Contents

45 Spectra in simplicial modules

Suppose A is a simplicial R-module and K is a pointed simplicial set.

The simplicial R-module $A \otimes K$ is defined by

$$A \otimes K = A \otimes_R \tilde{R}(K),$$

where $\tilde{R}(K) = R(K)/R(*)$ defines the reduced free R-module functor

$$\tilde{R} : s_*\mathbf{Set} \to s(R-\mathbf{Mod}).$$

(Compare with Section 15.)

There are natural isomorphisms

$$\tilde{R}(K \wedge L) \cong \tilde{R}(K) \otimes \tilde{R}(L) = K \otimes \tilde{R}(L),$$

and there is a natural map

$$\gamma : u(A) \wedge K \to u(A \otimes K).$$

Here,

$$u : s(R-\mathbf{Mod}) \to s_*\mathbf{Set}$$

is the forgetful functor, where $u(A)$ is the simplicial set underlying A, pointed by 0.

The functor u is right adjoint to $\tilde{R}$.

We frequently write A for both a simplicial R-module A and its underlying pointed simplicial set.

Lemma 45.1. *Suppose A is a simplicial abelian group.*

Then the canonical map

$$\eta : A \to \mathbf{hom}_*(S^1, A \otimes S^1)$$

is a weak equivalence.

Proof. Δ^1_* is the simplicial set Δ^1, pointed by the vertex 0.

There is a contracting homotopy $h : \Delta^1_* \wedge \Delta^1_* \to \Delta^1_*$ given by the picture

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 1 \end{tikzcd}$$

and this map h determines a contracting homotopy

$$h_* : \mathbf{hom}_*(\Delta^1_*, B) \otimes \Delta^1_* \to \mathbf{hom}_*(\Delta^1_*, B).$$

for all simplicial abelian groups B.

$B \otimes \Delta^1_*$ is a model for the cone on B, and there is a natural short exact sequence

$$0 \to B \to B \otimes \Delta^1_* \to B \otimes S^1 \to 0.$$

The homotopy h_* induces a composite morphism

$$\begin{tikzcd}
A \otimes \Delta^1_* \arrow[r, "\eta \otimes 1"] \arrow[ddr, "\gamma"'] & \mathbf{hom}_*(S^1, A \otimes S^1) \otimes \Delta^1_* \arrow[d] \\
& \mathbf{hom}_*(\Delta^1_*, A \otimes S^1) \otimes \Delta^1_* \arrow[d, "h_*"] \\
& \mathbf{hom}_*(\Delta^1_*, A \otimes S^1)
\end{tikzcd}$$

and there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\eta"] \arrow[d] & \mathbf{hom}_*(S^1, A \otimes S^1) \arrow[d] \\
A \otimes \Delta^1_* \arrow[r, "\simeq", "\gamma"'] \arrow[d] & \mathbf{hom}_*(\Delta^1_*, A \otimes S^1) \arrow[d] \\
A \otimes S^1 \arrow[r, "1"'] & A \otimes S^1
\end{tikzcd}$$

This is a comparison of fibre sequences, so the map η is a weak equivalence. $\square$

Compare with the proof of Lemma 31.1.

Corollary 45.2. *The natural map*

$$\varepsilon_A : \mathbf{hom}_*(S^1, A) \otimes S^1 \to A$$

induces isomorphisms in π_k *for* $k \geq 1$.

Write $\Omega A = \mathbf{hom}_*(S^1, A)$.

Proof. There is a diagram

$$\begin{tikzcd} \Omega A \arrow[r, "\eta", "\simeq"'] \arrow[dr, "1"'] & \Omega(\Omega A \otimes S^1) \arrow[d, "\Omega\varepsilon"] \\ & \Omega A \end{tikzcd}$$

Thus, $\Omega\varepsilon_A$ is a weak equivalence, so that ε_A has the claimed effect in homotopy groups. $\square$

A **spectrum** (or spectrum object) A in simplicial R-modules consists of simplicial R-modules A^n, $n \geq 0$, together with bonding maps

$$\sigma : S^1 \otimes A^n \to A^{n+1},\ n \geq 0.$$

A morphism $f : A \to B$ of spectrum objects consists of simplicial R-module maps $A^n \to B^n$, $n \geq 0$, which respect the bonding homomorphisms.

