Contents

32 Postnikov towers

Suppose X is a simplicial set, and that $x, y : \Delta^n \to X$ are n-simplices of X.

Say that x is k**-equivalent** to y and write $x \sim_k y$ if there is a commutative diagram

$$\begin{array}{ccc} \mathrm{sk}_k\,\Delta^n & \xrightarrow{i} & \Delta^n \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle y} \\ \Delta^n & \xrightarrow[x]{} & X \end{array}$$

or if

$$x|_{\mathrm{sk}_k\,\Delta^n} = y|_{sk_k \Delta^n}.$$

Write $X(k)_n$ for the set of equivalence classes of n-simplices of X_n mod k-equivalence.

Every morphism $\Delta^m \to \Delta^n$ induces a morphism $\mathrm{sk}_k\,\Delta^m \to \mathrm{sk}_k\,\Delta^n$. Thus, if $x \sim_k y$ then $\theta^*(x) \sim_k \theta^*(y)$.

The sets $X(k)_m$, $m \geq 0$, therefore assemble into a simplicial set $X(k)$.

The map

$$\pi_k : X \to X(k)$$

is the canonical surjection. It is natural in simplicial sets X, and is defined for $k \geq 0$.

$X(k)$ is the k^{th} **Postnikov section** of X.

If $x \sim_{k+1} y$ then $x \sim_k y$. It follows that there are natural commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\pi_{k+1}} & X(k+1) \\ & \searrow^{\pi_k} & \downarrow p \\ & & X(k) \end{array}$$

The system of simplicial set maps

$$X(0) \xleftarrow{p} X(1) \xleftarrow{p} X(2) \xleftarrow{p} \dots$$

is called the **Postnikov tower** of X.

The map $\pi_k : X_n \to X(k)_n$ of n-simplices is a bijection for $n \leq k$, since $\mathrm{sk}_k \Delta^n = \Delta^n$ in that case.

It follows that the induced map

$$X \to \varprojlim_k X(k)$$

is an isomorphism of simplicial sets.

Lemma 32.1. *Suppose X is a Kan complex. Then*

1) *$\pi_k : X \to X(k)$ is a fibration and $X(k)$ is a Kan complex for $k \geq 0$.*
2) *$\pi_k : X \to X(k)$ induces a bijection $\pi_0(X) \cong \pi_0 X(k)$ and isomorphisms*

$$\pi_i(X,x) \xrightarrow{\cong} \pi_i(X(k),x)$$

   for $1 \leq i \leq k$.

3) *$\pi_i(X(k),x) = 0$ for $i > k$.*

Proof. Suppose given a commutative diagram

$$\begin{array}{ccc} \Lambda^n_r & \xrightarrow{(x_0,\dots,\hat{x}_r,\dots,x_n)} & X \\ \downarrow & & \downarrow \pi_k \\ \Delta^n & \xrightarrow[{[y]}]{} & X(k) \end{array}$$

If $n \leq k$ the lift $y : \Delta^n \to X$ exists because π_k is an isomorphism in degrees $\leq k$.

If $n = k+1$ then $d_i(y) = d_i([y]) = x_i$ for $i \neq r$, so that the representative y is again a suitable lift.

If $n > k+1$ there is a simplex $x \in X_n$ such that $d_i x = x_i$ for $i \neq r$, since X is a Kan complex.

There is an identity $\mathrm{sk}_k(\Lambda^n_r) = \mathrm{sk}_k(\Delta^n)$ for since $n \geq k+2$, and it follows that $[x] = [y]$.

We have proved that π_k is a Kan fibration.

Generally, if $p : X \to Y$ is a surjective fibration and X is a Kan complex, then Y is a Kan complex (exercise).

It follows that all Postnikov sections $X(k)$ are Kan complexes.

