of the bisimplicial set $\underrightarrow{\text{holim}}_I X$.

The space $B(i/I)$ is contractible since the category i/I has an initial object. Thus, every projection

$$B(i/I) \times K \to K$$

is a weak equivalence.

The simplicial set $B(i/I) \times K$ is the homotopy colimit of the I diagram $\hom(i, \ ) \times K$ and the projection is isomorphic to the map

$$\underrightarrow{\text{holim}}_I(\hom(i, \ ) \times K) \to \varinjlim_I(\hom(i, \ ) \times K)$$

Thus, all diagrams $\hom(i, \ ) \times K$ are members of the class of I-diagrams X for which the map

$$\underrightarrow{\text{holim}}_I X \to \varinjlim_I X \tag{5}$$

is a weak equivalence.

Suppose given a pushout diagram

$$\begin{array}{ccc} \hom(i,\ )\times K & \longrightarrow & X \\ {\scriptstyle 1\times j}\downarrow & & \downarrow \\ \hom(i,\ )\times L & \longrightarrow & Y \end{array}$$

of I-diagrams, where j is a cofibration. Suppose also that the map (5) is a weak equivalence. Then the induced map

$$\underrightarrow{\text{holim}}_I Y \to \varinjlim_I Y$$

is a weak equivalence.

For this, the induced diagram

$$\begin{array}{ccc} \varinjlim_I(\hom(i,\ )\times K) & \longrightarrow & \varinjlim_I X \\ \downarrow & & \downarrow \\ \varinjlim_I(\hom(i,\ )\times L) & \longrightarrow & \varinjlim_I Y \end{array}$$

is a pushout, and one uses the glueing lemma to see the desired weak equivalence.

Suppose given a diagram of cofibrations of I-diagrams

$$X_0 \to X_1 \to \dots$$

such that all maps

$$\underrightarrow{\text{holim}}_I X_s \to \varinjlim_I X_s$$

are weak equivalences. Then the map

$$\underrightarrow{\mathrm{holim}}_I(\varinjlim_s X_s) \to \varinjlim_I(\varinjlim_s X_s)$$

is a weak equivalence.

In effect, the colimit and homotopy colimit functors commute, and filtered colimits preserve weak equivalences in $s\mathbf{Set}$.

A small object argument shows that, for every I-diagram Y, there is a trivial projective fibration $p : X \to Y$ such that X is projective cofibrant and the map (5) is a weak equivalence.

If Y is projective cofibrant, then Y is a retract of the covering X, so the map

$$\underrightarrow{\mathrm{holim}}_I Y \to \varinjlim_I Y$$

is a weak equivalence. □

Corollary 22.2. *Suppose $X : I \to s\mathbf{Set}$ is an I-diagram of simplicial sets, and let $\pi : U \to X$ be a projective cofibrant model of X. Then there are weak equivalences*

$$\underrightarrow{\mathrm{holim}}_I X \xleftarrow[\pi_*]{\simeq} \underrightarrow{\mathrm{holim}}_I U \xrightarrow{\simeq} \varinjlim_I U.$$

Proof. Generally, if $f : X \to Y$ is a weak equivalence of I-diagrams, then the induced maps

$$\bigsqcup_{i_0 \to \cdots \to i_n} X(i_0) \to \bigsqcup_{i_0 \to \cdots \to i_n} Y(i_0)$$

is a weak equivalence of simplicial sets for each vertical degree n, and it follows from Lemma 21.1 that the induced map

$$\underrightarrow{\text{holim}}_I X \to \underrightarrow{\text{holim}}_I Y$$

is a weak equivalence.

It follows that the map

$$\underrightarrow{\text{holim}}_I X \xleftarrow{\pi_*} \underrightarrow{\text{holim}}_I U$$

is a weak equivalence, and Lemma 22.1 shows that

$$\underrightarrow{\text{holim}}_I U \to \varinjlim_I U$$

is a weak equivalence. □

Homotopy limits

Each slice category I/i has a terminal object, so $B(I/i)$ is contractible, and the map

$$B(I/?) \to *$$

of I-diagrams is a weak equivalence.

If Z is an injective fibrant I-diagram, then the induced map

$$\varprojlim_I Z \cong \mathbf{hom}(*, Z) \to \mathbf{hom}(B(I/?), Z) =: \underleftarrow{\mathrm{holim}}_I Z$$

is a weak equivalence.

