Lecture 001 (September 12, 2014)

1 Categorical homotopy theory

Much of the material in this section was invented by Quillen to describe the K-theory spaces, and originally appeared in [5]. The theory has been refined and extended over the years (see [2], for example), and it is now part of the basic canon of Homotopy Theory. This section contains a short introduction to the theory.

1.1 Categories and simplicial sets

Recall that the finite ordinal number $\mathbf{n}$ is the finite poset

$$\mathbf{n} = \{0, 1, \dots, n\},$$

and that an ordinal number morphism $\theta : \mathbf{m} \to \mathbf{n}$ is an order-preserving function, aka. poset morphism. The ordinals $\mathbf{n}$, $n \geq 0$, and the morphisms between them form the category $\mathbf{\Delta}$ of finite ordinal numbers.

Suppose that C is a small category. Recall that the nerve (or classifying space) BC is the simplicial set with

$$BC_n = \hom(\mathbf{n}, C),$$

where the indicated morphisms are functors between small categories (every poset is a category). Note that the n-simplex (functor) $\sigma : \mathbf{n} \to C$ is completely determined by the images

$$\sigma(0) \to \sigma(1) \to \cdots \to \sigma(n)$$

of the string of relations (arrows)

$$0 \leq 1 \leq \cdots \leq n,$$

so that one tends to identify BC_n with strings of arrows

$$a_0 \to a_1 \to \cdots \to a_n$$

in the category C.

The simplicial structure map

$$\theta^* : BC_n \to BC_m$$

is precomposition with the ordinal number morphism $\theta : \mathbf{m} \to \mathbf{n}$: θ^* takes an n-simplex $\sigma : \mathbf{n} \to C$ to the composite

$$\mathbf{m} \xrightarrow{\theta} \mathbf{n} \xrightarrow{\sigma} C.$$

Example 1.1. 1) $B\mathbf{n} = \Delta^n$.

2) C is a groupoid if and only if BC is a Kan complex.

In a similar vein with the statement about groupoids, the functor $B : \mathbf{cat} \to s\mathbf{Set}$ is fully faithful: the function

$$B : \hom(C, D) \to \hom(BC, BD)$$

is a bijection (exercise). This means that a simplicial set map $BC \to BD$ can be identified with a functor $C \to D$. For this reason (and this is even fashionable), one can think of small categories as special types of simplicial sets.

Here's a fundamental observation:

Lemma 1.2. *Suppose that C and D are small categories. Then the projections $C \times D \to C$ and $C \times D \to D$ induce a natural isomorphism of simplicial sets*

$$B(C \times D) \xrightarrow{\cong} BC \times BD.$$

The proof is obvious, and is an exercise.

A natural transformation $h : f \to f'$ of functors $f, f' : C \to D$ between small categories can be identified with a functor

$$h : C \times \mathbf{1} \to D$$

or equivalently

$$h : C \to D^{\mathbf{1}},$$

($D^{\mathbf{1}}$ is the category of arrows in D) such that $h(a,0) = f(a)$ and $h(a,1) = f'(a)$. It follows from the Lemma that the functor h induces a simplicial set map

$$h : BC \times \Delta^1 \to BD,$$

which is a homotopy from $f : BC \to BD$ to $f' : BC \to BD$.

Corollary 1.3. *1) Suppose given functors*

$$f : C \leftrightarrows D : g$$

and natural transformations $\epsilon : fg \to 1_D$ *and* $\eta : 1_C \to gf$. *Then the induced simplicial set maps*

$$f : BC \leftrightarrows BD : g$$

define a homotopy equivalence.

2) Any adjoint pair of functors

$$f : C \leftrightarrows D : g$$

induces a homotopy equivalence $BC \simeq BD$.

3) Any equivalence of categories

$$f : C \leftrightarrows D : g$$

induces a homotopy equivalence $BC \simeq BD$.

4) Suppose that C has either an initial or terminal object. Then BC is contractible,

5) The simplicial sets $B(C/x)$ and $B(x/C)$ associated to the slice categories C/x and x/C are contractible.

