Contents

41 Suspensions and shift

The suspension $X \wedge S^1$ and the fake suspension ΣX of a spectrum X were defined in Section 35 — the constructions differ by a non-trivial twist of bonding maps.

The **loop** spectrum for X is the function complex object

$$\mathbf{hom}_*(S^1, X).$$

There is a natural bijection

$$\hom(X \wedge S^1, Y) \cong \hom(X, \mathbf{hom}_*(S^1, Y))$$

so that the suspension and loop functors are adjoint.

The **fake loop** spectrum ΩY for a spectrum Y consists of the pointed spaces ΩY^n, $n \geq 0$, with adjoint bonding maps

$$\Omega \sigma_* : \Omega Y^n \to \Omega^2 Y^{n+1}.$$

There is a natural bijection

$$\hom(\Sigma X, Y) \cong \hom(X, \Omega Y),$$

so the fake suspension functor is left adjoint to fake loops.

The adjoint bonding maps $\sigma_* : Y^n \to \Omega Y^{n+1}$ define a natural map

$$\gamma : Y \to \Omega Y[1].$$

for spectra Y.

The map $\omega : Y \to \Omega^\infty Y$ of the last section is the filtered colimit of the maps

$$Y \xrightarrow{\gamma} \Omega Y[1] \xrightarrow{\Omega\gamma[1]} \Omega^2 Y[2] \xrightarrow{\Omega^2\gamma[2]} \dots$$

Recall the statement of the Freudenthal suspension theorem (Theorem 34.2):

Theorem 41.1. *Suppose that a pointed space X is n-connected, where $n \geq 0$.*

Then the homotopy fibre F of the canonical map $\eta : X \to \Omega(X \wedge S^1)$ is $2n$-connected.

In particular, the suspension homomorphism

$$\pi_i X \to \pi_i(\Omega(X \wedge S^1)) \cong \pi_{i+1}(X \wedge S^1)$$

is an isomorphism for $i \leq 2n$ and is an epimorphism for $i = 2n+1$, provided that X is connected.

In general (ie. with no connectivity assumptions on Y), the space $S^n \wedge Y$ is $(n-1)$-connected, by Lemma 31.5 and Corollary 34.1.

Thus, the suspension homomorphism

$$\pi_{i+k}(S^{n+k} \wedge Y) \to \pi_{i+k+1}(S^{n+k+1} \wedge Y)$$

is an isomorphism if $i \leq 2n-2+k$, and it follows that the map

$$\pi_i(S^n \wedge Y) \to \pi^s_{i-n}(\Sigma^\infty Y)$$

is an isomorphism for $i \leq 2(n-1)$.

Here's an easy observation:

Lemma 41.2. *The natural map $\gamma : X \to \Omega X[1]$ is a stable equivalence if X is strictly fibrant.*

Proof. This is a cofinality argument, which uses the fact that $\Omega^\infty X$ is the filtered colimit of the system

$$X \to \Omega X[1] \to \Omega^2 X[2] \to \dots$$

□

Lemma 41.3. *Suppose that Y is a pointed space. Then the canonical map*

$$\eta : \Sigma^\infty Y \to \Omega\Sigma(\Sigma^\infty Y)$$

is a stable homotopy equivalence.

Proof. The map

$$\pi_k(S^n \wedge Y) \to \pi^s_{k-n}(\Sigma^\infty Y)$$

is an isomorphism for $k \leq 2(n-1)$.

Similarly (exercise), the map

$$\pi_k(\Omega(S^{n+1} \wedge X)) \to \pi^s_{k-n}(\Omega\Sigma(\Sigma^\infty X))$$

is an isomorphism for $k+1 \leq 2n$ or $k \leq 2n-1$. There is a commutative diagram

$$\begin{array}{ccc} \pi_k(S^n \wedge Y) & \xrightarrow{\cong} & \pi^s_{k-n}(\Sigma^\infty Y) \\ {\scriptstyle\cong}\downarrow & & \downarrow \\ \pi_k(\Omega(S^{n+1} \wedge Y)) & \xrightarrow[\cong]{} & \pi^s_{k-n}(\Omega\Sigma(\Sigma^\infty Y)) \end{array}$$

in which the indicated maps are isomorphisms for $k \leq 2(n-1)$.

