Contents

38 Spectra

The approach to stable homotopy that follows was introduced in a seminal paper of Bousfield and Friedlander [2], which appeared in 1978.

A **spectrum** X consists of pointed (level) simplicial sets X^n, $n \geq 0$, together with **bonding maps**

$$\sigma : S^1 \wedge X^n \to X^{n+1}.$$

A **map of spectra** $f : X \to Y$ consists of pointed maps $f : X^n \to Y^n$ which respect structure, in that the diagrams

$$\begin{array}{ccc} S^1 \wedge X^n & \xrightarrow{\sigma} & X^{n+1} \\ {\scriptstyle S^1 \wedge f} \downarrow & & \downarrow {\scriptstyle f} \\ S^1 \wedge Y^n & \xrightarrow[\sigma]{} & Y^{n+1} \end{array}$$

commute.

The category of spectra is denoted by **Spt**. This category is complete and cocomplete.

Examples:

1) Suppose Y is a pointed simplicial set. The **suspension spectrum** $\Sigma^\infty Y$ consists of the pointed simplicial sets

$$Y,\ S^1 \wedge Y,\ S^1 \wedge S^1 \wedge Y,\ \dots\ , S^n \wedge Y, \dots$$

where

$$S^n = S^1 \wedge \dots \wedge S^1$$

(n-fold smash power).

The bonding maps of $\Sigma^\infty Y$ are the canonical isomorphisms

$$S^1 \wedge S^n \wedge Y \cong S^{n+1} \wedge Y.$$

There is a natural bijection

$$\hom(\Sigma^\infty Y, X) \cong \hom(X, Y^0).$$

The suspension spectrum functor is left adjoint to the “level 0” functor $X \mapsto X^0$.

2) $S = \Sigma^\infty S^0$ is the **sphere spectrum**.

3) Suppose X is a spectrum and K is a pointed simplicial set.

The spectrum $X \wedge K$ has level spaces

$$(X \wedge K)^n = X^n \wedge K,$$

and bonding maps

$$\sigma \wedge K : S^1 \wedge X^n \wedge K \to X^{n+1} \wedge K.$$

There is a natural isomorphism

$$\Sigma^\infty K \cong S \wedge K.$$

3) $X \wedge S^1$ is the **suspension** of a spectrum X.

The **fake suspension** ΣX of X has level spaces $S^1 \wedge X^n$ and bonding maps

$$S^1 \wedge \sigma : S^1 \wedge S^1 \wedge X^n \to S^1 \wedge X^{n+1}.$$

Remark: There is a commutative diagram

$$\begin{array}{ccc}
S^1 \wedge S^1 \wedge X^n & \xrightarrow{S^1 \wedge \sigma} & S^1 \wedge X^{n+1} \\
{\scriptstyle \tau \wedge X^n} \downarrow & \nearrow & \\
S^1 \wedge S^1 \wedge X^n & & \cong \downarrow {\scriptstyle \tau} \\
{\scriptstyle S^1 \wedge \tau} \downarrow \cong & & \\
S^1 \wedge X^n \wedge S^1 & \xrightarrow[\sigma \wedge S^1]{} & X^{n+1} \wedge S^1
\end{array}$$

where τ flips adjacent smash factors:

$$\tau(x \wedge y) = y \wedge x.$$

The dotted arrow (bonding map induced by $\sigma \wedge S^1$) differs from $S^1 \wedge \sigma$ by precomposition by $\tau \wedge X^n$.

The flip $\tau : S^1 \wedge S^1 \to S^1 \wedge S^1$ is non-trivial: it is multiplication by -1 in $H_2(S^2)$.

Recall some definitions and results from Section 15:

Suppose X is a simplicial set.

Write $\tilde{\mathbb{Z}}(X)$ for the kernel of the map $\mathbb{Z}(X) \to \mathbb{Z}(*)$.

Then $H_n(X,\mathbb{Z}) = \pi_n(\mathbb{Z}(X),0)$ (see Theorem 15.4), and $\tilde{H}_n(X,\mathbb{Z}) = \pi_n(\tilde{\mathbb{Z}}(X),0)$ (reduced homology).

If X is pointed there is a natural isomorphism

$$\tilde{\mathbb{Z}}(X) \cong \mathbb{Z}(X)/\mathbb{Z}(*),$$

and there is a natural pointed map

$$h : X \xrightarrow{\eta} \mathbb{Z}(X) \to \tilde{\mathbb{Z}}(X)$$

(the Hurewicz map).