Proof. If C has an initial object 0 then the functors

$$0 : \mathbf{0} \leftrightarrows C : t$$

form an adjoint pair (the functor 0 which picks out the object 0 is left adjoint to the unique functor $t : C \to \mathbf{0}$. Thus $BC \simeq \Delta^0$.

Dually, if C has a terminal object $*$, then the functors

$$t : C \leftrightarrows \mathbf{0} : *$$

are adjoint.

$1 : x \to x$ is a terminal object for C/x and is an initial object for x/C. □

I prefer pictures of the homotopies when I can draw them. Suppose that C has a terminal object t. Then any string of morphisms in C (ie. any simplex of BC)

$$\sigma : a_0 \to a_1 \to \cdots \to a_n$$

fits into a commutative diagram

$$\begin{array}{ccccccc} a_0 & \longrightarrow & a_1 & \longrightarrow & \ldots & \longrightarrow & a_n \\ \downarrow & & \downarrow & & & & \downarrow \\ t & \xrightarrow[1]{} & t & \xrightarrow[1]{} & \ldots & \xrightarrow[1]{} & t \end{array}$$

which is a functor $h_\sigma : \mathbf{n} \times \mathbf{1} \to C$ and hence a simplicial set map $h_\sigma : \Delta^n \times \Delta^1 \to BC$. These maps h_σ respect the incidence relations of simplices of BC, and glue together to give the contracting homotopy $h : BC \times \Delta^1 \to BC$.

Different topic:

Lemma 1.4. *A simplicial set morphism $f : X \to BC$ is completely determined by its restriction*

$$\operatorname{sk}_2 X \subset X \xrightarrow{f} BC$$

to the 2-skeleton of X.

Proof. The simplicial set X is a colimit of its simplices in the sense that the canonical map

$$\varinjlim_{\Delta^n \to X} \Delta^n \to X$$

is an isomorphism. There is a corresponding isomorphism

$$\varinjlim_{\Delta^n \to X} \operatorname{sk}_2 \Delta^n \xrightarrow{\cong} \operatorname{sk}_2 X$$

since the 2-skeleton functor preserves colimits. Thus, it's enough to prove the Lemma for $X = \Delta^n$, but this is obvious: a functor $\mathbf{n} \to C$ is defined by the images of objects (vertices) and morphisms (1-simplices) subject to composition laws defined by 2-simplices. $\square$

Corollary 1.5. *A simplicial set morphism $f : X \to BC$ is completely determined by a the graph morphism determined by the vertex map $f : X_0 \to \mathrm{Ob}(C)$ and the edge map $f : X_1 \to \mathrm{Mor}(C)$, and the composition laws*

$$\begin{array}{ccc} f(\sigma(0)) & \xrightarrow{f(d_2\sigma)} & f(\sigma(1)) \\ & \searrow^{f(d_1\sigma)} & \downarrow{\scriptstyle f(d_0\sigma)} \\ & & f(\sigma(2)) \end{array}$$

determined by the 2-simplices σ of X.

The *path category* P_*X of a simplicial set X whose objects are the vertices X_0 of X, and whose morphisms are generated by the 1-simplices $\omega : d_1(\omega) \to d_0(\omega)$, subject to the relations

$$d_1(\sigma) = d_0(\sigma)d_2(\sigma)$$

for each 2-simplex σ.

Corollary 1.6. *The path category functor is* *left adjoint to the nerve functor: there are natural bijections*

$$\hom(P_*X, C) \cong \hom(X, BC)$$

Exercise 1.7. Show that the adjunction homomorphism $\epsilon : P_*BC \to C$ is an isomorphism of categories, for all small categories C.

Hint: show that the functor $\epsilon : P_*BC \to BC$ has a section $\sigma : C \to P_*BC$ which is the identity on objects, and every path

$$a_0 \to a_1 \to \cdots \to a_n$$

is in the class of its composite $a_0 \to a_n$.