It follows that the map

$$\pi^s_p(\Sigma^\infty Y) \to \pi^s_p(\Omega\Sigma(\Sigma^\infty Y))$$

is an isomorphism for $p \leq n-2$.

Finish by letting n vary. □

Remark: What we've really shown in Lemma 41.3 is that the composite

$$\Sigma^\infty X \xrightarrow{\eta} \Omega\Sigma(\Sigma^\infty X) \xrightarrow{\Omega j} \Omega F(\Sigma(\Sigma^\infty X))$$

is a natural stable equivalence.

Lemma 41.4. *Suppose that Y is a spectrum. Then the composite*

$$Y \xrightarrow{\eta} \Omega\Sigma Y \xrightarrow{\Omega j} \Omega F(\Sigma Y)$$

is a stable equivalence.

Proof. We show that the maps

$$L_n Y \xrightarrow{\eta} \Omega\Sigma L_n Y \xrightarrow{\Omega j} \Omega F(\Sigma L_n Y)$$

arising from the layer filtration for Y are stable equivalences.

In the layer filtration

$$L_n Y : \; Y^0, \dots, Y^n, S^1 \wedge Y^n, S^2 \wedge Y^n, \dots$$

the maps

$$(\Sigma^\infty Y^n[-n])^r \to L_n Y^r$$

are isomorphisms for $r \geq n$.

Thus, the maps

$$(\Omega F(\Sigma(\Sigma^\infty Y^n[-n])))^r \to \Omega F(\Sigma(L_n Y))^r$$

are weak equivalences for $r \geq n$, so that

$$\Omega F(\Sigma(\Sigma^{\infty} Y^n[-n])) \to \Omega F(\Sigma(L_n Y))$$

is a stable equivalence.

The map $\eta : X \to \Omega\Sigma X$ respects shifts, so Lemma 41.3 implies that the composite

$$\Sigma^{\infty} Y^n[-n] \to \Omega\Sigma(\Sigma^{\infty} Y^n[-n]) \to \Omega F(\Sigma(\Sigma^{\infty} Y^n[-n]))$$

is a stable equivalence. □

Theorem 41.5. *Suppose that X is a spectrum. Then the canonical map*

$$\sigma : \Sigma X \to X[1]$$

is a stable equivalence.

Proof. The map σ is adjoint to the map $\sigma_* : X \to \Omega X[1]$, so that there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta} & \Omega\Sigma X & \xrightarrow{\Omega j} & \Omega F(\Sigma X) \\ & \searrow^{\sigma_*} & \downarrow{\scriptstyle \Omega\sigma} & & \downarrow{\scriptstyle \Omega F\sigma} \\ & & \Omega X[1] & \xrightarrow[\Omega j]{} & \Omega F(X[1]) \end{array}$$

where $j : \Sigma X \to F(\Sigma X)$ is a strictly fibrant model.

The composite

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega j[1]} \Omega(FX)[1]$$

is a stable equivalence by Lemma 41.2, and the shifted map $j[1] : X[1] \to (FX)[1]$ is a strictly fibrant model of $X[1]$.

It follows that the composite

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega j} \Omega F(X[1])$$

is a stable equivalence.

The composite

$$X \xrightarrow{\eta} \Omega\Sigma X \xrightarrow{\Omega j} \Omega F(\Sigma X)$$

is a stable equivalence by Lemma 41.4.

