**Lecture 002** (September 17, 2014)

## 2 Exact categories

The material on exact categories which is presented here is taken (and adapted a little) from Quillen's "Higher Algebraic $K$-theory I" [1].

I'll start with the canonical example.

Suppose that $R$ is a unitary ring and let $\text{Mod}(R)$ of be the category of (left, right) $R$-modules. Let $\mathcal{P}(R)$ be the full subcategory of $\text{Mod}(R)$ whose objects are the finitely generated projectives.

The category $\mathcal{P}(R)$ is closed under extensions in $\text{Mod}(R)$ in the sense that if there is an exact sequence

$$0 \to P \to M \to Q \to 0$$

with $P, Q$ projective then $M$ is projective (because such a sequence splits). The category $\text{Mod}(R)$ is an abelian category.

Suppose that $\mathbf{E}$ is the class of exact sequences

$$0 \to P \overset{i}{\rightarrowtail} P'' \overset{p}{\twoheadrightarrow} P \to 0$$

with objects in $\mathcal{P}(R)$. The monomorphisms $i$ in such exact sequences are called *admissible mono-*

*morphisms* (typically denoted $P \rightarrowtail P'$) while the morphisms $p$ are *admissible epimorphisms* (and typically written $Q \twoheadrightarrow Q'$).

In the particular example of finitely generated projective modules over a ring $R$, the admissible monics and epis are the split monomorphisms and epimorphisms respectively.

**Example**: Not all monomorphisms of $\mathcal{P}(R)$ are admissible: the multiplication by $n$ map

$$\times n : \mathbb{Z} \to \mathbb{Z}$$

does not split.

The class of sequences $\mathbf{E}$ has the following properties (here, $\mathbf{M} = \mathcal{P}(R)$):

1) Any sequence in $\mathbf{M}$ which is isomorphic to a sequence in $\mathbf{E}$ is in $\mathbf{E}$.

   The canonical split exact sequence

   $$0 \to M \to M \oplus M' \to M' \to 0$$

   is in $\mathbf{E}$.

2) Admissible epis are closed under composition and under pullback by arbitrary maps of $\mathbf{M}$. Admissible monics are closed under composition and under pushout by arbitrary maps of $\mathbf{M}$.

3) Suppose that $M \to M'$ is a map which has a kernel in $\mathbf{M}$. If there is a map $N \to M$ such that the composite $N \to M \to M'$ is an admissible epi, then $M \to M'$ is an admissible epi. Dually for admissible monics, if $M'' \to M$ is a map which has a cokernel in $\mathbf{M}$ and there is a map $M \to K$ such that the composite $M'' \to M \to K$ is an admissible monic, then $M \to M''$ is an admissible monic.

These properties are easy to prove for $\mathbf{M} = \mathcal{P}(R)$. Split epis are clearly closed under composition and pullback, and dually for split monics. If the composite $N \to M \to M'$ is a split epi, then $M \to M'$ has a section and is therefore a split epi.

More generally we have the following:

**Definition**: An *exact category* $\mathbf{M}$ is a small additive subcategory of an abelian category $\mathbf{A}$ which is closed under extensions and is equipped with a family $\mathbf{E}$ of sequences

$$0 \to M \to M' \to M'' \to 0$$

in $\mathbf{M}$ which are exact in $\mathbf{A}$ (called the exact sequences of $\mathbf{M}$), such that properties 1), 2) and 3) hold.

**Exercise**: Suppose that $\mathbf{M}$ is an exact category. Show that the opposite category $\mathbf{M}^{op}$ is an exact category.

**Other examples**:

1) Any small abelian category, with all exact sequences.
2) $\mathbf{M}(R)$ = the category of finitely generated $R$-modules for a Noetherian ring $R$ (is closed under extensions), with all exact sequences.
3) $\mathcal{P}(X)$ = vector bundles on a scheme $X$ with exact sequences which are locally split.
4) $\mathbf{M}(X)$ = coherent sheaves on a Noetherian scheme $X$, with all exact sequences.

The vector bundles on a scheme $X$ are those coherent sheaves of $\mathcal{O}_X$-modules which are locally free for the Zariski topology. Equivalently a vector bundle is an $\mathcal{O}_X$ module which is finitely generated projective on all affine open patches. Admissible epis in this category are epimorphisms which split on affine patches, and admissible monomorphisms are monics which split on affine patches.