Here's the interesting thing to prove:

Proposition 22.3. *Suppose $p : X \to Y$ is a projective fibration (resp. trivial projective fibration). Then*

$$p_* : \underleftarrow{\mathrm{holim}}_I X \to \underleftarrow{\mathrm{holim}}_I Y$$

*is a fibration (resp. trivial fibration) of s**Set**.*

There are a few concepts involved in the proof of Proposition 22.3.

1) Every I-diagram Y has an associated cosimplicial space (aka. Δ-diagram in simplicial sets) $\prod^* Y$ with

$$\prod{}^n Y = \prod{}^* Y(\mathbf{n}) = \prod_{i_0 \to \cdots \to i_n} Y(i_n),$$

and with cosimplicial structure map $\theta_* : \prod^m Y \to \prod^n Y$ defined for an ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$

defined by the picture

$$\begin{array}{ccc} \prod_{\gamma: j_0 \to \cdots \to j_m} Y(j_m) & \xrightarrow{\theta_*} & \prod_{\sigma: i_0 \to \cdots \to i_n} Y(i_n) \\ {\scriptstyle pr_{\theta^*(\sigma)}} \downarrow & & \downarrow {\scriptstyle pr_\sigma} \\ Y(i_{\theta(m)}) & \longrightarrow & Y(i_n) \end{array}$$

in which the bottom horizontal map is induced by the morphism $i_{\theta(m)} \to i_n$ of I.

2) There is a cosimplicial space Δ consisting of the standard n-simplices and the maps between them, and there is a natural bijection

$$\hom(\Delta, \prod{}^* Y) \cong \hom(B(I/?), Y)$$

This bijection induces a natural isomorphism of simplicial sets

$$\mathbf{hom}(\Delta, \prod{}^* Y) \cong \mathbf{hom}(B(I/?), Y) = \underleftarrow{\text{holim}}_I Y.$$

Bousfield and Kan call this isomorphism "cosimplicial replacement of diagrams" in [1].

3) We also use the "matching spaces" $M^n Z$ for a cosimplicial space Z. Explicitly,

$$M^n Z \subset \prod_{i=0}^{n} Z^n$$

is the set of $(n+1)$-tuples $(z_0, \ldots, z_n)$ such that $s^j z_i = s^i z_{j+1}$ for $i \leq j$.

There is a natural simplicial set map

$$s : Z^{n+1} \to M^n Z$$

defined by $s(z) = (s^0 z, s^1 z, \dots, s^n z)$.

Lemma 22.4. *Suppose X is an I-diagram of sets. Then the map*

$$s : \prod{}^{n+1} X = \prod_{\sigma : i_0 \to \dots \to i_{n+1}} X(i_{n+1}) \to M^n \prod{}^{*} X$$

factors through a bijection

$$\prod_{\sigma : i_0 \to \dots \to i_{n+1} \in D(BI)_{n+1}} X(i_{n+1}) \xrightarrow{\cong} M^n \prod{}^{*} X,$$

where $D(BI)_{n+1}$ is the set of degenerate simplices in BI_{n+1}.

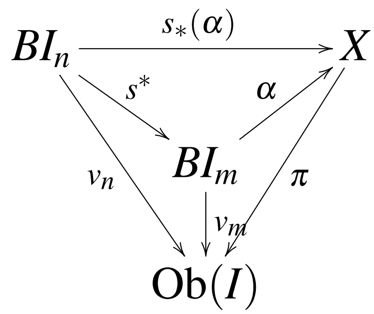
Proof. Write $X = \bigsqcup_{i \in \mathrm{Ob}(I)} X(i)$, and let $\pi : X \to \mathrm{Ob}(I)$ be the canonical map.

An element α of $\prod^m X$ is a commutative diagram

$$\begin{array}{ccc} BI_m & \xrightarrow{\alpha} & X \\ & {\scriptstyle v_m} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & \mathrm{Ob}(I) & \end{array}$$

where v_m is induced by the inclusion $\{m\} \subset \mathbf{m}$ of the vertex m.

If $s : \mathbf{m} \to \mathbf{n}$ is an ordinal number epimorphism then the diagram



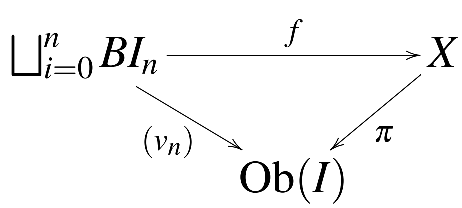
commutes.