If $n > k$, $x \in X_0 = X(k)_0$ and the picture

$$\begin{array}{ccc} \partial\Delta^n & & \\ \downarrow & \searrow^{x} & \\ \Delta^n & \xrightarrow[{[\alpha]}]{} & X(k) \end{array}$$

defines an element of $\pi_n(X(k), x)$, then all faces of the representative $\alpha : \Delta^n \to X$ and all faces of the element $x : \Delta^n \to X$ have the same k-skeleton, α and x have the same k-skeleton, and so $[\alpha] = [x]$.

We have proved statements 1) and 3). Statement 2) is an exercise. □

The fibration trick used in the proof of Lemma 32.1 is a special case of the following:

Lemma 32.2. *Suppose given a commutative diagram of simplicial set maps*

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ & {}_{q}\searrow & \downarrow{\scriptstyle \pi} \\ & & Z \end{array}$$

such that p and q are fibrations and p is surjective in all degrees.

Then π is a fibration.

Proof. The proof is an exercise. □

Remarks:

1) If X is a Kan complex, it follows from Lemma 32.1 and Lemma 32.2 that all maps

$$p : X(k+1) \to X(k)$$

in the Postnikov tower for X are fibrations.

2) There is a natural commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & B\pi(X) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_{1*}}\ \simeq \\ X(1) & \xrightarrow[\eta]{\simeq} & B\pi(X(1)) \end{array}$$

for Kan complexes X, in which the indicated maps η and π_{1*} are weak equivalences by Lemma 28.5

3) Suppose that X is a connected Kan complex. The fibre $F_n(X)$ of the fibration $\pi_n : X \to X(n)$ is the **n-connected cover** of X. The space $F_n(X)$ is n-connected, and the maps

$$\pi_k(F_n(X), z) \to \pi_k(X, z)$$

are isomorphisms for $k \geq n+1$, by Lemma 32.1.

The homotopy fibres of the map $\pi_1 : X \to X(1)$, equivalently of the map $X \to B(\pi(X))$ are the **universal covers** of X.

All universal covers of X are simply connected, and are weakly equivalent because X is connected.

More is true. Replace $\eta : X \to B\pi(X)$ by a fibration $p : Z \to B(\pi(X))$, and form the pullbacks

$$\begin{array}{ccc} p^{-1}(x) & \longrightarrow & Z \\ \downarrow & & \downarrow p \\ B(\pi(X)/x) & \longrightarrow & B\pi(X) \end{array}$$

All spaces $p^{-1}(x)$ are universal covers, and there are weak equivalences

$$\underrightarrow{\mathrm{holim}}_{x \in \pi(X)}\ p^{-1}(x) \xrightarrow{\simeq} Z \xleftarrow{\simeq} X.$$

Thus, every space X is a homotopy colimit of universal covers, indexed over its fundamental groupoid $\pi(X)$.

33 The Hurewicz Theorem

Suppose X is a pointed space.

The **Hurewicz map** for X is the composite

$$X \xrightarrow{\eta} \mathbb{Z}(X) \to \mathbb{Z}(X)/\mathbb{Z}(*)$$

where $*$ denotes the base point of X.

The homology groups of the quotient

$$\tilde{\mathbb{Z}}(X) := \mathbb{Z}(X)/\mathbb{Z}(*)$$

are the **reduced homology groups** of X, and one writes

$$\tilde{H}_n(X) = H_n(\tilde{\mathbb{Z}}(X)).$$

The reduced homology groups $\tilde{H}_n(X,A)$ are defined by

$$\tilde{H}_n(X,A) = H_n(\tilde{\mathbb{Z}}(X) \otimes A)$$

for any abelian group A.

The Hurewicz map is denoted by h. We have

$$h : X \to \tilde{\mathbb{Z}}(X).$$

Lemma 33.1. *Suppose that* π *is a group.*

The homomorphism

$$h_* : \pi_1(B\pi) \to \tilde{H}_1(B\pi)$$

is isomorphic to the homomorphism

$$\pi \to \pi/[\pi, \pi].$$

Proof. From the Moore chain complex $\mathbb{Z}(B\pi)$, the group $H_1(B\pi) = \tilde{H}_1(B\pi)$ is the free abelian group $\mathbb{Z}(\pi)$ on the elements of π modulo the relations $g_1g_2 - g_1 - g_2$ and $e = 0$.