Coherent sheaves are $\mathcal{O}_X$-modules which are finitely generated $R$-modules on all (Noetherian) affine open patches $\mathrm{Sp}(R) \subset X$. Obviously, $\mathcal{P}(X) \subset \mathbf{M}(X)$.

In particular, the axioms for vector bundles and coherent sheaves follow from the corresponding axioms for finitely generated projectives and finitely generated modules, respectively.

**Lemma 2.1.** *Suppose that* $\mathbf{M}$ *is an exact category. Then the isomorphisms of* $\mathbf{M}$ *are those maps which are both admissible monics and admissible epis.*

*Proof.* If $\theta : M \to N$ is an admissible epi and an admissible monic, then it's an epi and a monic in an abelian category and is therefore an isomorphism.

In effect, there is an exact sequence

$$0 \to M \xrightarrow{\theta} N \xrightarrow{p} P \to 0.$$

But then $\theta$ is an epimorphism so that $P \cong 0$. The identity map $1 : N \to N$ therefore factors through $\theta$: there is a unique map $\sigma : N \to M$ such that $\theta\sigma = 1_N$. But then $\theta\sigma\theta = \theta$ and $\theta$ is monic, so that $\sigma\theta = 1_M$.

The sequence

$$M \xrightarrow{1_M} M \to 0$$

is isomorphic to the canonical sequence

$$M \rightarrowtail M \oplus 0 \twoheadrightarrow 0,$$

and is therefore in $\mathbf{E}$, so that $1_M$ is an admissible monic. Similarly, $1_M$ is an admissible epi.

The classes of admissible monics and admissible epis are closed under isomorphism, and so the diagrams

$$\begin{array}{ccc} M & \xrightarrow{1_M} & M \\ & {\scriptstyle\theta}\searrow & \downarrow{\scriptstyle\theta}\ {\scriptstyle\cong} \\ & & N \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\theta} & N \\ {\scriptstyle\theta}\downarrow{\scriptstyle\cong} & \nearrow{\scriptstyle 1_N} & \\ N & & \end{array}$$

guarantee that an isomorphism $\theta$ is both an admissible epi and an admissible monic. $\square$

**Remark 2.2.** For each object $Q$ of $\mathbf{M}$ the sequences

$$0 \to 0 \rightarrowtail Q \overset{1}{\twoheadrightarrow} Q \to 0$$

and

$$0 \to Q \overset{1}{\rightarrowtail} Q \twoheadrightarrow 0 \to 0$$

are exact. Write $i_Q$ for the canonical admissible monic $0 \rightarrowtail Q$ and $p_Q$ for the canonical admissible epi $Q \twoheadrightarrow 0$.

**Lemma 2.3.** *Suppose given a diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \xrightarrow{i_K} & K & \xrightarrow{p_K} & K'' & \longrightarrow & 0 \\
 & & {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_3} & & \\
0 & \longrightarrow & E' & \xrightarrow{i_E} & E & \xrightarrow{p_E} & E'' & \longrightarrow & 0 \\
 & & {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_3} & & \\
0 & \longrightarrow & C' & \xrightarrow[i_C]{} & C & \xrightarrow[p_C]{} & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

*of exact sequences in* **M** *such that all vertical sequences are in* **E**. *If any two of the horizontal sequences are in* **E** *then so is the third.*

*Proof.* Suppose that the $E$ and $C$ sequences are in **E**. Then $j_2 i_K = i_E j_1$ is an admissible monic and $i_K$ has a cokernel in **M**, so that $i_K$ is an admissible monic and the $K$ sequence is in **E**.

Dually, if the $K$ and $E$ sequences are in **E** then the $C$ sequence is in **E**.

Suppose that the $K$ and $C$ sequences are in **E**. Suppose that

$$\begin{array}{ccc}
K' & \xrightarrow{i_K} & K \\
{\scriptstyle j_1}\downarrow & & \downarrow \\
E' & \longrightarrow & E' + K
\end{array}$$

is a pushout diagram. Then the map $E' \to E'+K$ is an admissible monic (since admissible monics are closed under pushout), and the comparison of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & E'+K & \longrightarrow & K'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle j_3} & & \\ 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \end{array}$$

forces the map $E'+K \to E$ to be monic. The sequence

$$0 \to E'+K \to E \xrightarrow{p_C q_2} C'' \to 0$$

is exact and $p_C q_2$ is an admissible epi, so the map $E'+K \to E$ is an admissible monic. The map $i_E : E' \to E$ is the composite

$$E' \to E'+K \to E$$

and is therefore an admissible monic. □

Lemma 2.3 is an exact category generator. Here are some examples, all associated to a fixed exact category $\mathbf{M}$:

1) Write $\mathrm{Ex}(\mathbf{M})$ for the category of exact sequences

$$0 \to P' \rightarrowtail P \twoheadrightarrow P' \to 0$$

in $\mathbf{E}$, with morphisms given by comparisons

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \rightarrowtail & P & \twoheadrightarrow & P'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q' & \rightarrowtail & Q & \twoheadrightarrow & Q'' & \longrightarrow & 0 \end{array}$$

in $\mathbf{M}$. An exact sequence of $\mathrm{Ex}(\mathbf{M})$ is a diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P' & \rightarrowtail & P & \twoheadrightarrow & P'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q' & \rightarrowtail & Q & \twoheadrightarrow & Q'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \rightarrowtail & N & \twoheadrightarrow & N'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

such that all vertical sequences are in $\mathbf{E}$.

2) $\mathrm{Mon}_n(\mathbf{M})$ is the category whose objects are all strings of admissible monics

$$P: \; P_1 \rightarrowtail P_2 \rightarrowtail \cdots \rightarrowtail P_n$$

in $\mathbf{M}$. The morphisms of $\mathrm{Mon}_n(\mathbf{M})$ are the obvious natural transformations, and an exact sequence

$$0 \to P \rightarrowtail Q \twoheadrightarrow N \to 0$$

of $\mathrm{Mon}_n(\mathbf{M})$ is an exact sequence of functors such that all component sequences

$$0 \to P_i \rightarrowtail Q_i \twoheadrightarrow N_i \to 0$$

are in $\mathbf{E}$.

3) $\mathrm{Epi}_n(\mathbf{M})$ is the category whose objects are all strings

$$Q: \ Q_1 \twoheadrightarrow Q_2 \twoheadrightarrow \cdots \twoheadrightarrow Q_n$$

The morphisms of $\mathrm{Epi}_n(\mathbf{M})$ are natural transformations, and an exact sequence

$$0 \to P \rightarrowtail Q \twoheadrightarrow N \to 0$$

of $\mathrm{Epi}_n(\mathbf{M})$ is a exact sequence of functors which is componentwise in $\mathbf{E}$.

4) $\mathrm{Iso}_n(\mathbf{M})$ is the category whose objects are all strings

$$Q: \ Q_0 \xrightarrow{\cong} Q_1 \xrightarrow{\cong} \dots \xrightarrow{\cong} Q_n$$

of isomorphisms of length $n$ (note the change of string length compared to the definitions above). The morphisms of $\mathrm{Iso}_n(\mathbf{M})$ are natural transformations, and an exact sequence

$$0 \to P \rightarrowtail Q \twoheadrightarrow N \to 0$$

of $\mathrm{Iso}_n(\mathbf{M})$ is a exact sequence of functors which is componentwise in $\mathbf{E}$. In this case, it suffices to require only that some sequence

$$0 \to P_i \to Q_i \to N_i \to 0$$

is in $\mathbf{E}$.

One doesn't need Lemma 2.3 for Example 4), but it is required for Examples 1)–3).

**Example**: $\mathrm{Mon}(\mathbf{M}) = \mathrm{Mon}_1(\mathbf{M})$ is the category of admissible monomorphisms $A \rightarrowtail B$ and exact sequences.

$\mathrm{Mon}(\mathbf{M})$ is closed under extensions by Lemma 2.3.

Any sequence of $\mathrm{Mon}(\mathbf{M})$ which is isomorphic to an exact sequence is exact, because this is so in $\mathbf{M}$.

Admissible epis are closed under composition. Consider the picture

$$\begin{array}{ccccccc} K_1 & \rightarrowtail & A_1 & \twoheadrightarrow & A_2 & \twoheadrightarrow & A_3 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_2 & \rightarrowtail & B_1 & \twoheadrightarrow & B_2 & \twoheadrightarrow & B_3 \end{array}$$

where $K_1$ and $K_2$ are the kernels of the composite admissible epis $A_1 \twoheadrightarrow A_3$ and $B_1 \twoheadrightarrow B_3$, respectively. Then the map $K_1 \to K_2$ is an admissible monic by Lemma 2.3. Dually, admissible monics are closed under composition.