If A is a simplicial abelian group, there is a natural simplicial map

$$\gamma : S^1 \wedge A \to \tilde{\mathbb{Z}}(S^1) \otimes A =: S^1 \otimes A,$$

defined by $x \wedge a \mapsto x \otimes a$.

4) The **Eilenberg-Mac Lane spectrum** $H(A)$ associated to a simplicial abelian group A consists of the spaces

$$A,\ S^1 \otimes A,\ S^2 \otimes A,\ \dots$$

with bonding maps

$$S^1 \wedge (S^n \otimes A) \xrightarrow{\gamma} S^1 \otimes (S^n \otimes A) \cong S^{n+1} \otimes A.$$

5) Suppose X is a spectrum and K is a pointed simplicial set.

The spectrum $\mathbf{hom}_*(K,X)$ has

$$\mathbf{hom}_*(K,X)^n = \mathbf{hom}_*(K,X^n),$$

with bonding map

$$S^1 \wedge \mathbf{hom}_*(K,X^n) \to \mathbf{hom}_*(K,X^{n+1})$$

adjoint to the composite

$$S^1 \wedge \mathbf{hom}_*(K,X^n) \wedge K \xrightarrow{S^1 \wedge ev} S^1 \wedge X^n \xrightarrow{\sigma} X^{n+1}.$$

There is a natural bijection

$$\hom(X \wedge K, Y) \cong \hom(X, \mathbf{hom}_*(K,Y)).$$

Suppose X is a spectrum and $n \in \mathbb{Z}$.

The **shifted spectrum** $X[n]$ has

$$X[n]^m = \begin{cases} * & m+n < 0 \\ X^{m+n} & m+n \geq 0 \end{cases}$$

Examples: $X[-1]^0 = *$ and $X[-1]^n = X^{n-1}$ for $n \geq 1$.

$X[1]^n = X^{n+1}$ for all $n \geq 0$.

Remarks: 1) The bonding maps define a natural map

$$\Sigma X \to X[1].$$

We'll see later that this map is a stable equivalence, and that there is a stable equivalence $\Sigma X \simeq X \wedge S^1$.

2) There is a natural bijection

$$\hom(X[n], Y) \cong \hom(X, Y[-n])$$

and a stable equivalence $X[n][-n] \to X$, so that all shift operators are invertible in the stable category.

3) There is a natural bijection

$$\hom(\Sigma^\infty K[-n], Y) \cong \hom(K, Y^n)$$

for $n \geq 0$, so that the n^{th} level functor $Y \to Y^n$ has a left adjoint.

4) The n^{th} **layer** $L_n X$ of a spectrum X consists of the spaces

$$X^0, \ \dots \ , X^n, \ S^1 \wedge X^n, \ S^2 \wedge X^n, \ \dots$$

There are obvious maps $L_n X \to L_{n+1} X \to X$ and a natural isomorphism

$$\varinjlim_n L_n X \cong X.$$

The functor $X \mapsto L_n X$ is left adjoint to truncation up to level n.

The system of maps

$$\Sigma^\infty X^0 = L_0 X \to L_1 X \to \dots$$

is called the **layer filtration** of X.

Here's an exercise: show that there are natural pushout diagrams

$$\begin{array}{ccc} \Sigma^\infty(S^1 \wedge X^n)[-n-1] & \longrightarrow & L_n X \\ {\scriptstyle \sigma_*}\downarrow & & \downarrow \\ \Sigma^\infty X^{n+1}[-n-1] & \longrightarrow & L_{n+1} X \end{array}$$

39 Strict model structure

A map $f : X \to Y$ is a **strict (levelwise) weak equivalence** (resp. **strict (levelwise) fibration**) if all maps $f : X^n \to Y^n$ are weak equivalences (resp. fibrations) of pointed simplicial sets.

A **cofibration** is a map $i : A \to B$ such that

1) $i : A^0 \to B^0$ is a cofibration of (pointed) simplicial sets, and

2) all maps

$$(S^1 \wedge B^n) \cup_{(S^1 \wedge A^n)} A^{n+1} \to B^{n+1}$$

are cofibrations.

Exercise: Show that all cofibrations are levelwise cofibrations.

Given spectra X,Y, the function complex $\mathbf{hom}(X,Y)$ is a simplicial set with

$$\mathbf{hom}(X,Y)_n = \hom(X \wedge \Delta^n_+, Y).$$

Recall that $\Delta^n_+ = \Delta^n \sqcup \{*\}$ is the simplex with a disjoint base point attached.