The map $\Omega F\sigma$ is therefore a stable equivalence, so Lemma 41.2 implies that

$$F\sigma : F(\Sigma X) \to F(X[1])$$

is a stable equivalence. □

Here's another, still elementary but much fussier result:

Theorem 41.6. *The functors $X \mapsto X \wedge S^1$ and $X \mapsto \Sigma X$ are naturally stably equivalent.*

Sketch Proof: ([2], Lemma 1.9, p.7) The isomorphisms $\tau : S^1 \wedge X^n \to X^n \wedge S^1$ and the bonding maps $\sigma \wedge S^1$ together define a spectrum with the space

$S^1 \wedge X^n$ in level n, and with bonding maps $\tilde{\sigma}$ defined by the diagrams

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge X^n & \xrightarrow{\tilde{\sigma}} & S^1 \wedge X^{n+1} \\ {\scriptstyle S^1\wedge\tau}\downarrow\cong & & \cong\downarrow{\scriptstyle\tau} \\ S^1 \wedge X^n \wedge S^1 & \xrightarrow[\sigma\wedge S^1]{} & X^{n+1} \wedge S^1 \end{array}$$

There are commutative diagrams

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge X^n & \searrow^{S^1\wedge\sigma} & \\ {\scriptstyle \tau\wedge X^n}\downarrow & & S^1 \wedge X^{n+1} \\ S^1 \wedge S^1 \wedge X^n & \nearrow_{\tilde{\sigma}} & \end{array}$$

Composing then gives a diagram

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge S^1 \wedge X^n & \searrow^{(S^1\wedge\sigma)(S^1\wedge S^1\wedge\sigma)} & \\ {\scriptstyle (3,2,1)\wedge X^n}\downarrow & & S^1 \wedge X^{n+2} \\ S^1 \wedge S^1 \wedge S^1 \wedge X^n & \nearrow_{\tilde{\sigma}\cdot(S^1\wedge\tilde{\sigma})} & \end{array}$$

where $(3,2,1)$ is induced on the smash factors making up S^3 by the corresponding cyclic permutation of order 3.

The spaces $S^1 \wedge X^0, S^1 \wedge X^2, \ldots$ and the respective composite bonding maps $(S^1 \wedge \sigma)(S^1 \wedge S^1 \wedge \sigma)$ and $\tilde{\sigma} \cdot (S^1 \wedge \tilde{\sigma})$ define "partial" spectrum structures from which the stable homotopy types of the original spectra can be recovered.

The self map $(3,2,1)$ of the 3-sphere S^3 has degree 1 and is therefore homotopic to the identity.

This homotopy can be used to describe a telescope construction (see [2], p.11-15, and the next section) which is stably equivalent to both of these partial spectra. □

Remark: The proof of Theorem 41.6 that is sketched here is essentially classical. See Prop. 10.53 of [3] for a more modern alternative.

Corollary 41.7. *1) The functors $X \mapsto X[1]$, $X \mapsto \Sigma X$ and $X \mapsto X \wedge S^1$ are naturally stably equivalent.*

2) The functors $X \mapsto X[-1]$, $X \mapsto \Omega X$ and $X \mapsto \mathbf{hom}_*(S^1, X)$ are naturally stably equivalent.

Proof. Lemma 41.2 implies that the composite

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega j[1]} \Omega F X[1]$$

is a stable equivalence for all spectra X, where $j : X \to FX$ is a strictly fibrant model.

Shift preserves stable equivalences, so the induced composite

$$X[-1] \xrightarrow{\sigma_*[-1]} \Omega X \xrightarrow{\Omega j} \Omega F X$$

is a stable equivalence.

The natural stable equivalence $\Sigma Y \simeq Y \wedge S^1$ induces a natural stable equivalence

$$\Omega X \simeq \mathbf{hom}_*(S^1, X)$$

for all strictly fibrant spectra X. $\square$

In other words, the suspension and loop functors (real or fake) are equivalent to shift functors, and define equivalences $\mathrm{Ho}(\mathbf{Spt}) \to \mathrm{Ho}(\mathbf{Spt})$ of the stable category.

42 The telescope construction

Observe that a spectrum Y is cofibrant if and only if all bonding maps $\sigma : S^1 \wedge Y^n \to Y^{n+1}$ are cofibrations.