Admissible epis are closed under base change in Mon($\mathbf{M}$), since admissible epis are closed under base change in $\mathbf{M}$. Dually, admissible epis of Mon($\mathbf{M}$) are closed under cobase change.

Suppose given maps

$$\begin{array}{ccccc} P_1 & \xrightarrow{g_1} & M_1 & \xrightarrow{f_1} & N_1 \\ \downarrow & & \downarrow & & \downarrow \\ P_2 & \xrightarrow[g_2]{} & M_2 & \xrightarrow[f_2]{} & N_2 \end{array}$$

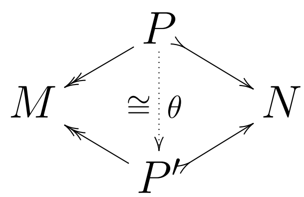
such that the map $(f_1, f_2)$ has a kernel $K_1 \rightarrowtail K_2$ in Mon($\mathbf{M}$), and such that the composites $P_1 \to N_1$ and $P_2 \to N_2$ are admissible epis. Then the maps $f_1$ and $f_2$ are admissible epis of $\mathbf{M}$ and hence define an admissible epi of Mon($\mathbf{M}$). The dual statement for admissible epis is has a similar proof.

## 3 The $Q$ construction

Suppose that $\mathbf{M}$ is an exact category. Define an equivalence relation on all pictures

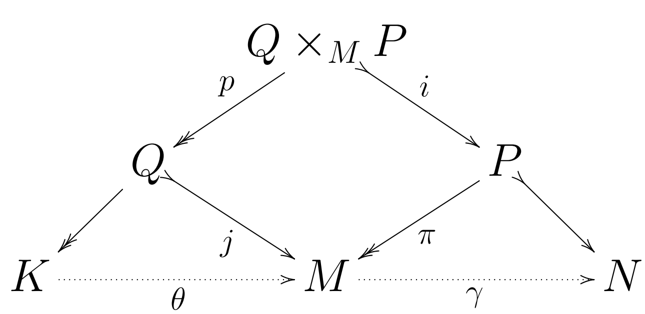
$$M \twoheadleftarrow P \rightarrowtail N$$

by saying that the top and bottom pictures in the diagram




are equivalent if the displayed isomorphism exists, making the diagram commute.

The category $Q\mathbf{M}$ has for objects all objects of $\mathbf{M}$. The morphisms $M \to M''$ are the equivalence classes of the pictures above. Composition is defined by pullback:




If the displayed pictures represent the classes $\theta$ and $\gamma$, then the outer composites represent $\gamma \cdot \theta$.

For the definition of composition to make sense you need to know that $i$ is an admissible monic. Recall that the classes of admissible epis and admissible monics are closed under composition, and admissible epis are closed under pullback (so that $p$ is an admissible epi). In the diagram

$$\begin{array}{ccccccccc} & & Q \times_M P & \xrightarrow{i} & P & & & & \\ & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} & & & & \\ 0 & \longrightarrow & Q & \underset{j}{\rightarrowtail} & M & \underset{\pi'}{\twoheadrightarrow} & M' & \longrightarrow & 0 \end{array}$$

the bottom sequence is exact, and $i$ is the kernel of $\pi'\pi$. The composite $\pi'\pi$ is an admissible epi, so that $i$ is an admissible monic. We have shown:

**Lemma 3.1.** *The class of admissible monics in an exact category* $\mathbf{M}$ *is closed under pullback along admissible epis. Dually, the class of admissible epis is closed under pushout along admissible monics.*

The pullback construction preserves isomorphisms of the defining pictures, so that the composition is well defined. The pictures

$$M \overset{1_M}{\twoheadleftarrow} M \overset{1_M}{\rightarrowtail} M$$

represent the identity morphisms for $Q\mathbf{M}$.

Pullback diagrams of the form

$$\begin{array}{ccc} P & \rightarrowtail & P' \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ Q & \rightarrowtail & Q' \end{array}$$

are pushout diagrams, and conversely. Square diagrams like this which are both pullbacks ("cartesian") and pushouts ("cocartesian") are often said to be *bicartesian*.