Proposition 39.1. *With these definitions, the category* **Spt** *of spectra satisfies the axioms for a proper closed simplicial model category.*

This model structure is also cofibrantly generated.

Proof. Suppose given a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\beta]{} & Y \end{array}$$

where i is a cofibration and p is a strict fibration and strict weak equivalence.

The lifting θ^0 exists in the diagram

$$\begin{array}{ccc} A^0 & \xrightarrow{\alpha} & X^0 \\ {\scriptstyle i}\downarrow & \overset{\theta^0}{\nearrow} & \downarrow{\scriptstyle p} \\ B^0 & \xrightarrow[\beta]{} & Y^0 \end{array}$$

and then θ^1 exists in the diagram

$$\begin{array}{ccc} (S^1 \wedge B^0) \cup_{(S^1 \wedge A^0)} A^1 & \xrightarrow{(\theta^0_*,\alpha)} & X^1 \\ {\scriptstyle cof}\downarrow & \overset{\theta^1}{\nearrow} & \downarrow{\scriptstyle p} \\ B^1 & \xrightarrow[\beta]{} & Y^1 \end{array}$$

Proceed inductively to show that the lifting problem can be solved.

The lifting problem is solved in a similar way if i is a trivial cofibration and p is a strict fibration. We have proved the lifting axiom **CM4**.

Suppose that $f : X \to Y$ is a map of spectra, and find a factorization

$$\begin{array}{ccc} X^0 & \xrightarrow{i^0} & Z^0 \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p^0} \\ & & Y^0 \end{array}$$

in level 0, where i^0 is a cofibration and p^0 is a fibration.

Form the diagram

$$\begin{array}{ccccc} S^1 \wedge X^0 & \longrightarrow & & & X^1 \\ {\scriptstyle S^1 \wedge i^0}\downarrow & & & \overset{i_*}{\swarrow} & \\ S^1 \wedge Z^0 & \longrightarrow & (S^1 \wedge Z^0) \cup X^1 & & \downarrow{\scriptstyle f} \\ {\scriptstyle S^1 \wedge p^0}\downarrow & & & \underset{f_*}{\searrow} & \\ S^1 \wedge Y^0 & \longrightarrow & & & Y^1 \end{array}$$

and find a factorization

$$\begin{array}{ccc} (S^1 \wedge Z^0) \cup X^1 & \xrightarrow{j} & Z^1 \\ & \searrow^{f_*} & \downarrow{\scriptstyle p^1} \\ & & Y^1 \end{array}$$

where j is a cofibration and p^1 is a trivial fibration. Write $i^1 = j \cdot i_*$.

We have factorized f as a cofibration followed by a trivial fibration up to level 1. Proceed inductively to show that $f = p \cdot i$ where p is a trivial strict fibration and i is a cofibration.

The other factorization statement has the same proof, giving **CM5**.

The simplicial model structure is inherited from pointed simplicial sets, as is properness (exercise).

The generating sets for the cofibrations and trivial cofibrations, respectively are the maps

$$\Sigma^\infty(\Lambda^n_k)_+[-m] \to \Sigma^\infty\Delta^n_+[-m]$$

and

$$\Sigma^\infty(\partial\Delta^n)_+[-m] \to \Sigma^\infty\Delta^n_+[-m]$$

respectively. □

Suppose X is a pointed simplicial set, and recall that the loop space ΩX is defined by

$$\Omega X = \mathbf{hom}_*(S^1, X).$$

The construction only makes homotopy theoretic sense (ie. preserves weak equivalences) if X is fibrant — in that case there are isomorphisms

$$\pi_{i+1}(X,*) \cong \pi_i(\Omega X,*),\ i \geq 0,$$

of simplicial homotopy groups ($*$ is the base point for X), by a standard long exact sequence argument (see Section 31).

If X is not fibrant, then ΩX is most properly a derived functor:

$$\Omega X := \Omega X_f$$

where $j : X \to X_f$ is a fibrant model for X in the sense that j is a weak equivalence and X_f is fibrant.

This construction can be made functorial, since $\mathbf{sSet}_*$ has functorial fibrant replacements.

There is a natural bijection

$$\hom(Z \wedge S^1, X) \cong \hom(Z, \Omega X).$$

so that every morphism $f : Z \wedge S^1 \to X$ has a uniquely determined adjoint $f_* : Z \to \Omega X$.