The composite

$$\pi \xrightarrow{\cong} \pi_1(B\pi) \xrightarrow{h_*} H_1(B\pi)$$

is the canonical map. □

Consequence: If A is an abelian group, the map

$$h_* : \pi_1(BA) \to \tilde{H}_1(BA)$$

is an isomorphism.

Lemma 33.2. *Suppose* X *is a connected pointed space.*

Then $\eta : X \to B\pi(X)$ *induces an isomorphism*

$$H_1(X) \xrightarrow{\cong} H_1(B\pi(X)).$$

Proof. The homotopy fibre F of η is simply connected, so $H_1(F) = 0$ by Lemma 33.1 (or otherwise — exercise).

It follows that $E_2^{0,1} = 0$ in the (general) Serre spectral sequence for the fibre sequence

$$F \to X \to B\pi(X)$$

Thus, $E_\infty^{0,1} = 0$, while $E_2^{1,0} = E_\infty^{1,0} = H_1(B\pi(X))$.

The edge homomorphism

$$H_1(X) \to H_1(B\pi(X)) = E_\infty^{1,0}$$

is therefore an isomorphism. □

The proof of the following result is an exercise:

Corollary 33.3. *Suppose X is a connected pointed Kan complex.*

The Hurewicz homomorphism

$$h_* : \pi_1(X) \to \tilde{H}_1(X)$$

is an isomorphism if $\pi_1(X)$ is abelian.

The following result gives the relation between the path-loop fibre sequence and the Hurewicz map.

Lemma 33.4. *Suppose Y is an n-connected pointed Kan complex, with $n \geq 1$.*

For $2 \leq i \leq 2n$ there is a commutative diagram

$$\begin{array}{ccc} \pi_i(Y) & \xrightarrow[\cong]{\partial} & \pi_{i-1}(\Omega Y) \\ {\scriptstyle h*}\downarrow & & \downarrow{\scriptstyle h*} \\ \tilde{H}_i(Y) & \xrightarrow[d_i]{\cong} & \tilde{H}_{i-1}(\Omega Y) \end{array}$$

Proof. Form the diagram

$$\begin{array}{ccccc} \tilde{\mathbb{Z}}(\Omega Y) & = & \tilde{\mathbb{Z}}(\Omega Y) & \xleftarrow{h} & \Omega Y \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathbb{Z}}(C\Omega Y) & \longrightarrow & \tilde{\mathbb{Z}}(PY) & \xleftarrow{h} & PY \\ \downarrow & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p} \\ \tilde{\mathbb{Z}}(\Sigma\Omega Y) & \xrightarrow[\varepsilon_*]{} & \tilde{\mathbb{Z}}(Y) & \xleftarrow[h]{} & Y \end{array}$$

By replacing p_* by a fibration one finds a comparison diagram of fibre sequences and there is an induced diagram

$$\begin{array}{ccccc} \pi_i(Y) & & \xrightarrow{\qquad\partial\qquad} & & \pi_{i-1}(\Omega Y) \\ {\scriptstyle h_*}\downarrow & & & & \downarrow{\scriptstyle h_*} \\ \tilde{H}_i(Y) & \xleftarrow[\varepsilon_*]{\cong} & \tilde{H}_i(\Sigma\Omega Y) & \xrightarrow[\partial]{\cong} & \tilde{H}_{i-1}(\Omega Y) \end{array}$$

The bottom composite is the transgression d_i by Corollary 31.4. $\square$

Theorem 33.5 (Hurewicz Theorem)**.** *Suppose X is an n-connected pointed Kan complex, and that $n \geq 1$.*

Then the Hurewicz homomorphism

$$h_* : \pi_i(X) \to \tilde{H}_i(X)$$

is an isomorphism if $i = n+1$ and is an epimorphism if $i = n+2$.