The **telescope** TX for a spectrum X is a natural cofibrant replacement, equipped with a natural strict equivalence $s : TX \to X$.

The construction is an iterated mapping cylinder. We find natural trivial cofibrations

$$X^k \xrightarrow{j_k} CX^k \xrightarrow{\alpha_k} TX^k,\ k \leq n,$$

and $t_k : TX^k \to X^k$ such that $t_k \cdot (\alpha_k \cdot j_k) = 1$ and the maps t_k define a strict weak equivalence of spectra $t : TX \to X$.

- $X^0 = CX^0 = TX^0$ and j_0 and α_0 are identities,
- CX^n is the mapping cylinder for $\sigma : S^1 \wedge X^n \to X^{n+1}$, meaning that there is a pushout diagram

$$\begin{array}{ccc} S^1 \wedge X^n & \xrightarrow{\sigma} & X^{n+1} \\ {\scriptstyle d^0}\downarrow & & \downarrow{\scriptstyle j_{n+1}} \\ (S^1 \wedge X^n) \wedge \Delta^1_+ & \xrightarrow[\zeta_{n+1}]{} & CX^{n+1} \end{array}$$

for each n.

Write σ_* for the composite

$$S^1 \wedge X^n \xrightarrow{d^1} (S^1 \wedge X^n) \wedge \Delta^1_+ \xrightarrow{\zeta_{n+1}} CX^{n+1}$$

and observe that σ_* is a cofibration.

The projection map

$$s : (S^1 \wedge X^n) \wedge \Delta^1_+ \to S^1 \wedge X^n$$

satisfies $s \cdot d^0 = 1$ and induces a map $s_{n+1} : CX^{n+1} \to X^{n+1}$ such that $s_{n+1} \cdot j_{n+1} = 1$. Further $s_{n+1} \cdot \sigma_* = \sigma$.

- Form the pushout diagram

$$\begin{array}{ccc} S^1 \wedge X^n & \xrightarrow{\sigma_*} & CX^{n+1} \\ {\scriptstyle S^1\wedge j_n}\downarrow & & \\ S^1 \wedge CX^n & & \downarrow{\scriptstyle \alpha_{n+1}} \\ {\scriptstyle S^1\wedge \alpha_n}\downarrow & & \\ S^1 \wedge TX^n & \xrightarrow[\tilde{\sigma}]{} & TX^{n+1} \end{array}$$

Then $\tilde{\sigma}$ is a cofibration, and the maps j_{n+1}, α_{n+1} are trivial cofibrations.

The maps $S^1 \wedge t_n$ and s_{n+1} together induce $t_{n+1} : TX^{n+1} \to X^{n+1}$ such that $t_{n+1} \cdot (\alpha_{n+1} \cdot j_{n+1}) = 1$, and the $t_k : TX^k \to X^k$ define a map of spectra up to level $n+1$.

The projection maps s can be replaced with homotopies $h : (S^1 \wedge X^n) \wedge \Delta^1_+ \to Z^n$ in the construction above, giving the following:

Lemma 42.1. *Suppose X is a spectrum with bonding maps $\sigma : S^1 \wedge X^n \to X^{n+1}$. Suppose X' is a spectrum with the same objects as X, with bonding maps $\sigma' : S^1 \wedge X^n \to X^{n+1}$. Suppose $j : X' \to Z$*

is a map of spectra such that there are homotopies

$$\begin{array}{ccc} S^1 \wedge X^n & & \\ {\scriptstyle d^1}\downarrow & \searrow^{j\sigma'} & \\ (S^1 \wedge X^n) \wedge \Delta^1_+ & \xrightarrow{h} & Z^{n+1} \\ {\scriptstyle d^0}\uparrow & \nearrow_{j\sigma} & \\ S^1 \wedge X^n & & \end{array}$$

Then the homotopies h define a map $h_* : TX \to Z$, giving a morphism

$$X \underset{\simeq}{\xleftarrow{t}} TX \xrightarrow{h_*} Z$$

from X to Z in the stable category.