1.2 The fundamental groupoid

Write GC for the free groupoid on a small category C. We then have the following:

Corollary 1.8. *The functor $X \mapsto GP_*X$ is left adjoint to the nerve functor $G \mapsto BG$ defined on groupoids G.*

Recall from the homotopy theory course (Lecture 010, Section 28, to be precise, or see [1]) that GP_*X is one of the equivalent models for the fundamental groupoid $\pi(X)$ of a simplicial set X.

The idea behind the equivalence with $\pi(X)$ is that one can show that the functor GP_* takes the anodyne extensions $\Lambda^n_k \subset \Delta^n$ to strong deformation retractions of groupoids, and hence takes weak equivalences of simplicial sets to weak equivalences (in fact, equivalences: exercise) of groupoids. Then one shows that GP_*Y is isomorphic to the traditional fundamental groupoid $\pi|Y|$ if Y is a Kan complex.

The fundamental groupoid functor $X \mapsto \pi(X)$ is left adjoint to the nerve functor $G \mapsto BG$ for groupoids G. The Van Kampen Theorem is a trivial consequence: every pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

induces a pushout diagram of groupoids

$$\begin{array}{ccc} \pi(A) & \longrightarrow & \pi(X) \\ \downarrow & & \downarrow \\ \pi(B) & \longrightarrow & \pi(Y). \end{array}$$

The key point about the fundamental groupoid for K-theory is the following:

Lemma 1.9. *Suppose that C is a small category. Then there is a natural isomorphism*

$$GP_*(BC) \cong GC.$$

Proof. The required isomorphism is obtained by applying the free groupoid functor G to the natural isomorphism

$$\epsilon : P_* BC \xrightarrow{\cong} C$$

of Exercise 1.7. □

1.3 Homotopy colimits

Nerves of groupoids have relatively simple structures defined by path components and automorphism groups (this is why they are extensively studied), but nerves of more general categories certainly do not.

Recall that every simplicial set X is a homotopy colimit of its simplices, in the sense that the simplices (maps) $\Delta^n \to X$ induce a weak equivalence

$$\underrightarrow{\text{holim}}_{\Delta^n \to X} \Delta^n \xrightarrow{\simeq} X, \tag{1}$$

where the homotopy colimit is indexed over the *simplex category* $\mathbf{\Delta}/X$ of X.

The objects of $\mathbf{\Delta}/X$ are the simplices $\Delta^n \to X$, and its morphisms are all commutative diagrams (incidence relations)

$$\begin{array}{ccc} \Delta^m & & \\ \downarrow & \searrow & \\ & & X \\ & \nearrow & \\ \Delta^n & & \end{array}$$

of simplicial set maps.

The displayed map in (1) is an equivalence, because fibres over fixed simplices are nerves of categories which have initial objects (exercise).

The simplicial sets Δ^n are contractible (the posets $\mathbf{n}$ have initial and terminal objects — take your pick), so the natural transformation $\Delta^n \to *$ of functors on $\mathbf{\Delta}/X$ is a weak equivalence, and hence induces a weak equivalence

$$\underrightarrow{\mathrm{holim}}_{\,\Delta^n \to X}\ \Delta^n \xrightarrow{\simeq} \underrightarrow{\mathrm{holim}}_{\,\Delta^n \to X}\ * = B(\mathbf{\Delta}/X). \tag{2}$$

It follows that there are natural weak equivalences

$$B(\mathbf{\Delta}/X) \xleftarrow{\simeq} \underrightarrow{\mathrm{holim}}_{\,\Delta^n \to X}\ \Delta^n \xrightarrow{\simeq} X.$$

In particular, every simplicial set X is naturally weakly equivalent to the nerve of its simplex category. This statement is a theorem of Kan and Thurston [4], from the 1970s.

Homotopy colimits are discussed at some length in the homotopy theory course [3] (particularly in Lecture 008), and in [1].