We can say that a spectrum X consists of pointed simplicial sets $X^n, n \geq 0$, and **adjoint bonding maps** $\sigma_* : X^n \to \Omega X^{n+1}$

Here are two constructions::

1) There is a natural (levelwise) fibrant model $j : Y \to FY$ in the strict model structure for **Spt**.

2) Suppose X is a spectrum. Set

$$\Omega^\infty X^n = \varinjlim \left(X^n \xrightarrow{\sigma_*} \Omega X^{n+1} \xrightarrow{\Omega\sigma_*} \Omega^2 X^{n+2} \to \dots \right).$$

The comparison diagram

$$\begin{array}{ccccccc} X^n & \xrightarrow{\sigma_*} & \Omega X^{n+1} & \xrightarrow{\Omega\sigma_*} & \Omega^2 X^{n+2} & \longrightarrow & \cdots \\ {\scriptstyle \sigma_*}\downarrow & & \downarrow{\scriptstyle \Omega\sigma_*} & & \downarrow{\scriptstyle \Omega^2\sigma_*} & & \\ \Omega X^{n+1} & \xrightarrow[\Omega\sigma_*]{} & \Omega^2 X^{n+2} & \xrightarrow[\Omega^2\sigma_*]{} & \Omega^3 X^{n+3} & \longrightarrow & \cdots \end{array}$$

determines a spectrum structure $\Omega^\infty X$ and a natural map $\omega : X \to \Omega^\infty X$.

The adjoint bonding map

$$\Omega^\infty X^n \xrightarrow{\sigma_*} \Omega(\Omega^\infty X^{n+1})$$

is an isomorphism (exercise).

Write $QY = \Omega^{\infty}FY$ and let $\eta : Y \to QY$ be the composite

$$Y \xrightarrow{j} FY \xrightarrow{\omega} \Omega^{\infty}FY = QY.$$

The spectrum QY is the **stabilization** of Y.

Say that a map $f : X \to Y$ is a **stable equivalence** if the map $f_* : QX \to QY$ is a strict equivalence.

Remarks:

1) All spaces QY^n are fibrant (NB: this is a special property of "ordinary" spectra), and the map $\sigma_* : QY^n \to \Omega QY^{n+1}$ is an isomorphism.

2) All QY^n are H-spaces with groups $\pi_0 QY^n$ of path components. All induced maps $f_* : QX^n \to QY^n$ are H-maps.

It follows that the maps $f_* : QX^n \to QY^n$ are weak equivalences (or that f is a stable equivalence) if and only if all maps

$$\pi_i(QX^n, *) \to \pi_i(QY^n, *)$$

based at the distinguished base point are isomorphisms.

Define the **stable homotopy groups** $\pi_k^s Y$, $k \in \mathbb{Z}$ by

$$\pi_k^s Y = \varinjlim_{n+k \geq 0} (\cdots \to \pi_{n+k} FY^n \to \pi_{n+k+1} FY^{n+1} \to \dots),$$

where the maps of homotopy groups are induced by the maps $\sigma_* : FY^n \to \Omega FY^{n+1}$.

There are isomorphisms

$$\pi_k(QY^n, *) \cong \pi^s_{k-n} Y,$$

so $f : X \to Y$ is a stable equivalence if and only if f induces an isomorphism in all stable homotopy groups.

The strict model structure on the category of spectra **Spt** and the stablization functor Q fits into a general framework.

Suppose **M** is a right proper closed model category with a functor $Q : \mathbf{M} \to \mathbf{M}$, and suppose there is a natural map $\eta_X : X \to QX$.

Say that a map $f : X \to Y$ of **M** is a ***Q*-equivalence** if the induced map $Qf : QX \to QY$ is a weak equivalence of **M**.

***Q*-cofibrations** are cofibrations of **M**.

A ***Q*-fibration** is a map which has the RLP wrt all maps which are cofibrations and Q-equivalences.

Here are some conditions:

A4 The functor Q preserves weak equivalences of **M**.

A5 The maps $\eta_{QX}, Q(\eta_X) : QX \to QQX$ are weak equivalences of **M**.

A6′ Q-equivalences are stable under pullback along Q-fibrations.