The degeneracies $s_i : BI_n \to BI_{n+1}$ take values in DBI_{n+1} and the simplicial identities $s_i s_j = s_{j+1} s_i$, $i \leq j$ determine a coequalizer

$$\bigsqcup_{i \leq j} BI_{n-1} \rightrightarrows \bigsqcup_{i=0}^{n} BI_n \to DBI_{n+1}.$$

Write p_1, p_2 for the maps defining the coequalizer.

An element of $M_n \prod^* X$ is a map



fibred over $\mathrm{Ob}(I)$, such that $f \cdot p_1 = f \cdot p_2$. It follows that f factors uniquely through a function $DBI_{n+1} \to X$, fibred over $\mathrm{Ob}(I)$. □

Proof of Proposition 22.3. By an adjointness argument and cosimplicial replacement of diagrams, showing that the map $\underleftarrow{\text{holim}}_I X \to \underleftarrow{\text{holim}}_I Y$ has the RLP wrt an inclusion $i : K \subset L$ of simplicial sets amounts to solving a lifting problem

$$\begin{array}{ccc} \Delta \times K & \longrightarrow & \prod^* X \\ {\scriptstyle 1\times i}\downarrow & \nearrow & \downarrow \\ \Delta \times L & \longrightarrow & \prod^* Y \end{array}$$

in cosimplicial spaces.

One solves such lifting problems inductively in cosimplicial degrees by solving lifting problems

$$\begin{array}{ccc} (L \times \partial\Delta^{n+1}) \cup (K \times \Delta^{n+1}) & \longrightarrow & \prod^{n+1} X \\ \downarrow & \nearrow & \downarrow {\scriptstyle (p,s)} \\ L \times \Delta^{n+1} & \longrightarrow & \prod^{n+1} Y \times_{M^n \prod^* Y} M^n \prod^* X \end{array}$$

By Lemma 22.4, solving this lifting problem amounts to solving lifting problems

$$\begin{array}{ccc} (L \times \partial\Delta^{n+1}) \cup (K \times \Delta^{n+1}) & \longrightarrow & X(i_{n+1}) \\ \downarrow & \nearrow & \downarrow {\scriptstyle p} \\ L \times \Delta^{n+1} & \longrightarrow & Y(i_{n+1}) \end{array}$$

one for each non-degenerate simplex $\sigma : i_0 \to \cdots \to i_{n+1}$ of BI_{n+1}. This can be done if either $K \subset L$ is anodyne or if p is trivial, since p is a projective fibration. $\square$

Corollary 22.5. *Suppose X is a projective fibrant I-diagram and that $X \to Z$ is an injective fibrant model of X. Then there are weak equivalences*

$$\underleftarrow{\mathrm{holim}}_I X \xrightarrow{\simeq} \underleftarrow{\mathrm{holim}}_I Z \xleftarrow{\simeq} \varprojlim_I Z.$$

Example: Every bisimplicial set X is a functor

$$X : \Delta^{op} \to s\mathbf{Set}.$$

The homotopy colimit $\underrightarrow{\mathrm{holim}}_{\Delta^{op}} X$ is defined by the coend (ie. colimit of all diagrams)

$$\begin{array}{ccc} B(\mathbf{m}/\Delta^{op}) \times X_n & \xrightarrow{1\times\theta^*} & B(\mathbf{m}/\Delta^{op}) \times X_m \\ {\scriptstyle \theta^*\times 1}\downarrow & & \\ B(\mathbf{n}/\Delta^{op}) \times X_n & & \end{array}$$

and therefore by the coend

$$\begin{array}{ccc} B(\Delta/\mathbf{m}) \times X_n & \xrightarrow{1\times\theta^*} & B(\Delta/\mathbf{m}) \times X_m \\ {\scriptstyle \theta\times 1}\downarrow & & \\ B(\Delta/\mathbf{n}) \times X_n & & \end{array}$$

There is a natural map of cosimplicial categories

$$h : \Delta/\mathbf{n} \to \mathbf{n}$$

(the "last vertex map") which takes an object $\alpha : \mathbf{k} \to \mathbf{n}$ to $\alpha(k) \in \mathbf{n}$.

This map induces a morphism of coends

$$B(\Delta/\mathbf{n}) \times X_n \xrightarrow{h\times 1} \Delta^n \times X_n,$$

and therefore induces a natural map

$$h_* : \varinjlim\nolimits_{\Delta^{op}} X \to d(X).$$

Note: here the colimit symbol stands for $\underrightarrow{\mathrm{holim}}_{\Delta^{op}}$.