Recall that if $Y : I \to s\mathbf{Set}$ is a small diagram taking values in simplicial sets, then $\underrightarrow{\text{holim}}_I\, Y$ can be identified up to weak equivalence with the diagonal of a bisimplicial set given in horizontal degree n by

$$\bigsqcup_{i_0 \to \cdots \to i_n} X(i_0). \tag{3}$$

It's clear from this description that the homotopy colimit for the terminal functor $* : I \to s\mathbf{Set}$ is the bisimplicial set

$$\bigsqcup_{i_0 \to \cdots \to i_n} *,$$

which has diagonal BI (this, by the way, is how one defines the nerve functor in exotic settings [2], as the homotopy colimit of the one-point diagram). Also, any natural transformation $Y \to Y'$ induces a map

$$\underrightarrow{\text{holim}}_I\, Y \to \underrightarrow{\text{holim}}_I\, Y'.$$

In particular the canonical transformation $Y \to *$ induces a natural map

$$\pi : \underrightarrow{\text{holim}}_I\, Y \to BI.$$

The map in (2) is an example of this map.

For the record, the horizontal simplicial set in vertical degree m for the bisimplicial set (3) which defines the homotopy colimit $\underrightarrow{\text{holim}}_I X$ is the nerve of the translation category $E_I X_m$ associated to the set-valued functor $X_m : I \to \mathbf{Set}$.

Any functor $Y : I \to \mathbf{Set}$ has a *translation category* $E_I Y$: the objects of $E_I Y$ are all pairs (i, x) with $i \in \text{Ob}(I)$ and $x \in Y(i)$, and a morphism $\alpha : (i, x) \to (j, y)$ is a morphism $\alpha : i \to j$ of I such that $\alpha_*(x) = y$.

$$\begin{aligned} B(E_I Y)_n &= \{(i_0, x_0) \to (i_1, x_1) \to \cdots \to (i_n, x_n)\} \\ &= \bigsqcup_{i_0 \to \cdots \to i_n} Y(i_0), \end{aligned}$$

because all x_i in the string are determined by x_0.

The following is clear, from the corresponding result for bisimplicial sets:

Lemma 1.10. *Any natural transformation $Y \to Y'$ of I-diagrams which consists of weak equivalences $Y(i) \xrightarrow{\simeq} Y'(i)$, $i \in \text{Ob}(I)$ induces a weak equivalence*

$$\underrightarrow{\text{holim}}_I Y \xrightarrow{\simeq} \underrightarrow{\text{holim}}_I Y'.$$

This is why the map in (2) is a weak equivalence.

Beyond this, one of the most important things that one can say about homotopy colimits in general is the following result of Quillen:

Theorem 1.11. *Suppose that $X : I \to s\mathbf{Set}$ is a diagram of equivalences in the sense that all morphisms $\alpha : i \to j$ of I induce weak equivalences $\alpha_* : X(i) \xrightarrow{\simeq} X(j)$. Then for each object i of I the pullback diagram*

$$\begin{array}{ccc} X(i) & \longrightarrow & \underrightarrow{\text{holim}}_I\ X \\ \downarrow & & \downarrow \pi \\ \Delta^0 & \xrightarrow[i]{} & BI \end{array}$$

is homotopy cartesian.

This result is proved in Section 23 (Lecture 008) of [3]. See also [1].

The overall idea of proof is to find a factorization

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{i} & BI \\ & {\scriptstyle j}\searrow^{\simeq} \quad \nearrow{\scriptstyle p} & \\ & A & \end{array}$$

with p a fibration and i a trivial cofibration, and show that the induced map

$$X(i) \xrightarrow{j_*} A \times_{BI} \underrightarrow{\text{holim}}_I X$$

is a weak equivalence. One does this by showing that for all iterated pullback diagrams

$$\begin{array}{ccccc} pb_1 & \longrightarrow & pb_2 & \longrightarrow & \underrightarrow{\text{holim}}_I X \\ \downarrow & & \downarrow & & \downarrow \pi \\ \Lambda^n_k & \longrightarrow & \Delta^n & \longrightarrow & BI \end{array}$$

the induced map $pb_1 \to pb_2$ is a weak equivalence.