The proof of the Hurewicz Theorem requires some preliminary observations about Eilenberg-Mac Lane spaces:

The **good truncation** $T_m C$ for a chain complex C is the chain complex

$$C_0 \xleftarrow{\partial} \dots \xleftarrow{\partial} C_{m-1} \xleftarrow{\partial *} C_m/\partial(C_{m+1}) \leftarrow 0 \dots$$

The canonical map

$$C \to T_m(C)$$

induces isomorphisms $H_i(C) \cong H_i(T_m(C))$ for $i \leq m$, while $H_i(T_m(C)) = 0$ for $i > m$.

The isomorphism $H_m(C) \cong H_m(T_m(C))$ is the "goodness". It means that the functor $C \mapsto T_m(C)$ preserves homology isomorphisms.

It follows that the composite

$$Y \xrightarrow{h_*} \tilde{\mathbb{Z}}(Y) \cong \Gamma N \tilde{\mathbb{Z}}(Y) \to \Gamma T_m N \tilde{\mathbb{Z}}(Y)$$

is a weak equivalence for a space Y of type $K(A,m)$, if A is abelian.

For this, we need to show that h_* induces an isomorphism $\pi_m(Y) \to \tilde{H}_m(Y)$.

This seems like a special case of the Hurewicz theorem, but it is true for $m = 1$ by Corollary 33.3, and then true for all $m \geq 1$ by an inductive argument that uses Lemma 33.4.

We have shown that there is a weak equivalence $Y \to B$ where B is a simplicial abelian group of type $K(A,m)$.

It is an exercise to show that B is weakly equivalent as a simplicial abelian group to the simplicial abelian group

$$K(A,m) = \Gamma(A(m)).$$

Proof of Theorem 33.5. The space $X(n+1)$ is an Eilenberg-Mac Lane space of type $K(A,n+1)$, where $A = \pi_{n+1}(X)$.

The Hurewicz map

$$h_* : \pi_m(Y) \to \tilde{H}_m(Y)$$

is an isomorphism for all spaces Y of type $K(A,m)$, for all $m \geq 1$.

We know from Lemma 29.5 and the remarks above that there is an isomorphism

$$H_{m+1}(Y) = 0$$

for all spaces Y of type $K(A,m)$, for all $m \geq 2$.

It follows that

$$H_{n+2}(X(n+1)) = 0.$$

Now suppose that F is the homotopy fibre of the map $\pi_{n+1} : X \to X(n+1)$.

There are diagrams

$$\begin{array}{ccc} \pi_{n+1}(X) & \xrightarrow{\cong} & \pi_{n+1}(X(n+1)) \\ {\scriptstyle h_*}\downarrow & & \cong\downarrow{\scriptstyle h_*} \\ \tilde{H}_{n+1}(X) & \longrightarrow & \tilde{H}_{n+1}(X(n+1)) \end{array} \qquad \begin{array}{ccc} \pi_{n+2}(F) & \xrightarrow{\cong} & \pi_{n+2}(X) \\ {\scriptstyle h_*}\downarrow & & \downarrow{\scriptstyle h_*} \\ \tilde{H}_{n+2}(F) & \longrightarrow & \tilde{H}_{n+2}(X) \end{array}$$

The Serre spectral sequence for the fibre sequence

$$F \to X \to X(n+1)$$

is used to show that

1) the map $H_{n+1}(X) \to H_{n+1}(X(n+1))$ is an isomorphism since F is $(n+1)$-connected, and

2) the map $H_{n+2}(F) \to H_{n+2}(X)$ is surjective, since $H_{n+2}(X(n+1)) = 0$.

The isomorphism statement in the Theorem is a consequence of statement 1).

It follows that the map $h_* : \pi_{n+2}(F) \to \tilde{H}_{n+2}(F)$ is an isomorphism since F is $(n+1)$-connected.