Suppose that $i : M \rightarrowtail N$ is an admissible monic of $\mathbf{M}$, and let $i_! : M \to N$ be the morphism of $Q\mathbf{M}$ which is represented by the picture

$$M \overset{1_M}{\twoheadleftarrow} N \overset{i}{\rightarrowtail} N.$$

**Exercise 3.2.** Write $\text{Mon}(\mathbf{M})$ for the subcategory of admissible monics of $\mathbf{M}$. Show that the function

$$\text{Mon}(\mathbf{M})(M, N) \to Q\mathbf{M}(M, N)$$

defined by $i \mapsto i_!$ is injective and defines a functor $\text{Mon}(\mathbf{M}) \to Q\mathbf{M}$.

If $p : P \to N$ is an admissible epi, let $p^! : N \to P$ be the morphism of $Q\mathbf{M}$ which is represented by the picture

$$N \overset{p}{\twoheadleftarrow} P \overset{1_P}{\rightarrowtail} P.$$

**Exercise 3.3.** Write $\mathrm{Epi}(\mathbf{M})$ for the subcategory of admissible epis of $\mathbf{M}$. Show that the function

$$\mathrm{Epi}(P,N) \to Q\mathbf{M}(N,P)$$

defined by $p \mapsto p^!$ is injective and defines a (contravariant) functor $\mathrm{Epi}(\mathbf{M})^{op} \to Q\mathbf{M}$.

**Exercise 3.4.** 1) Suppose that the morphism $\theta : M \to N$ of $Q\mathbf{M}$ is represented by the picture

$$M \overset{p}{\twoheadleftarrow} P \overset{i}{\rightarrowtail} N.$$

Then $\theta = i_! \cdot p^!$.

2) Suppose given a pullback diagram

$$\begin{array}{ccc} M' & \overset{i'}{\rightarrowtail} & P \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ M & \underset{i}{\rightarrowtail} & N \end{array}$$

in $\mathbf{M}$, where $p$ is an admissible epi and $i$ is an admissible monic. Then $p^! \cdot i_! = i'_! \cdot p'^!$ in $Q\mathbf{M}$.

**Lemma 3.5.** *Suppose given functors*

$$h_m : \mathrm{Mon}(\mathbf{M}) \to C, \quad h_e : \mathrm{Epi}(\mathbf{M})^{op} \to C$$

*such that*

- *$h_m$ and $h_e$ coincide on objects*

- *For any pullback square*

$$\begin{array}{ccc} M' & \overset{i'}{\rightarrowtail} & P \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ M & \underset{i}{\rightarrowtail} & N \end{array}$$

  *in* $\mathbf{M}$ *we have the relation*

$$h_e(p)h_m(i) = h_m(i)h_e(p')$$

  *in* $C$*.*

*Then there is a unique functor* $h : Q\mathbf{M} \to C$ *such that* $h(i_!) = h_m(i)$ *and* $h(p^!) = h_e(p)$*.*

*Proof.* If the morphism $\theta : M \to N$ of $Q\mathbf{M}$ is represented by the picture

$$M \overset{p}{\twoheadleftarrow} P \overset{i}{\rightarrowtail} N,$$

then $\theta = i_! \cdot p^!$ and any functor $g : Q\mathbf{M} \to C$ satisfying the criteria of the Lemma must satisfy

$$g(\theta) = g(i_! p^!) = g(i_!)g(p^!) = h_m(i)h_e(p),$$

so that $g$ is uniquely determined if it exists.

It remains to show that the assignment

$$\theta \mapsto h_m(i)h_e(p)$$

is independent of the representing picture.

Every isomorphism $\theta : M \to N$ of $\mathbf{M}$ is both an admissible epi and an admissible monic, and there is a pullback square

$$\begin{array}{ccc} M & \xrightarrow{1_M} & M \\ {\scriptstyle 1_M}\downarrow & & \downarrow{\scriptstyle \theta} \\ M & \xrightarrow[\theta]{} & N \end{array}$$

The morphism $\theta$ is invertible in both $\mathrm{Epi}(\mathbf{M})$ and $\mathrm{Mon}(\mathbf{M})$ so that the morphisms $h_e(\theta)$ and $h_m(\theta)$ are invertible in $C$. The compatibility condition for the functors $h_e$ and $h_m$ and the square above together imply that $h_e(\theta)h_m(\theta) = 1$ in $C$, so that $h_e(\theta) = h_m(\theta)^{-1}$.