Remark 1.12. Here's some culture: the category I consists of a set $\mathrm{Ob}(I)$ of objects and a set $\mathrm{Mor}(I)$ of morphisms, source and target maps $s, t : \mathrm{Mor}(I) \to \mathrm{Ob}(I)$ identity arrows $e : \mathrm{Ob}(I) \to \mathrm{Mor}(I)$ and a law of composition

$$\mathrm{Mor}(I) \times_{\mathrm{Ob}(I)} \mathrm{Mor}(I) \to \mathrm{Mor}(I)$$

which is associative and respects identities (I'll leave it to the reader to render these last two properties as commutative diagrams of functions). A functor $X : I \to s\mathbf{Set}$ consists of simplicial sets $X(i)$, $i \in \mathrm{Ob}(I)$, and functions $\alpha_* : X(i) \to X(j)$, $\alpha : i \to j$ in $\mathrm{Mor}(I)$ which respect identities and the composition law. Write

$$X = \bigsqcup_{i \in \mathrm{Ob}(I)} X(i),$$

and observe that the functions $X(i) \to *$ define a simplicial set map $\pi : X \to \mathrm{Ob}(I)$. Then in this notation, the functor X is completely captured by an "action"

$$\begin{array}{ccc} \mathrm{Mor}(I) \times_s X & \xrightarrow{m} & X \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Mor}(I) & \xrightarrow[t]{} & \mathrm{Ob}(I) \end{array} \tag{4}$$

which respects composition and identities in a way the reader can describe. Here, the pullback diagram

$$\begin{array}{ccc} \mathrm{Mor}(I) \times_s X & \longrightarrow & X \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Mor}(I) & \xrightarrow[s]{} & \mathrm{Ob}(I) \end{array}$$

defines the fibre product object in (4). This is the "internal" description of a functor X on the category I.

Then the functor $X : I \to s\mathbf{Set}$ is a diagram of equivalences if and only if the diagram (4) is homotopy cartesian (exercise), and the statement of Theorem 1.11 is equivalent to the assertion that the pullback

$$\begin{array}{ccc} X & \longrightarrow & \underrightarrow{\text{holim}}_{I} X \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \text{Ob}(I) & \longrightarrow & BI \end{array}$$

is homotopy cartesian.

Here's some more culture: suppose that $f : X \to Y$ is a map of simplicial sets, and form the pullback diagrams

$$\begin{array}{ccc} f^{-1}(\sigma) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \Delta^n & \xrightarrow[\sigma]{} & Y \end{array}$$

as σ varies through the simplices of Y. In this way, we get a functor $\sigma \mapsto f^{-1}(\sigma)$ defined on the simplex category $\mathbf{\Delta}/Y$, and the maps $f^{-1}(\sigma) \to X$ define a map

$$\underrightarrow{\text{holim}}_{\,\Delta^n \xrightarrow{\sigma} Y} f^{-1}(\sigma) \to X$$

Lemma 1.13. . *The map*

$$\underrightarrow{\text{holim}}_{\,\Delta^n \xrightarrow{\sigma} Y} f^{-1}(\sigma) \to X$$

is a weak equivalence, for any map $f : X \to Y$ of simplicial sets.

The assertion that the map

$$\underrightarrow{\text{holim}}_{\,\Delta^n \to X} \Delta^n \to X$$

is a weak equivalence is a special case of Lemma 1.13.

The Lemma is easy enough to prove, with the same technique: one shows that the fibres of the map

$$\underrightarrow{\mathrm{holim}}_{\Delta^n \xrightarrow{\sigma} Y} f^{-1}(\sigma)_m \to X_m$$

are nerves of categories which have initial objects and are therefore contractible.

Lemma 1.13 is the foundation of all discussions of the Serre spectral sequence.