$\mathbf{Spt}(R)$ is the corresponding category. This category is complete and cocomplete.

The maps

$$\gamma : S^1 \wedge u(A^n) \to u(S^1 \otimes A^n)$$

give the pointed simplicial sets $u(A^n)$ the structure of a spectrum, and define a **forgetful** functor

$$u : \mathbf{Spt}(R) \to \mathbf{Spt}.$$

The reduced free R-module functor $\tilde{R}$ determines a left adjoint to u. Explictly,

$$(\tilde{R}X)^n = \tilde{R}(X^n),$$

and the bonding morphisms are the composites

$$S^1 \otimes \tilde{R}(X^n) \cong \tilde{R}(S^1 \wedge X^n) \xrightarrow{\sigma_*} \tilde{R}(X^{n+1}).$$

A map $f : A \to B$ of spectrum objects is a **stable equivalence** (respectively **stable fibration**) if the underlying map $u(f) : uA \to uB$ of spectra is a stable equivalence (respectively stable fibration).

A **cofibration** in $\mathbf{Spt}(R)$ is a map which has the LLP wrt all morphisms which are stable fibrations and stable equivalences.

By adjointness, if $A \to B$ is a cofibration of spectra, then the induced map $\tilde{R}(A) \to \tilde{R}(B)$ is a cofibration of spectrum objects.

Lemma 45.3. *The functor $\tilde{R} : \mathbf{Spt} \to \mathbf{Spt}(R)$ preserves stable equivalences.*

Proof. The functor $\tilde{R}$ preserves level equivalences, so it suffices to show that if $A \to B$ is a stably trivial cofibration of spectra, then $\tilde{R}(A/B) \to 0$ is a stable equivalence.

Thus, it suffices to show that $\tilde{R}(X) \to 0$ is a stable equivalence if $X \to *$ is a stable equivalence. We can assume that X is level fibrant.

Since X is level fibrant, the assumption that $X \to *$ is a stable equivalence implies that all spaces $\Omega^\infty X^n$ are contractible. Thus, if $K \subset X^n$ is a finite subcomplex of X^n, there is a $k \geq 0$ such that the composite

$$S^k \wedge K \to S^k \wedge X^n \xrightarrow{\sigma^k} X^{n+k}$$

is homotopically trivial. This means that the induced map

$$S^k \otimes \tilde{R}(K) \to S^k \otimes \tilde{R}(X^n) \to \tilde{R}(X^{n+k})$$

is also homotopically trivial, and so the morphism

$$\Sigma^\infty \tilde{R}(K)[-n] \to \tilde{R}(X)$$

induces 0 in all stable homotopy groups. Every element in $\pi_k^s(\tilde{R}(X))$ is in the image of such a map, so all stable homotopy groups of $\tilde{R}(X)$ are 0. $\square$

Suppose that $i : A \to B$ is a level monomorphism in $\mathbf{Spt}(R)$ (as are all level cofibrations). Then there is a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{\pi} B/A \to 0$$

and the map π is a level surjection, hence a level fibration. In particular, the sequence is a level fibre sequence, and so there is a long exact sequence

$$\dots \pi^s_{k+1}(B/A) \xrightarrow{\partial} \pi^s_k A \xrightarrow{i_*} \pi^s_k(B) \xrightarrow{\pi_*} \pi^s_k(B/A) \to \dots$$

Theorem 45.4. *With these definitions, the category* $\mathbf{Spt}(R)$ *of spectrum objects in simplicial R-modules has the structure of a proper closed simplicial model category.*

Proof. The category **Spt** is cofibrantly generated (Lemma 44.2). Thus, a map $p : A \to B$ is a stable fibration if and only if it has the right lifting property with respect to the maps

$$\tilde{R}(U) \to \tilde{R}(V)$$

induced by a set J of stably trivial cofibrations $U \to V$.