Finally, suppose given a commutative diagram

$$\begin{array}{ccccc} & & P & & \\ & \stackrel{p}{\swarrow} & & \stackrel{i}{\searrow} & \\ M & & \cong \downarrow \theta & & N \\ & \underset{p'}{\nwarrow} & & \underset{i'}{\nearrow} & \\ & & P' & & \end{array}$$

in $\mathbf{M}$. Then

$$\begin{aligned} h_m(i)h_e(p) &= h_m(i')h_m(\theta)h_e(p) \\ &= h_m(i')h_e(\theta)^{-1}h_e(p) \\ &= h_m(i')h_e(p'), \end{aligned}$$

and so the assignment $\theta \mapsto h_m(i)h_e(p)$ is well defined. $\square$

**Definition**: Suppose that $\mathbf{M}$ and $\mathbf{N}$ are exact categories. An *exact functor* $f : \mathbf{M} \to \mathbf{N}$ is an additive functor which takes exact sequences of $\mathbf{M}$ to exact sequences of $\mathbf{N}$.

Exact functors abound in nature. Here are some examples:

1) The inclusions $\mathcal{P}(R) \subset \mathbf{M}(R), \mathcal{P}(X) \subset \mathbf{M}(X)$ for Noetherian rings $R$ and schemes $X$, respectively, are exact.

2) Every ring homomorphism $f : R \to S$ (respectively scheme homomorphism $g : Y \to X$) induces an exact functor $f^* : \mathcal{P}(R) \to \mathcal{P}(S)$ with $P \mapsto P \otimes_R S$ (respectively $g^* : \mathcal{P}(X) \to \mathcal{Q}(Y)$ with $P \mapsto g^*(P)$)

3) If $g : R \to S$ is a flat morphism of Noetherian rings then the assignment $N \mapsto N \otimes_R S$ defines an exact functor $\mathbf{M}(R) \to \mathbf{M}(S)$. Similarly, if $f : Y \to X$ is a flat morphism of Noetherian schemes, then inverse image defines an exact functor $g^* : \mathbf{M}(X) \to \mathbf{M}(Y)$ for coherent sheaves.

4) Suppose that $f : R \to S$ gives $S$ the structure of a finitely generated $R$-module, where $R$ is

Noetherian. Then every finitely generated $S$-module is finitely generated as an $R$-module, so that restriction of scalars defines an exact functor $f_* : \mathbf{M}(S) \to \mathbf{M}(R)$, which we may as well call the *transfer*. Similarly, if $g : Y \to X$ is a finite morphism of schemes and $X$ is Noetherian, then restriction of scalars defines a transfer morphism $g_* : \mathbf{M}(Y) \to \mathbf{M}(X)$.

**Exercise 3.6.** Every exact functor $f : \mathbf{M} \to \mathbf{N}$ induces a functor $f : Q\mathbf{M} \to Q\mathbf{N}$.

You can either take this to be completely obvious, or apply Lemma 3.5.

Observe that every natural isomorphism $f \cong g$ of exact functors $\mathbf{M} \to \mathbf{N}$ induces a natural isomorphism $f_* \cong g_*$ of induced functors $Q\mathbf{M} \to Q\mathbf{N}$, and hence a homotopy $f_* \simeq g_*$ of the associated simplicial set maps $BQ\mathbf{M} \to BQ\mathbf{N}$.

An *exact equivalence* is a pair of exact functors

$$f : \mathbf{M} \leftrightarrows \mathbf{N} : g$$

which forms an equivalence of categories in the usual sense that there are natural isomorphisms $fg \xrightarrow{\cong} 1_{\mathbf{N}}$ and $gf \xrightarrow{\cong} 1_{\mathbf{M}}$. Every such exact equivalence determines a homotopy equivalence $BQ\mathbf{M} \simeq BQ\mathbf{N}$.

**Exercise 3.7.** Let $\mathbf{M}$ be an exact category, and define a functor

$$f : \mathbf{M} \to \mathrm{Iso}_n(\mathbf{M})$$

by sending an object $P$ to the identity string

$$P \xrightarrow{1} P \xrightarrow{1} \dots \xrightarrow{1} P.$$

Define a functor

$$g : \mathrm{Iso}_n(\mathbf{M}) \to \mathbf{M}$$

by sending the string

$$P_0 \xrightarrow{\cong} P_1 \xrightarrow{\cong} \dots \xrightarrow{\cong} P_n$$

to the object $P_0$. Show that the functors

$$f : \mathbf{M} \leftrightarrows \mathrm{Iso}_n(\mathbf{M}) : g$$

form an exact equivalence.