1.4 Theorem B, and A

Lemma 1.13 has a categorical analog. Suppose that $f : C \to D$ is a functor between small categories. The slice categories D/x and the canonical functors $D/x \to D$ are categorical standins for the simplices of a simplicial set (slice categories are contractible), and one forms the pullbacks

$$\begin{array}{ccc} f/x & \longrightarrow & C \\ \downarrow & & \downarrow \\ D/x & \longrightarrow & D \end{array}$$

Here, f/x is the category whose objects are morphisms $\tau : f(y) \to x$ in D, and whose morphisms

$\tau \to \tau'$ are morphisms $\alpha : y \to y'$ of C such that the diagrams

$$\begin{array}{ccc} f(y) & & \\ {\scriptstyle f(\alpha)}\downarrow & \searrow^{\tau} & \\ & & x \\ & \nearrow_{\tau'} & \\ f(y') & & \end{array}$$

commute.

The assignment $x \mapsto f/x$ defines a diagram $D \to$ **cat**, and the functors $f/x \to C$ define a map

$$\underrightarrow{\mathrm{holim}}_{\,x \in D}\ B(f/x) \to BC.$$

I believe that Quillen was the first to observe the following:

Lemma 1.14. *The map*

$$\underrightarrow{\mathrm{holim}}_{\,x \in D}\ B(f/x) \to BC.$$

is a weak equivalence.

Lemma 1.14 (like its simplicial set analog) is easy to prove: a bisimplex in bidegree (n, m) in the bisimplicial set which computes

$$\underrightarrow{\mathrm{holim}}_{\,x \in D}\ B(f/x)$$

is a triple

$$(x_0 \to \cdots \to x_m, f(x_m) \to y_0, y_0 \to \cdots \to y_n)$$

consisting of a string x of arrows of C of length m, a string y of arrows of D of length n and a morphism $f(x_m) \to y_0$ of D, and the fibre over x in vertical degree m is the nerve of the category $f(x_m)/D$, which is contractible.

Lemma 1.14 and Theorem 1.11 are the two major components of the proof of the following result of Quillen:

Theorem 1.15 (Theorem B)**.** *Suppose that $f : C \to D$ is a functor between small categories such that every morphism $x \to x'$ of D induces a weak equivalence $B(f/x) \to B(f/x')$. Then all induced pullback diagrams*

$$\begin{array}{ccc} B(f/x) & \longrightarrow & BC \\ \downarrow & & \downarrow f_* \\ B(D/x) & \longrightarrow & BD \end{array}$$

are homotopy cartesian.

This result appears (with its proof) as Theorem 23.5 (Lecture 008) of [3], and also in [1].

The proof is a picture of bisimplicial set maps:

$$\begin{array}{ccccc}
B(f/x) & \longrightarrow & \sqcup_{x_0\to\cdots\to x_n} B(f/x_0) & \xrightarrow{\simeq} & BC \\
\downarrow & \mathbf{II} & \downarrow & \mathbf{III} & \downarrow \\
B(D/x) & \longrightarrow & \sqcup_{x_0\to\cdots\to x_n} B(D/x_0) & \xrightarrow[\simeq]{} & BD \\
\simeq\downarrow & \mathbf{I} & \downarrow\simeq & & \\
* & \longrightarrow & \sqcup_{x_0\to\cdots\to x_n} * & &
\end{array}$$

Then $\mathbf{I}+\mathbf{II}$ is homotopy cartesian by Theorem 1.11, so $\mathbf{II}$ is homotopy cartesian and $\mathbf{II}+\mathbf{III}$ is homotopy cartesian, both by formal nonsense.

Theorem B has a little brother, namely Quillen's Theorem A:

Theorem 1.16 (Theorem A)**.** *Suppose that* $f : C \to D$ *is a functor between small categories such that all spaces* $B(f/x)$, $x \in \mathrm{Ob}(D)$ *are weakly equivalent to a point. Then the map* $f_* : BC \to BD$ *is a weak equivalence.*

Theorem A follows from Theorem B, but it is much easier to prove directly: the map $BC \to BD$ is weakly equivalent to the map in homotopy colimits over D which is induced by the natural transformation

$$B(f/x) \to B(D/x),$$

and this natural transformation is a sectionwise weak equivalence if all spaces $B(f/x)$ are weakly equivalent to a point.

Theorem A is important in its own right, particularly in homology of groups — see [6], for example.

References

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