All induced maps $\tilde{R}(U) \to \tilde{R}(V)$ are stable equivalences by Lemma 45.3.

The class of level inclusions which are stable equivalences is closed under pushout, by a long exact sequence argument.

It follows from a (transfinite) small object argument that every map $f : A \to B$ in $\mathbf{Spt}(R)$ has a factorization

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "f"'] & C \arrow[d, "p"] \\ & D \end{tikzcd}$$

where j is a stably trivial cofibration which has the LLP wrt all fibrations and p is a fibration.

The proof of the other statement of the factorization axiom **CM5** uses the fact (Lemma 40.4) that a map $p : A \to B$ is a stable fibration and a stable equivalence if and only if it has the right lifting property with respect to all morphisms

$$\tilde{R}(\Sigma^{\infty}\partial\Delta^{n}_{+}[k]) \to \tilde{R}(\Sigma^{\infty}\Delta^{n}_{+}[k]).$$

If $i : A \to B$ is a stably trivial cofibration, then i is a retract of a map which has the LLP wrt all fibrations, on account of a factorization for i in the style displayed above. Thus, every stably trivial cofibration has the LLP wrt all fibrations, proving **CM4**.

The function complex $\mathbf{hom}(A,B)$ is the simplicial R-module with n-simplices

$$\mathbf{hom}(A,B)_n = \{A \otimes \Delta^n \to B\}.$$

There is a natural isomorphism

$$\mathbf{hom}(\tilde{R}(K), A) \cong \mathbf{hom}(K, u(A)),$$

so that Quillen's axiom **SM7** follows from the corresponding statement for spectra. Thus, $\mathbf{Spt}(R)$ has a simplicial model structure.

Right properness follows from right properness for **Spt**, and left properness is proved by comparing long exact sequences. □

Here are some things to notice:

0) Every spectrum object in simplicial R-modules is level fibrant.

1) The forgetful functor u and its left adjoint $\tilde{R}$ determine a Quillen adjunction

$$\tilde{R} : \mathbf{Spt} \leftrightarrows \mathbf{Spt}(R) : u$$

If $R = \mathbb{Z}$ the canonical map $X \to u(\mathbb{Z}(X))$ is the **Hurewicz homomorphism** for spectra.

2) There is a Quillen adjunction

$$\Sigma^{\infty} : s(R - \mathbf{Mod}) \leftrightarrows \mathbf{Spt}(R) : \text{0-level}$$

where

$$(\Sigma^{\infty} A)^n = S^n \otimes A$$

(suspension spectrum) and the "0-level" functor is defined by $B \mapsto B^0$.

We also write $H(A) = \Sigma^\infty A$, and call it an Eilenberg-Mac Lane spectrum.

3) Suppose that A is a simplicial R-module, and consider the suspension spectrum object $\Sigma^\infty A$.

The bonding maps $S^1 \otimes S^n \otimes A \to S^{n+1} \otimes A$ are canonical isomorphisms, with adjoints

$$S^n \otimes A \to \mathbf{hom}(S^1, S^1 \otimes S^n \otimes A)$$

given by adjunction maps η.

All of these maps η are weak equivalences by Lemma 45.1, and so $\Sigma^\infty A$ is stably fibrant, ie. $u(\Sigma^\infty A)$is an Ω-spectrum. It also follows that there are isomorphisms

$$\pi_n^s(A) = \begin{cases} \pi_n(A) & \text{if } n \geq 0, \text{ and} \\ 0 & \text{if } n < 0. \end{cases}$$

In particular,

$$\pi_n^s(\tilde{R}(\Sigma^\infty(X))) \cong \pi_n^s(\Sigma^\infty \tilde{R}(X))$$

coincides with the reduced homology group $\tilde{H}_n(X, R)$ for $n \geq 0$ and is 0 otherwise.