If $j : X' \to Z$ is a strict weak equivalence then the map h_* is a strict weak equivalence.

Remarks:

1) The construction of Lemma 42.1 is natural, and hence applies to diagrams of spectra.

Suppose that $i \mapsto X_i$ and $i \mapsto X_i'$ are spectrum valued functors defined on an index category I such that $X_i^n = X_i'^n$ for all $i \in I$. Let $j : X' \to Z$ be a natural choice of strict fibrant model for the diagram X' and suppose finally that there are natural homotopies

$$\begin{array}{ccc} S^1 \wedge X_i^n & & \\ {\scriptstyle d^1}\downarrow & \searrow^{j\sigma'} & \\ (S^1 \wedge X_i^n) \wedge \Delta^1_+ & \xrightarrow{h} & Z_i^n \\ {\scriptstyle d^0}\uparrow & \nearrow_{j\sigma} & \\ S^1 \wedge X_i^n & & \end{array}$$

where σ and σ' are the bonding maps for X and X' respectively.

Then the homotopies h canonically determine a natural strict equivalence $h_* : TX \to Z$, and there are natural strict equivalences

$$X \leftarrow TX \xrightarrow{h_*} Z \xleftarrow{j} X'.$$

2) Suppose given S^2-spectra $X(1)$ and $X(2)$ having objects $S^1 \wedge X^{2n}$ and bonding maps

$$\sigma_1, \sigma_2 : S^2 \wedge S^1 \wedge X^{2n} = S^3 \wedge X^{2n} \to S^1 \wedge X^{2n+2}$$

respectively, such that the diagram

$$\begin{array}{ccc} S^3 \wedge X^{2n} & \searrow^{\sigma_1} & \\ {\scriptstyle c\wedge 1}\downarrow & & S^1 \wedge X^{2n+2} \\ S^3 \wedge X^{2n} & \nearrow_{\sigma_2} & \end{array}$$

commutes, where c is induced by the cyclic permutation $(3,2,1)$.

The map c has degree 1 and is therefore the identity in the homotopy category.

Choose a strict fibrant model $j: X(2) \to FX(2)$ in S^2-spectra for $X(2)$. Then

$$j \cdot \sigma_1 \simeq j \cdot \sigma_2 : S^3 \wedge X^{2n} \to F(S^1 \wedge X^{2n+2}),$$

and it follows that there are strict equivalences

$$X(1) \xleftarrow{t} TX(1) \xrightarrow{h_*} FX(2) \xleftarrow{j} X(2).$$

If $X(1)$ and $X(2)$ are the outputs of functors defined on spectra (eg. the comparison of fake and real suspension in Theorem 41.6), then these equivalences are natural.

43 Fibrations and cofibrations

Suppose $i: A \to X$ is a levelwise cofibration of spectra with cofibre $\pi: X \to X/A$.

Suppose $\alpha: S^r \to X^n$ represents a homotopy element such that the composite

$$S^r \xrightarrow{\alpha} X^n \xrightarrow{\pi} X^n/A^n$$

represents $0 \in \pi_r(X/A)^n$.

Comparing cofibre sequences gives a diagram

$$\begin{array}{ccccccc}
S^r & \longrightarrow & CS^r & \longrightarrow & S^1 \wedge S^r & \xrightarrow{\simeq} & S^1 \wedge S^r \\
{\scriptstyle\alpha}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle S^1 \wedge \alpha} \\
X^n & \xrightarrow[\pi]{} & (X/A)^n & \longrightarrow & S^1 \wedge A^n & \xrightarrow{S^1 \wedge i} & S^1 \wedge X^n \\
 & & & & {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\
 & & & & A^{n+1} & \xrightarrow[i]{} & X^{n+1}
\end{array}$$

where $CS^r \simeq *$ is the cone on S^r.

It follows that the image of $[\alpha]$ under the suspension map

$$\pi_r X^n \to \pi_{r+1} X^{n+1}$$

is in the image of the map $\pi_{r+1} A^{n+1} \to \pi_{r+1} X^{n+1}$.