Claim: This map h_* is a weak equivalence.

Both functors involved in h preserve levelwise weak equivalences in X, so we can assume that X is projective cofibrant. If Y is a Kan complex, then the induced map

$$\mathbf{hom}(d(X),Y) \to \mathbf{hom}(\underrightarrow{\mathrm{holim}}_{\Delta^{op}} X, Y)$$

can be identified up to isomorphism with the map

$$\mathbf{hom}(X,\mathbf{hom}(\Delta,Y)) \to \mathbf{hom}(X,\mathbf{hom}(B(\Delta/?),Y)). \tag{6}$$

The map

$$\mathbf{hom}(\Delta,Y) \to \mathbf{hom}(B(\Delta/?),Y)$$

is a weak equivalence of projective fibrant simplicial spaces, so the map in (6) is a weak equivalence since X is projective cofibrant.

This is true for all Kan complexes Y, so h_* is a weak equivalence as claimed.

Example: Suppose Y is an injective fibrant cosimplicial space. Then the weak equivalence h induces a weak equivalence

$$\mathbf{hom}(\Delta,Y) \xrightarrow{h_*} \mathbf{hom}(B(\Delta/?),Y) = \underleftarrow{\mathrm{holim}}_{\Delta} Y.$$

This is also true if the cosimplicial space Y is Bousfield-Kan fibrant [1] in the sense that all maps

$$s : Y^{n+1} \to M_n Y$$

are fibrations — see [1, X.4] or [2]. Every injective fibrant cosimplicial space is fibrant in this sense.

Following [1], the space $\mathbf{hom}(\Delta, Y)$ is usually denoted by $\mathrm{Tot}(Y)$.

23 Applications, Quillen's Theorem B

Suppose $p : X \to Y$ is a map of simplicial sets, and choose pullbacks

$$\begin{array}{ccc} p^{-1}(\sigma) & \longrightarrow & X \\ \downarrow & & \downarrow p \\ \Delta^n & \xrightarrow[\sigma]{} & Y \end{array}$$

for all simplices $\sigma : \Delta^n \to Y$ of the base Y.

A morphism $\alpha : \sigma \to \tau$ in Δ/Y of Y induces a simplicial set map $p^{-1}(\sigma) \to p^{-1}(\tau)$, and we have a functor

$$p^{-1} : \Delta/Y \to s\mathbf{Set}.$$

The maps $p^{-1}(\sigma) \to X$ induce maps of simplicial sets

$$\omega : \bigsqcup_{\sigma_0 \to \cdots \to \sigma_n} p^{-1}(\sigma_0) \to X$$

or rather a morphism of bisimplicial sets

$$\omega : \underrightarrow{\operatorname{holim}}_{\sigma:\Delta^n \to Y} p^{-1}(\sigma) \to X.$$

Lemma 23.1. *The bisimplicial set map*

$$\omega : \underrightarrow{\operatorname{holim}}_{\sigma:\Delta^n \to Y} p^{-1}(\sigma) \to X$$

is a diagonal weak equivalence.

Proof. The simplicial set Y is a colimit of its simplices in the sense that the canonical map

$$\varinjlim_{\Delta^n \to Y} \Delta^n \to Y$$

is an isomorphism. The pullback functor is exact, so the canonical map

$$\varinjlim_{\Delta^n \to Y} p^{-1}(\sigma) \to X$$

is an isomorphism.

Take $\tau \in X_m$. Then fibre $\omega^{-1}(\tau)$ over τ for the simplicial set map

$$\omega : \bigsqcup_{\sigma_0 \to \cdots \to \sigma_n} p^{-1}(\sigma_0)_m \to X_m$$

is the nerve of a category C_τ whose objects consist of pairs (σ, y), where $\sigma : \Delta^n \to Y$ is a simplex of Y and $y \in p^{-1}(\sigma)_m$ such that $y \mapsto \tau$ under the map $p^{-1}(\sigma) \to X$.

A morphism $(\sigma, y) \to (\gamma, z)$ of C_τ is a map $\sigma \to \gamma$ of the simplex category Δ/Y such that $y \mapsto z$ under the map $p^{-1}(\sigma) \to p^{-1}(\gamma)$.

There is an element $x_\tau \in p^{-1}(p(\tau))$ such that $x_\tau \mapsto \tau \in X$ and $x_\tau \mapsto \iota_m \in \Delta^m$. The element $(p(\tau), x_\tau)$ is initial in C_τ (exercise), and this is true for all $\tau \in X_m$, so the map ω is a weak equivalence in each vertical degree m.