Recall that there is a natural map

$$\gamma : u(A) \wedge K \to u(A \otimes K)$$

for pointed simplicial sets K and simplicial R-modules R. In simplicial degree n it is the obvious function

$$\bigvee_{K_n - *} A_n \to \bigoplus_{K_n - *} A_n.$$

The construction can be iterated, meaning that there are commutative diagrams

$$\begin{tikzcd} L \wedge u(A) \wedge K \arrow[r, "1 \wedge \gamma"] \arrow[d, "\gamma \wedge 1"'] & L \wedge u(A \otimes K) \arrow[d, "\gamma"] \\ u(L \otimes A) \wedge K \arrow[r, "\gamma"'] & u(L \otimes A \otimes K) \end{tikzcd}$$

The map γ may therefore promoted to the spectrum level, so there is a natural map

$$\gamma : u(B) \wedge K \to u(B \otimes K)$$

for spectrum objects B and pointed simplicial sets K.

Theorem 45.5. *The map*

$$\gamma : u(B) \wedge K \to u(B \otimes K)$$

is a stable equivalence for all spectrum objects B and pointed simplicial sets K.

Proof. The simplicial set K has a (pointed) skeletal decomposition $\mathrm{sk}_n K \subset K$, and there are pushout diagrams

$$\begin{tikzcd}
\bigvee_{x\in NK_n} \partial\Delta^n_+ \arrow[r] \arrow[d] & \mathrm{sk}_{n-1} K \arrow[d] \\
\bigvee_{x\in NK_n} \Delta^n_+ \arrow[r] & \mathrm{sk}_n K
\end{tikzcd}$$

of pointed simplicial sets.

Smashing with $u(B)$ gives a homotopy cocartesian diagram, which can be compared to the diagram of spectra underlying the pushout diagram

$$\begin{tikzcd}
\bigoplus_{x\in NK_n}(B\otimes\partial\Delta^n_+) \arrow[r] \arrow[d] & B\otimes \mathrm{sk}_{n-1} B \arrow[d] \\
\bigoplus_{x\in NK_n}(B\otimes\Delta^n_+) \arrow[r] & B\otimes \mathrm{sk}_n K
\end{tikzcd}$$

in $\mathbf{Spt}(R)$ via the map γ. The underlying diagram of spectra is homotopy cocartesian since both vertical maps have the same cofibres.

Inductively, one assumes that

$$u(B)\wedge \mathrm{sk}_{n-1} K \to u(B\otimes \mathrm{sk}_{n-1} K)$$

is a stable equivalence for all K. The statement for 0-skeleta is a consequence of additivity (Corollary 43.3), with a filtered colimit argument.

It therefore suffices to show that the map

$$\gamma : u(B)\wedge(\bigvee_{NK_n} \Delta^n_+) \to u(B\otimes(\bigvee_{NK_n}\Delta^n_+)).$$

is a stable equivalence. By additivity, this reduces to the statement that

$$\gamma : u(B) \wedge \Delta^n_+ \to u(B \otimes \Delta^n_+)$$

is a stable equivalence.

Both displayed functors preserve homotopy equivalences, so this particular instance of γ is equivalent to

$$\gamma : u(B) \wedge S^0 \to u(B \otimes S^0),$$

which is an isomorphism. □

Example: There is a natural isomorphism

$$H_n(X,R) \cong \pi^s_n(H(R) \wedge X).$$

Here $H(R)$ is the Eilenberg-Mac Lane spectrum $\tilde{R}(\mathbf{S}) = \Sigma^\infty R(S^0)$; it's also the sphere spectrum for $\mathbf{Spt}(R)$.