We have proved the following:

Lemma 43.1. *Suppose $A \to X \to X/A$ is a level cofibre sequence of spectra.*

Then the sequence

$$\pi_k^s A \to \pi_k^s X \to \pi_k^s (X/A)$$

is exact.

Corollary 43.2. *Any levelwise cofibre sequence*

$$A \to X \to X/A$$

induces a long exact sequence

$$\ldots \xrightarrow{\partial} \pi_k^s A \to \pi_k^s X \to \pi_k^s (X/A) \xrightarrow{\partial} \pi_{k-1}^s A \to \ldots \tag{1}$$

The sequence (1) is the **long exact sequence** in stable homotopy groups for a level cofibre sequence of spectra.

Proof. The map $X/A \to A \wedge S^1$ in the Puppe sequence induces the boundary map

$$\pi_k^s(X/A) \to \pi_k^s(A \wedge S^1) \cong \pi_k^s(A[1]) \cong \pi_{k-1}^s A,$$

since $A \wedge S^1$ is naturally stably equivalent to the shifted spectrum $A[1]$ by Corollary 41.7. □

Corollary 43.3. *Suppose that X and Y are spectra. Then the inclusion*

$$X \vee Y \to X \times Y$$

is a natural stable equivalence.

Proof. The sequence

$$0 \to \pi_k^s X \to \pi_k^s(X \vee Y) \to \pi_k^s Y \to 0$$

arising from the level cofibration $X \subset X \vee Y$ is split exact, as is the sequence

$$0 \to \pi_k^s X \to \pi_k^s(X \times Y) \to \pi_k^s Y \to 0$$

arising from the fibre sequence $X \to X \times Y \to Y$.

It follows that the map $X \vee Y \to X \times Y$ induces an isomorphism in all stable homotopy groups. □

Corollary 43.4. *The stable homotopy category* $\mathrm{Ho}(\mathbf{Spt})$ *is additive.*

Proof. The sum of two maps $f, g : X \to Y$ is represented by the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \xleftarrow{\simeq} Y \vee Y \xrightarrow{\nabla} Y.$$

□

Corollary 43.5. *Suppose that*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ C & \xrightarrow[j]{} & D \end{array}$$

is a pushout in **Spt** *where i is a levelwise cofibration. Then there is a long exact sequence in stable homotopy groups*

$$\dots \xrightarrow{\partial} \pi_k^s A \xrightarrow{(i,\alpha)} \pi_k^s C \oplus \pi_k^s B \xrightarrow{j-\beta} \pi_k^s D \xrightarrow{\partial} \pi_{k-1}^s A \to \dots \tag{2}$$

The sequence (2) is the **Mayer-Vietoris sequence** for the cofibre square.

The boundary map $\partial : \pi_k^s D \to \pi_{k-1}^s A$ is the composite

$$\pi_k^s D \to \pi_k^s (D/C) = \pi_k^s (B/A) \xrightarrow{\partial} \pi_{k-1}^s A.$$

Lemma 43.6. *Suppose*

$$A \xrightarrow{i} X \xrightarrow{\pi} X/A$$

is a level cofibre sequence in **Spt**, *and let F be the strict homotopy fibre of the map π.*

Then the map $i_* : A \to F$ is a stable equivalence.

Proof. Choose a strict fibration $p : Z \to X/A$ such that $Z \to *$ is a strict weak equivalence.

Form the pullback

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi_*} & Z \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\pi]{} & X/A \end{array}$$

Then $\tilde{X}$ is the homotopy fibre of π and the maps $i : A \to X$ and $* : A \to Z$ together determine a map $i_* : A \to \tilde{X}$. We show that i_* is a stable equivalence.