The surjectivity statement of the Theorem is then a consequence of statement 2). □

34 Freudenthal Suspension Theorem

Here's a first consequence of the Hurewicz Theorem (Theorem 33.5):

Corollary 34.1. *Suppose X is an n-connected space where* $n \geq 0$*.*

Then the suspension $\Sigma(X)$ *is* $(n+1)$*-connected.*

Proof. The case $n = 0$ has already been done, as an exercise. Suppose that $n \geq 1$.

Then ΣX is at least simply connected since X is connected, and $\tilde{H}_k(\Sigma X) = 0$ for $k \leq n+1$.

Thus, the first non-vanishing homotopy group $\pi_r(\Sigma X)$ is in degree at least $n+2$. □

Theorem 34.2. *[Freudenthal Suspension Theorem] Suppose X is an n-connected pointed Kan complex where $n \geq 0$.*

The homotopy fibre F of the canonical map

$$\eta : X \to \Omega\Sigma X$$

is $2n$-connected.

Remark: "The canonical map" in the statement of the Theorem is actually the "derived" map, meaning the composite

$$X \to \Omega(\Sigma X) \xrightarrow{j_*} \Omega(\Sigma X_f),$$

where $j : \Sigma X \to \Sigma X_f$ is a fibrant model, ie. a weak equivalence such that ΣX_f is fibrant.

Proof. In the triangle identity

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\Sigma\eta} & \Sigma\Omega\Sigma(X) \\ & {}_{1}\searrow & \downarrow{\scriptstyle \varepsilon} \\ & & \Sigma X \end{array}$$

the space ΣX is $(n+1)$-connected (Corollary 34.1) so that the map ε induces isomorphisms

$$\tilde{H}_i(\Sigma\Omega\Sigma X) \xrightarrow{\cong} \tilde{H}_i(\Sigma X)$$

for $i \leq 2n+2$, by Corollary 31.4.

It follows that η induces isomorphisms

$$\tilde{H}_i(X) \xrightarrow{\cong} \tilde{H}_i(\Omega\Sigma X) \qquad (1)$$

for $i \leq 2n+1$.

In the diagram

$$\begin{array}{ccc} \pi_{n+1}(X) & \xrightarrow{\eta_*} & \pi_{n+1}(\Omega\Sigma X) \\ h\downarrow & & \cong\downarrow h \\ H_{n+1}(X) & \xrightarrow[\eta_*]{\cong} & H_{n+1}(\Omega\Sigma X) \end{array}$$

the indicated Hurewicz map is an isomorphism for $n > 0$ since $\pi_1(\Omega\Sigma X)$ is abelian (Corollary 33.3), while the map $h : \pi_1(X) \to H_1(X)$ is surjective by Lemma 33.1 and Lemma 33.2. It follows that $\eta_* : \pi_{n+1}(X) \to \pi_{n+1}(\Omega\Sigma X)$ is surjective, so F is n-connected.

A Serre spectral sequence argument for the fibre sequence

$$F \to X \xrightarrow{\eta} \Omega\Sigma X$$

shows that that $\tilde{H}_i(F) = 0$ for $i \leq 2n$, so the Hurewicz Theorem implies that F is $2n$-connected.

In effect, $E_2^{i,0} \cong E_\infty^{i,0}$ for $i \leq 2n+1$ and $E_\infty^{p,q} = 0$ for $q > 0$ and $p+q \leq 2n+1$, all by the isomorphisms in (1).

It follows that the first non-vanishing $H_k(F)$ is in degree greater than $2n$. $\square$

Example: The *suspension homomorphism*

$$\Sigma : \pi_i(S^n) \to \pi_i(\Omega(S^{n+1})) \cong \pi_{i+1}(S^{n+1})$$

is an isomorphism if $i \leq 2(n-1)$ and is an epimorphism if $i = 2n-1$.

In effect, the homotopy fibre of $S^n \to \Omega S^{n+1}$ is $(2n-1)$-connected.

In particular, the maps $\Sigma : \pi_{n+k}(S^n) \to \pi_{n+1+k}(S^{n+1})$ are isomorphisms (ie. the groups stabilize) for $n \geq k+2$, ie. $n+k \leq 2n-2$.

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