Theorem 40.1 (Bousfield-Friedlander)**.** *Suppose given a right proper closed model category* **M** *with a functor* $Q : \mathbf{M} \to \mathbf{M}$ *and natural map* $\eta : X \to QX$ *as above. Suppose the Q-equivalences, cofibrations and Q-fibrations satisfy the axioms* **A4***,* **A5** *and* **A6′***.*

Then **M***, together with these three classes of maps, has the structure of a right proper closed model category.*

Proposition 40.2. *The category* **Spt** *of spectra and the stabilization functor Q satisfy the axioms* **A4***,* **A5** *and* **A6′***.*

For the proof of Proposition 40.2, the condition **A4** is a consequence of the following:

Lemma 40.3. *Suppose I is a filtered category, and suppose given a natural transformation* $f : X \to Y$

of functors $X, Y : I \to$ **sSet** *such that each component map $f_i : X_i \to Y_i$ is a weak equivalence.*

Then the map $f_* : \varinjlim_i X_i \to \varinjlim_i Y_i$ is a weak equivalence.

Proof. Exercise. □

To verify condition **A5**, consider the diagram

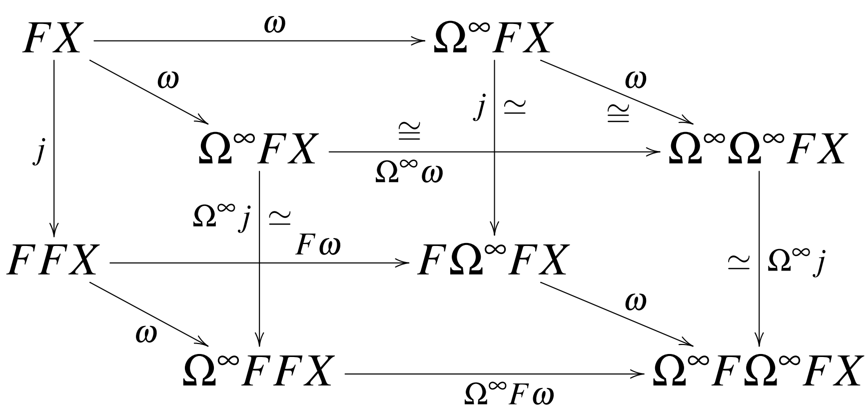
$$\begin{array}{ccccc}
X & \xrightarrow{j} & FX & \xrightarrow{\omega} & \Omega^\infty FX \\
{\scriptstyle j}\downarrow & & \simeq\downarrow{\scriptstyle j} & & \simeq\downarrow{\scriptstyle j} \\
FX & \underset{\simeq}{\xrightarrow{Fj}} & FFX & \xrightarrow{F\omega} & F\Omega^\infty FX \\
{\scriptstyle \omega}\downarrow & & {\scriptstyle \omega}\downarrow & & \downarrow{\scriptstyle \omega} \\
\Omega^\infty FX & \underset{\Omega^\infty Fj}{\xrightarrow{\simeq}} & \Omega^\infty FFX & \underset{\Omega^\infty F\omega}{\longrightarrow} & \Omega^\infty F\Omega^\infty FX
\end{array}$$

The indicated maps are strict weak equivalences, so it suffices to show that $\Omega^\infty F\omega$ and

$$\omega : F\Omega^\infty FX \to \Omega^\infty F\Omega^\infty FX$$

are strict weak equivalences.

Here's another picture:



It's an exercise to show that $\Omega^\infty\omega$ is an isomorphism: actually

$$\omega = \Omega^\infty\omega : \Omega^\infty FX \to \Omega^\infty\Omega^\infty FX.$$

But then the required maps are strict equivalences.

To verify **A6′**, use the fact that every strict fibre sequence $F \to X \to Y$ induces a long exact sequence

$$\cdots \to \pi_k^s F \to \pi_k^s X \to \pi_k^s Y \xrightarrow{\partial} \pi_{k-1}^s F \to \cdots$$

(exercise). "Right properness" follows from an exact sequence comparison.

This completes the proof of Proposition 40.2

The model structure on **Spt** arising from the Bousfield-Friedlander Theorem via Proposition 40.2 and Theorem 40.1 is called the **stable model structure** for the category of spectra.

The homotopy category $\mathrm{Ho}(\mathbf{Spt})$ is the **stable category**.

This is traditional usage, but also a misnomer, because there are many stable categories.

The proof of Theorem 40.1 is accomplished with a series of lemmas.

Recall that $\mathbf{M}$ is a right proper closed model category with functor $Q : \mathbf{M} \to \mathbf{M}$ and natural transformation $\eta : X \to QX$ such that the following conditions hold:

A4 The functor Q preserves weak equivalences of $\mathbf{M}$.