Pull back the cofibre square

$$\begin{array}{ccc} A & \longrightarrow & * \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \xrightarrow[\pi]{} & X/A \end{array}$$

along the fibration p to find a (levelwise) cofibre square

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & U \\ {\scriptstyle \tilde{i}}\downarrow & & \downarrow \\ \tilde{X} & \longrightarrow & Z \end{array}$$

A Mayer-Vietoris sequence argument (Corollary 43.5) implies that the map $\tilde{A} \to \tilde{X} \times U$ is a stable equivalence.

From the fibre square

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & U \\ \downarrow & & \downarrow \\ A & \longrightarrow & * \end{array}$$

we see that the map $\tilde{A} \to A \times U$ is a stable equivalence.

The map $i_* : A \to \tilde{X}$ induces a section $\theta : A \to \tilde{A}$ of the map $\tilde{A} \to A$ which composes with the projection $\tilde{A} \to U$ to give the trivial map $* : A \to U$.

Thus, there is a commutative diagram

$$\begin{array}{ccccc} & & A & \xrightarrow{i_*} & \tilde{X} \\ & {\scriptstyle (1_A,*)}\swarrow & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle (1_{\tilde{X}},*)} \\ A \times U & \xleftarrow{\simeq} & \tilde{A} & \xrightarrow{\simeq} & \tilde{X} \times U \\ & {\scriptstyle pr}\searrow & \downarrow & \swarrow & \\ & & U & & \end{array}$$

and it follows that A is the stable fibre of the map $\tilde{A} \to U$, so i_* is a stable equivalence. $\square$

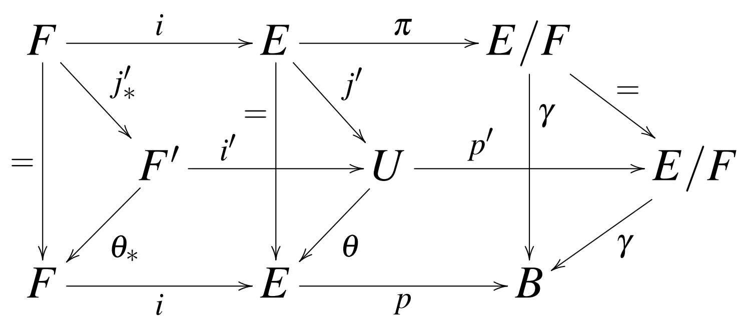
Lemma 43.7. *Suppose that*

$$F \xrightarrow{i} E \xrightarrow{p} B$$

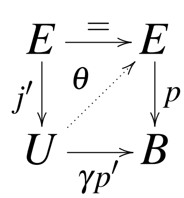
is a strict fibre sequence, where i is a level cofibration.

Then the map $E/F \to B$ is a stable equivalence.

Proof. There is a diagram



where p' is a strict fibration, j' is a cofibration and a strict equivalence, and θ exists by a lifting property:



The map j'_* is a stable equivalence by Lemma 43.6, so θ_* is a stable equivalence.

The map θ is a strict equivalence, and a comparison of long exact sequences shows that γ is a stable equivalence. □

Remark: Lemma 43.6 and Lemma 43.7 together say that fibre and cofibre sequences coincide in the stable category.

44 Cofibrant generation

We will show that the stable model structure on **Spt** is cofibrantly generated.

This means that there are sets I and J of stably trivial cofibrations and cofibrations, such that $p : X \to Y$ is a stable fibration (resp. stably trivial fibration) if and only if it has the RLP wrt all members of the set I (resp. all members of J).

Recall that a map $p : X \to Y$ is a stably trivial fibration if and only if it is a strict fibration and a strict weak equivalence.

Thus p is a stably trivial fibration if and only if it has the RLP wrt all maps

$$\Sigma^\infty \partial \Delta^n_+[m] \to \Sigma^\infty \Delta^n_+[m].$$

We have found our set of maps J.

It remains to find a set of stably trivial cofibrations I which generates the full class of stably trivial cofibrations. We do this in a sequence of lemmas.

Say that a spectrum A is **countable** if all consituent simplicial sets A^n are countable in the sense that they have countably many simplices in each degree — see Section 11.