Finish the proof by using Lemma 21.1. □

Here's a first consequence, originally due to Kan and Thurston [4]:

Corollary 23.2. *There are natural weak equivalences*

$$B(\Delta/X) \xleftarrow{\simeq} \underrightarrow{\operatorname{holim}}_{\Delta^n \to X} \Delta^n \xrightarrow{\simeq} X$$

for each simplicial set X.

Proof. The map

$$\underrightarrow{\operatorname{holim}}_{\Delta^n \to X} \Delta^n \to B(\Delta/X)$$

is induced by the weak equivalence of diagrams $\Delta^n \to *$ on the simplex category.

The other map is a weak equivalence, by Lemma 23.1 applied to the identity map $X \to X$. $\square$

Suppose $f : C \to D$ is a functor between small categories, and consider the pullback squares of functors

$$\begin{array}{ccc} f/d & \longrightarrow & C \\ \downarrow & & \downarrow \\ D/d & \longrightarrow & D \end{array}$$

for $d \in \mathrm{Ob}(D)$.

Here, f/d is the category whose objects are pairs (c,α) where $c \in \mathrm{Ob}(C)$ and $\alpha : f(c) \to d$ is a morphism of D.

A morphism $\gamma : (c,\alpha) \to (c',\beta)$ is a morphism $\gamma : c \to c'$ of C such that the diagram

$$\begin{array}{ccc} f(c) & & \\ & \searrow^{\alpha} & \\ {\scriptstyle f(\gamma)}\downarrow & & d \\ & \nearrow_{\beta} & \\ f(c') & & \end{array}$$

commutes in D.

Any morphism $d \to d'$ of D induces a functor $f/d \to f/d'$, and there is a D-diagram in simplicial sets $d \mapsto B(f/d)$.

The forgetful functors $f/d \to C$ (with $(c, \alpha) \mapsto c$ define a map of bisimplicial sets

$$\omega : \bigsqcup_{d_0 \to \cdots \to d_n} B(f/d_0) \to BC.$$

Then we have the following categorical analogue of Lemma 23.1:

Lemma 23.3 (Quillen [5]). *The map ω induces a weak equivalence of diagonal simplicial sets.*

Proof. The homotopy colimit in the statement of the Lemma is the bisimplicial set with (n,m)-bisimplices consisting of pairs

$$(c_0 \to \cdots \to c_m, f(c_m) \to d_0 \to \cdots \to d_n)$$

of strings of arrows in C and D, respectively.

The fibre of ω over the m-simplex $c_0 \to \cdots \to c_m)$ is the nerve $B(f(c_m)/D)$, which is contractible.

This is true for all elements of BC_m so ω is a weak equivalence in each vertical degree m, and is therefore a diagonal weak equivalence. □

Now here's what we're really after:

Theorem 23.4 (Quillen). *Suppose $X : I \to s\mathbf{Set}$ is a diagram such that each map $i \to j$ of I induces a weak equivalence $X(i) \to X(j)$.*

Then all pullback diagrams

$$\begin{array}{ccc} X(i) & \longrightarrow & \underrightarrow{\text{holim}}_I X \\ \downarrow & & \downarrow \pi \\ \Delta^0 & \xrightarrow[i]{} & BI \end{array}$$

are homotopy cartesian.

Functors $X : I \to s\mathbf{Set}$ which take all morphisms of I to weak equivalences of simplicial sets are **diagrams of equivalences**.

If $f : I \to J$ is a functor between small categories and $X : J \to s\mathbf{Set}$ is a J-diagram of simplicial sets, then the diagram

$$\begin{array}{ccc} \underrightarrow{\text{holim}}_I \, Xf & \longrightarrow & \underrightarrow{\text{holim}}_J \, X \\ \pi \downarrow & & \downarrow \pi \\ BI & \xrightarrow[f_*]{} & BJ \end{array}$$

is a pullback (exercise).

In particular, the diagram in the statement of the Theorem is a pullback.

Proof. There are two tricks in this proof:

- Factor the map $i : \Delta^0 \to BI$ as the composite

$$\begin{array}{ccccc} \Delta^0 & & \xrightarrow{i} & & BI \\ & {}_j\searrow & & \nearrow_p & \\ & & U & & \end{array}$$

such that p is a fibration and j is a trivial cofibration, and show that the induced map $X(i) \to U \times_{BI} \underrightarrow{\text{holim}}_I X$ is a weak equivalence.