A5 The maps $\eta_{QX}, Q(\eta_X) : QX \to QQX$ are weak equivalences of $\mathbf{M}$.

A6′ Q-equivalences are stable under pullback along Q-fibrations.

Lemma 40.4. *A map $p : X \to Y$ is a Q-fibration and a Q-equivalence if and only if it is a trivial fibration of* $\mathbf{M}$.

Proof. Every trivial fibration p has the RLP wrt all cofibrations, and is therefore a Q-fibration. p is also a Q-equivalence, by **A4**.

Suppose that $p : X \to Y$ is a Q-fibration and a Q-equivalence.

There is a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle \pi} \\ & & Y \end{array}$$

where j is a cofibration and π is a trivial fibration of **M**.

π is a Q-equivalence by **A4**, so j is a Q-equivalence.

There is a diagram

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[\pi]{} & Y \end{array}$$

since j is a cofibration and a Q-equivalence and p is a Q-fibration.

Then p is a retract of π and is therefore a trivial fibration of **M**. $\square$

Lemma 40.5. *Suppose $p : X \to Y$ is a fibration of* **M** *and the maps $\eta : X \to QX$, $\eta : Y \to QY$ are weak equivalences of* **M**.

Then p is a Q-fibration.

Proof. Consider the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[\beta]{} & Y \end{array}$$

There is a diagram

$$\begin{array}{ccccc} QA & & \xrightarrow{Q\alpha} & & QX \\ & \searrow^{j_\alpha} & & \nearrow^{p_\alpha} & \\ {\scriptstyle Qi}\big\downarrow & & Z & & \big\downarrow{\scriptstyle Qp} \\ & & \big\downarrow{\scriptstyle \pi} & & \\ QB & & \longrightarrow & & QY \\ & \searrow_{j_\beta} & & \nearrow_{p_\beta} & \\ & & W & & \end{array}$$

where j_α, j_β are trivial cofibrations of $\mathbf{M}$ and p_α, p_β are fibrations.

There is an induced diagram

$$\begin{array}{ccccc} A & \longrightarrow & Z \times_{QX} X & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & {\scriptstyle \pi_*}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & W \times_{QY} Y & \longrightarrow & Y \end{array}$$

and the lifting problem is solved if we can show that π_* is a weak equivalence.

But there is finally a diagram

$$\begin{array}{ccccc} QA & \xrightarrow{j_\alpha} & Z & \xleftarrow{pr} & Z \times_{QX} X \\ {\scriptstyle Qi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi_*} \\ QB & \xrightarrow[j_\beta]{} & W & \xleftarrow[pr]{} & W \times_{QY} Y \end{array}$$

The maps Qi, j_α and j_β are weak equivalences of $\mathbf{M}$ so that π is a weak equivalence.

The maps pr are weak equivalences by right properness of $\mathbf{M}$ and the assumptions on p.

It follows that π_* is a weak equivalence of $\mathbf{M}$. $\square$

Lemma 40.6. *Every map $f : QX \to QY$ has a factorization $f = q \cdot j$, where j is a cofibration and Q-equivalence and q is a Q-fibration.*

Proof. f has a factorization $f = q \cdot j$ where j is a trivial cofibration and q is a fibration of $\mathbf{M}$.

j is a Q-equivalence by **A4**, and q is a Q-fibration by Lemma 40.5.

In effect, there is a diagram

$$\begin{array}{ccccc}
QX & \xrightarrow[\simeq]{j} & Z & \xrightarrow{p} & QY \\
{\scriptstyle \eta}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \eta}\downarrow & & {\scriptstyle \simeq}\downarrow{\scriptstyle \eta} \\
QQX & \xrightarrow[Qj]{\simeq} & QZ & \xrightarrow[Qp]{} & QQY
\end{array}$$

so $\eta : Z \to QZ$ is a weak equivalence of $\mathbf{M}$. $\square$

Lemma 40.7. *Every map $f : X \to Y$ has a factorization $f = q \cdot j$, where j is a cofibration and Q-equivalence and q is a Q-fibration.*

Proof. The map $f_* : QX \to QY$ has a factorization

$$\begin{array}{ccccc} QX & & \xrightarrow{f_*} & & QY \\ & {}_i\searrow & & \nearrow_p & \\ & & X & & \end{array}$$

where p is a Q-fibration and i is a cofibration and a Q-equivalence, by Lemma 40.6.