It follows from Lemma 11.2 that a countable spectrum A has countable stable homotopy groups.

The following "bounded cofibration lemma" is the analogue of Lemma 11.3 for the category of spectra.

Lemma 44.1. *Suppose given level cofibrations of spectra*

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i]{} & Y \end{array}$$

such that A is countable and j is a stable equivalence.

Then there is a countable subobject $B \subset Y$ such that $A \subset B \subset Y$ and the map $B \cap X \to B$ is a stable equivalence.

Proof. The map $B \cap X \to B$ is a stable equivalence if and only if all stable homotopy groups

$$\pi_n^s(B/(B \cap X))$$

vanish, by Corollary 43.2.

Write $A_0 = A$. Y is a filtered colimit of its countable subobjects, and the countable set of elements of the homotopy groups $\pi_n^s(A_0/(A_0 \cap X))$ vanish in $\pi_n^s(A_1/(A_1 \cap X))$ for some countable subobject $A_1 \subset X$ with $A_0 \subset A_1$.

Repeat the construction inductively to find countable subcomplexes

$$A = A_0 \subset A_1 \subset A_2 \subset \dots$$

of Y such that all induced maps

$$\pi_n^s(A_i/(A_i \cap X)) \to \pi_n^s(A_{i+1}/(A_{i+1} \cap X))$$

are 0. Set $B = \cup_i A_i$. Then B is countable and all groups $\pi_n^s(B/(B \cap X))$ vanish. □

Consider the set of all stably trivial level cofibrations $j : C \to D$ with D countable, and find a factorization

$$\begin{array}{ccc} C & \xrightarrow{in_j} & E_j \\ & \searrow^{j} & \downarrow{\scriptstyle p_j} \\ & & D \end{array}$$

for each such j such that in_j is a stably trivial cofibration and p_j is a stably trivial fibration.

Make fixed choices of the factorizations $j = p_j \cdot in_j$, and let I be the set of all stably trivial cofibrations in_j.

Lemma 44.2. *The set I generates the class of stably trivial cofibrations.*

Proof. Suppose given a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

where j is a cofibration, f is a stable equivalence and B is countable.

Then f has a factorization $f = q \cdot i$ where i is a stably trivial cofibration and q is a stably trivial fibration.

There is a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ & & \downarrow{\scriptstyle i} \\ {\scriptstyle j}\downarrow & \overset{\theta}{\nearrow} & Z \\ & & \downarrow{\scriptstyle q} \\ B & \longrightarrow & Y \end{array}$$

where the lift θ exists since j is a cofibration and q is a stably trivial fibration.

The image $\theta(B)$ of B is a countable subobject of Z, so Lemma 44.1 says that there is a subobject $D \subset Z$ such that D is countable and the level cofibration $j : D \cap X \to D$ is a stable equivalence.

What we have, then, is a factorization

$$\begin{array}{ccccc} A & \longrightarrow & D \cap X & \longrightarrow & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & D & \longrightarrow & Y \end{array}$$

of the original diagram, such that j is a countable, stably trivial level countable.

We can further assume (by lifting to E_j) that the original diagram has a factorization

$$\begin{array}{ccccc} A & \longrightarrow & D \cap X & \longrightarrow & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle in_j} & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & E_j & \longrightarrow & Y \end{array}$$

where the map in_j is a member of the set I.

Now suppose that $i : U \to V$ is a stably trivial cofibration. Then i has a factorization

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & W \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle q} \\ & & V \end{array}$$

where α is a member of the saturation of I and q has the RLP wrt all members of I.

But then q has the RLP wrt all countable cofibrations by the construction above, so that q has the RLP wrt all cofibrations.

In particular, there is a diagram

$$\begin{array}{ccc} U & \xrightarrow{j} & W \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle q} \\ V & \xrightarrow[1]{} & V \end{array}$$

so that i is a retract of j. □

Remark: Compare the proof of Lemma 44.2 with the proof of Lemma 11.5 — they are the same.

References

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