More generally, the groups

$$E_*(X) = \pi^s_*(E \wedge X)$$

are the E**-homology groups** of the space X, for a spectrum E.

Given a chain complex D in Ch_+, define the shifted complex $D[k]$ by

$$D[k]_p = \begin{cases} D_{k+p} & \text{if } p > 0, \\ \ker(\partial : D_k \to D_{k-1}) & \text{if } p = 0. \end{cases}$$

For $n \geq 0$, $D[-n]$ shifts up ("suspends") n times while $D[n]$ is the good truncation of a shift down.

There are two suspension constructions for simplicial R-modules:

- the standard suspension $S^1 \otimes A = \tilde{R}(S^1) \otimes A$,
- the Eilenberg-Mac Lane (or Kan) suspension $\overline{W}A = \Gamma(NA[-1])$.

There is an alternative construction for $\overline{W}A$, as follows.

Every simplicial abelian group can be written as a coequalizer

$$\bigoplus_{\theta:\mathbf{m}\to\mathbf{n}} A_n \otimes \Delta^m_+ \rightrightarrows \bigoplus_{n\geq 0} A_n \otimes \Delta^n_+ \to A$$

There is a pointed cosimplicial set $\mathbf{n} \mapsto \Delta^{n+1}_*$, where Δ^{n+1}_* is Δ^{n+1} pointed by 0, and $\theta : \mathbf{m} \to \mathbf{n}$ induces

$\theta_* : \mathbf{m+1} \to \mathbf{n+1}$ which is defined by

$$\theta_*(j) = \begin{cases} 0 & j = 0, \\ \theta(j-1)+1 & j > 0. \end{cases}$$

The simplicial set maps $d^0 : \Delta^n \to \Delta^{n+1}$ determine a map of cosimplicial spaces, and a pointwise monomorphism of cosimplicial simplicial modules

$$\tilde{R}(\Delta^n_+) \to \tilde{R}(\Delta^{n+1}_*)$$

One checks that there is an isomorphism of cosimplicial chain complexes

$$N(\tilde{R}\Delta^{n+1}_* / N\tilde{R}\Delta^n_+) \cong N\tilde{R}\Delta^n_+[-1]$$

that is natural in ordinal numbers $\mathbf{n}$ (exercise).

Thus, $\Gamma NA[-1]$ is defined by the coequalizer

$$\bigoplus_{\theta:\mathbf{m}\to\mathbf{n}} A_n \otimes N\tilde{R}\Delta^m_+[-1] \rightrightarrows \bigoplus_{n\geq 0} A_n \otimes N\tilde{R}\Delta^n_+[-1] \to \Gamma NA[-1]$$

There is a natural short exact sequence

$$0 \to A \xrightarrow{d^0} CA \to \overline{W}A \to 0$$

where the "cone" CA is defined by the coequalizer

$$\bigoplus_{\theta:\mathbf{m}\to\mathbf{n}} A_n \otimes \Delta^{m+1}_* \rightrightarrows \bigoplus_{n\geq 0} A_n \otimes \Delta^{n+1}_* \to CA$$

The inclusion $d^0 : \Delta^n \to \Delta^{n+1}$ contracts to the vertex $0 \in \mathbf{n+1}$, via the homotopy

$$h : \Delta^n_+ \wedge \Delta^1_* \to \Delta^{n+1}_*$$

(Δ^1 is pointed by 0) which is given by the picture

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & 0 \arrow[d] \\ 1 \arrow[r] & 2 \arrow[r] & \dots \arrow[r] & n+1 \end{tikzcd}$$

The homotopies h form a map of cosimplicial spaces, and hence determine a natural map

$$A \otimes \Delta^1_* \to CA,$$

which in turn induces a natural map

$$h : S^1 \otimes A \to \overline{W}A.$$

This map h is a natural equivalence, since $A \otimes \Delta^1_*$ and CA are both contractible.