Form the diagram

$$\begin{array}{ccccc} X & \xrightarrow{i_*} & Z \times_{QY} Y & \xrightarrow{p_*} & Y \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta_*} & & \downarrow{\scriptstyle \eta} \\ QX & \xrightarrow[i]{} & Z & \xrightarrow[p]{} & QY \end{array}$$

The maps η are Q-equivalence by **A5**, so η_* is a Q-equivalence by **A6**$'$. It follows that i_* is a Q-equivalence.

The map i_* has a factorization

$$\begin{array}{ccccc} X & & \xrightarrow{i_*} & & Z \times_{QY} Y \\ & {}_j\searrow & & \nearrow_\pi & \\ & & W & & \end{array}$$

where j is a cofibration and π is a trivial strict fibration.

Then π is a Q-equivalence and a Q-fibration by Lemma 40.4, so j is a Q-equivalence, and the composite $p_* \cdot \pi$ is a Q-fibration. $\square$

Proof of Theorem 40.1. The non-trivial closed model statements are the lifting axiom **CM4** and the factorization axiom **CM5**.

CM5 is a consequence of Lemma 40.4 and Lemma 40.7. **CM4** follows from Lemma 40.4.

The right properness of the model structure is the statement **A6**$'$. □

Say that the model structure on **M** given by Theorem 40.1 is the ***Q*-structure**.

Lemma 40.8. *Suppose that, in addition to the assumptions of Theorem 40.1, that the model structure* **M** *is left proper.*

Then the Q-structure on **M** *is left proper.*

Proof. Suppose given a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \xrightarrow[f_*]{} & B \cup_A C \end{array}$$

where f is a Q-equivalence and i is a cofibration. We must show that f_* is a Q-equivalence (see Definition 17.4).

Find a factorization

$$
\begin{array}{ccc}
A & \xrightarrow{j} & D \\
& {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi} \\
& & C
\end{array}
$$

where j is a cofibration and π is a trivial fibration of $\mathbf{M}$.

The map $\pi_* : B \cup_A D \to B \cup_A C$ is a weak equivalence of $\mathbf{M}$ by left properness for $\mathbf{M}$, so π_* is a Q-equivalence by **A4**.

j is a Q-equivalence as well as a cofibration, so that $j_* : B \to B \cup_A D$ is a cofibration and a Q-equivalence.

Then the composite $f_* = \pi_* \cdot j_*$ is a Q-equivalence.

$\square$

Here's the other major abstract result in this game, again from [2]:

Theorem 40.9. *Suppose the model category* $\mathbf{M}$ *and the functor* Q *satisfy the conditions for Theorem 40.1*

Then a map $p : X \to Y$ *of* $\mathbf{M}$ *is a stable fibration if and only if the following conditions hold:*

1) p *is a fibration of* $\mathbf{M}$*, and*

2) the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & QX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Qp} \\ Y & \xrightarrow[\eta]{} & QY \end{array}$$

is homotopy cartesian in **M**.

Corollary 40.10. *1) An object X of* **M** *is Q-fibrant if and only if it is fibrant and the map $\eta : X \to QX$ is a weak equivalence of* **M**.

2) A spectrum X is stably fibrant if and only if it is strictly fibrant and all adjoint bonding maps $\sigma_* : X^n \to \Omega X^{n+1}$ are weak equivalences of pointed simplicial sets.

Fibrant spectra are often called **Ω-spectra**.

Corollary 40.11. *Suppose given a diagram*

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Y & \xrightarrow[\simeq]{} & Y' \end{array}$$

in which p, p' are fibrations and the horizontal maps are weak equivalences of **M**.

Then p is a Q-fibration if and only if p' is a Q-fibration.

Proof of Theorem 40.9. Suppose $p : X \to Y$ is a fibration of $\mathbf{M}$, and that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & QX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Qp} \\ Y & \xrightarrow[\eta]{} & QY \end{array}$$

is homotopy cartesian in $\mathbf{M}$.

Then Qp has a factorization

$$\begin{array}{ccc} QX & \xrightarrow{i} & Z \\ & {\scriptstyle Qp}\searrow & \downarrow{\scriptstyle q} \\ & & QY \end{array}$$

where i is a trivial cofibration and q is a fibration. Then q is a Q-fibration by Lemma 40.5.