- Use the fact that pullback along a simplicial set map is exact (so it preserves all colimits and monomorphisms), to reduce to showing that every composite $\Lambda^n_k \subset \Delta^n \to BI$ induces a weak equivalence

$$\Lambda^n_k \times_{BI} \underrightarrow{\text{holim}}_I X \to \Delta^n \times_{BI} \underrightarrow{\text{holim}}_I X.$$

To finish off, the map $\Delta^n \to BI$ is induced by a functor $\sigma : \mathbf{n} \to I$, so there is an isomorphism

$$\underrightarrow{\text{holim}}_{\mathbf{n}} X\sigma \cong \Delta^n \times_{BI} \underrightarrow{\text{holim}}_I X.$$

The composite functor $X\sigma$ is a diagram of equivalences, and so the initial object $0 \in \mathbf{n}$ determines a natural transformation

$$X\sigma(0) \to X\sigma$$

of $\mathbf{n}$-diagrams defined on a constant diagram which is a weak equivalence of diagrams.

The induced weak equivalence

$$B\mathbf{n} \times X(\sigma(0)) \cong \underrightarrow{\text{holim}}_{\mathbf{n}} X(\sigma(0)) \to \underrightarrow{\text{holim}}_{\mathbf{n}} X\sigma$$

pulls back to a weak equivalence

$$\Lambda^n_k \times X(\sigma(0)) \cong \Lambda^n_k \times_{B\mathbf{n}} \underrightarrow{\text{holim}}_{\mathbf{n}} X(\sigma(0)) \to \Lambda^n_k \times_{B\mathbf{n}} \underrightarrow{\text{holim}} X\sigma.$$

It follows that there is a commutative diagram

$$\begin{array}{ccc} \Lambda^n_k \times X(\sigma(0)) & \xrightarrow{\simeq} & \Delta^n \times X(\sigma(0)) \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ \Lambda^n_k \times_{BI} \underrightarrow{\operatorname{holim}}_I X & \longrightarrow & \Delta^n \times_{BI} \underrightarrow{\operatorname{holim}}_I X. \end{array}$$

so the bottom horizontal map is a weak equivalence. □

It's hard to overstate the importance of Theorem 23.4.

The conditions for the Theorem are always satisfied, for example, by diagrams defined on groupoids. In particular, if G is a group and X is a space carrying a G-action, then there is a fibre sequence

$$X \to EG \times_G X \to BG$$

defined by the Borel construction, aka. the homotopy colimit for the action of G on X.

Theorem 23.4 first appeared as a lemma in the proof of Quillen's "Theorem B" in [5].

Theorem B is the homotopy-theoretic starting point for Quillen's description of higher algebraic K-theory:

Theorem 23.5 (Quillen)**.** *Suppose $f : C \to D$ is a functor between small categories such that all* *morphisms $d \to d'$ of D induce weak equivalences $B(f/d) \to B(f/d')$.*

Then all diagrams

$$\begin{array}{ccc} B(f/d) & \longrightarrow & BC \\ \downarrow & & \downarrow{\scriptstyle f_*} \\ B(D/d) & \longrightarrow & BD \end{array}$$

of simplicial set maps are homotopy cartesian.

Proof. Form the diagram

$$\begin{array}{ccccc} B(f/d) & \longrightarrow & \underset{\overrightarrow{d \in D}}{\text{holim}}\, B(f/d) & \xrightarrow{\simeq} & BC \\ \downarrow & \mathbf{I} & \downarrow & \mathbf{II} & \downarrow \\ B(D/d) & \longrightarrow & \underset{\overrightarrow{d \in D}}{\text{holim}}\, B(D/d) & \xrightarrow[\simeq]{} & BD \\ \simeq\downarrow & \mathbf{III} & \downarrow\simeq & & \\ \Delta^0 & \xrightarrow[d]{} & BD & & \end{array}$$

The indicated horizontal maps are weak equivalences by Lemma 23.3, while the indicated vertical maps are weak equivalences since the spaces $B(D/d)$ are contractible.

Theorem 23.4 says that the composite diagram $\mathbf{I}+\mathbf{III}$ is homotopy cartesian, so Lemma 18.5 (Lecture 07) implies that $\mathbf{I}$ is homotopy cartesian. It follows, again from Lemma 18.5, that the composite $\mathbf{I}+\mathbf{II}$ is homotopy cartesian. □

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