The map h is even a natural homotopy equivalence, since the cosimplicial objects $S^1 \otimes \tilde{R}(\Delta_+)$ and $\overline{W}\tilde{R}(\Delta_+)$ are projective cofibrant.

For this last claim, we use Corollary 26.4, and its proof to show that the cosimplicial map

$$\tilde{R}(\Delta^n_+) \to \tilde{R}(\Delta^{n+1}_*)$$

is a projective cofibration.

Write g for the natural homotopy inverse for f.

Every spectrum object $\sigma : S^1 \otimes A^n \to A^{n+1}$ in simplicial R-modules determines a "Kan" spectrum object

$$\overline{W}A^n \xrightarrow{g} S^1 \otimes A^n \xrightarrow{\sigma} A^{n+1}$$

and hence a spectrum object

$$\tilde{\sigma} : NA^n[-1] \cong N(\overline{W}A^n) \to NA^{n+1}$$

in chain complexes.

Let $\sigma_* : A^n \to \Omega A^{n+1}$ be the adjoint of σ.

Corollary 45.2 says that the evaluation map

$$ev : \Omega A^{n+1} \otimes S^1 \to A^{n+1}$$

is a homology isomorphism above degree 0, and further that there is an induced equivalence

$$ev_*[1] : N\Omega A^{n+1} \to NA^{n+1}[1]$$

(as $\mathbb{Z}$-graded chain complexes), on account of the diagram

$$\begin{tikzcd}
N(S^1 \otimes \Omega A^{n+1}) \arrow[r, "Nev"] & NA^{n+1} \\
N(\overline{W}\Omega A^{n+1}) \arrow[u, "Ng"] \arrow[r, "\cong"'] & N(\Omega A^{n+1})[-1] \arrow[u, "ev_*"']
\end{tikzcd}$$

There is, finally, a natural commutative diagram of chain complex maps

$$\begin{tikzcd} NA^n \arrow[r, "N\sigma_*"] \arrow[dr, "\sigma"'] & N\Omega A^{n+1} \arrow[d, "ev_*[1]", "\simeq"'] \\ & NA^{n+1}[1] \end{tikzcd}$$

which defines the map σ.

Identify all chain complexes NA^n with $\mathbb{Z}$-graded chain complexes, and let QNA be the colimit of the diagram

$$NA^0 \xrightarrow{\sigma} NA^1[1] \xrightarrow{\sigma[1]} NA^2[2] \xrightarrow{\sigma[2]} \dots$$

Then one can show the following:

Proposition 46.1. *A map $f : A \to B$ is a stable equivalence of spectrum objects in simplicial R-modules if and only if the induced map $f_* : QNA \to QNB$ is a quasi-isomorphism of $\mathbb{Z}$-graded chain complexes.*

One can go further [1], to show that the Dold-Kan equivalence induces a Quillen equivalence

$$N : \mathbf{Spt}(R) \leftrightarrows Ch(R) : \Gamma$$

of the stable model structure on $\mathbf{Spt}(R)$, with the model structure on the category $Ch(R)$ of $\mathbb{Z}$-graded chain complexes of R-modules of Section 3, from the beginning of the course.

The weak equivalences in $Ch(R)$ are the quasi-isomorphisms, and the fibrations are the surjective homomorphisms of chain complexes.

This equivalence further induces an equivalence of the stable homotopy category for $\mathbf{Spt}(R)$ with the full derived category $\mathrm{Ho}(Ch(R))$ for chain complexes of R-modules.

This is the start of a long story — see also [1], [2].

References

[1] J. F. Jardine. Presheaves of chain complexes. *K-Theory*, 30(4):365–420, 2003. Special issue in honor of Hyman Bass on his seventieth birthday. Part IV.

[2] J.F. Jardine. *Local Homotopy Theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2015.