Factorize the weak equivalence $\theta : X \to Y \times_{QY} Z$ (the square is homotopy cartesian) as

$$\begin{array}{ccc} X & \xrightarrow{i} & W \\ & {\scriptstyle \theta}\searrow & \downarrow{\scriptstyle \pi} \\ & & Y \times_{QY} Z \end{array}$$

where π is a trivial fibration of $\mathbf{M}$ and i is a trivial cofibration.

Then $q_* \cdot \pi$ is a Q-fibration (Lemma 40.4), and the lifting exists in the diagram

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ W & \xrightarrow[q_*\pi]{} & Y \end{array}$$

Thus, p is a retract of a Q-fibration, and is therefore a Q-fibration.

Conversely, suppose that $p : X \to Y$ is a Q-fibration, and factorize $Qp = q \cdot i$ as above.

The map $\eta_* : Y \times_{QY} Z \to Z$ is a Q-equivalence by **A6**$'$, so θ is a Q-equivalence.

The picture

$$\begin{array}{ccc} X & \xrightarrow{\theta} & Y \times_{QY} Z \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q_*} & \\ & Y & \end{array}$$

is a weak equivalence of fibrant objects in the category $\mathbf{M}/Y$ of objects fibred over Y, for the Q-structure on $\mathbf{M}$.

The usual category of fibrant objects trick (see Section 13) implies that θ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & V \\ & {\scriptstyle \theta}\searrow & \downarrow{\scriptstyle \pi} \\ & & Y \times_{QY} Z \end{array}$$

in $\mathbf{Spt}/Y$, where π is a Q-fibration and a Q-equivalence, and i is a section of a map $V \to X$ which is a Q-fibration and a Q-equivalence.

Thus, π and i are weak equivalences of $\mathbf{M}$ by Lemma 40.4, so that θ is a weak equivalence of $\mathbf{M}$. $\square$

Write

$$A \otimes K = A \wedge K_+,$$

for a spectrum A and a simplicial set K.

Lemma 40.12. *Suppose $i : A \to B$ is a stably trivial cofibration of spectra.*

Then all induced maps

$$(B \otimes \partial\Delta^n) \cup (A \otimes \Delta^n) \to B \otimes \Delta^n$$

are stably trivial cofibrations.

Quillen's axiom **SM7** for the stable model structure on **Spt** follows easily: if $j : K \to L$ is a cofibration of simplicial sets and $i : A \to B$ is a cofibration of spectra, then the induced map

$$(B \otimes K) \cup (A \otimes L) \subset B \otimes L$$

is a cofibration which is a stable equivalence if either i is a stable equivalence (Lemma 40.12) or j is a weak equivalence of simplicial sets (use the simplicial model axiom for the strict structure).

Proof of Lemma 40.12. It suffices to show that

$$i \otimes \partial\Delta^n : A \otimes \partial\Delta^n \to B \otimes \partial\Delta^n$$

is a stable equivalence.

There is a pushout diagram

$$\begin{array}{ccc} A \otimes \partial\Delta^{n-1} & \longrightarrow & A \otimes \Lambda^n_k \\ \downarrow & & \downarrow \\ A \otimes \Delta^{n-1} & \longrightarrow & A \otimes \partial\Delta^n \end{array}$$

There is also a corresponding diagram for B and an obvious comparison.

The simplicial sets Λ^n_k and Δ^{n-1} are both weakly equivalent to a point, so it suffices to show that the comparison

$$i \otimes \partial\Delta^{n-1} : A \otimes \partial\Delta^{n-1} \to B \otimes \partial\Delta^{n-1}$$

is a stable equivalence.

This is the inductive step in an argument that starts with the case

$$i \otimes \partial\Delta^1 : A \otimes \partial\Delta^1 \to B \otimes \partial\Delta^1$$

and this map is isomorphic to the map

$$i \wedge i : A \wedge A \to B \wedge B.$$

Finally, a wedge (coproduct) of stably trivial cofibrations is stably trivial. □

Note: Bousfield gives a different proof of the Lemma 40.12 in [1]. The result is also mentioned in Remark X.4.7 (on p.496) of [3], without proof.

References

[1] A. K. Bousfield. On the telescopic homotopy theory of spaces. *Trans. Amer. Math. Soc.*, 353(6):2391–2426 (electronic), 2001.

[2] A. K. Bousfield and E. M. Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II*, volume 658 of *Lecture Notes in Math.*, pages 80–130. Springer, Berlin, 1978.

[3] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.