## 4 Fundamental groupoid of $Q\mathbf{M}$ and $K_0(\mathbf{M})$

Suppose that $\mathbf{M}$ is an exact category. The group $K_0(\mathbf{M})$ is the free group generated by the isomorphism classes $[P]$ of objects $P$ of $\mathbf{M}$, subject to the relations

$$[P] = [P'] + [P''],$$

one for every exact sequence

$$0 \to P' \rightarrowtail P \twoheadrightarrow P'' \to 0$$

of $\mathbf{E}$.

An exact functor $f : \mathbf{M} \to \mathbf{N}$ induces a group homomorphism

$$f_* : K_0(\mathbf{M}) \to K_0(\mathbf{N}).$$

**Theorem 4.1.** *There is an equivalence of groupoids*

$$GQ\mathbf{M} \simeq K_0(\mathbf{M}).$$

The following is a consequence of Theorem 4.1 and Lemma 1.9 of Lecture 001.

**Corollary 4.2.** *The space $BQ\mathbf{M}$ is connected, and there is an isomorphism of groups*

$$\pi_1(BQ\mathbf{M}, 0) \cong K_0(\mathbf{M}).$$

**Remark 4.3.** You don't need Theorem 4.1 to see that $BQ\mathbf{M}$ is connected. This assertion is a triviality, given the presence of the admissible monics $i_P : 0 \rightarrowtail P$ for all objects $P$ of $\mathbf{M}$.

*Proof of Theorem 4.1.* We define a functor $\phi : Q\mathbf{M} \to K_0(\mathbf{M})$ by using Lemma 3.5; this functor inverts all morphisms, and hence factors through a functor $\phi_* : GQ\mathbf{M} \to K_0(\mathbf{M})$.

Suppose that $p : P \twoheadrightarrow Q$ is an admissible epi, and let $\phi_e(p) = [K]$ where

$$0 \to K \rightarrowtail P \stackrel{p}{\twoheadrightarrow} Q \to 0$$

is an exact sequence of $\mathbf{E}$. If $q : Q \twoheadrightarrow N$ is another admissible epi, then there is a diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(p) & \longrightarrow & \ker(qp) & \longrightarrow & \ker(q) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow & PB & \downarrow & & \\
0 & \longrightarrow & \ker(p) & \longrightarrow & P & \xrightarrow{p} & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle qp} & & \downarrow{\scriptstyle q} & & \\
0 & \longrightarrow & 0 & \longrightarrow & N & \xrightarrow{1} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(the square $PB$ is a pullback). It follows that

$$\phi_e(qp) = \phi_e(p) + \phi_e(q)$$

in $K_0(\mathbf{M})$. Also $\phi_e(1_P) = 0$, so we have defined a functor $\phi_e : \mathrm{Epi}(\mathbf{M})^{op} \to K_0(\mathbf{M})$.

Let $\phi_m : \mathrm{Mon}(\mathbf{M}) \to K_0(\mathbf{M})$ be the constant functor which takes all admissible monics $i$ to $[0]$.

For any pullback square

$$\begin{array}{ccc} M' & \overset{i'}{\rightarrowtail} & P \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ M & \underset{i}{\rightarrowtail} & N \end{array}$$

we have the relations

$$\phi_e(p) + \phi_m(i) = \phi_e(p) = \phi_e(p') = \phi_m(i') + \phi_e(p'),$$

since $p$ and $p'$ have the same kernel. Lemma 3.5 therefore implies that there is a functor $\phi : Q\mathbf{M} \to K_0(\mathbf{M})$ which is defined by $\phi_e$ and $\phi_m$.

We can now define a functor

$$\psi : K_0(\mathbf{M}) \to GQ\mathbf{M},$$

by sending a class $[Q]$ to the class of the path

$$0 \xrightarrow{p_Q^!} Q \xleftarrow{i_{Q!}} 0,$$

or $i_{Q!}^{-1} p_Q^!$ in $GQ\mathbf{M}$. If there is an exact sequence

$$0 \to K \overset{i}{\rightarrowtail} P \overset{p}{\twoheadrightarrow} Q \to 0$$

then there is a commutative diagram

$$\begin{array}{ccccccc} & & & P & & & \\ & & {\scriptstyle p}\swarrow & & \nwarrow{\scriptstyle i} & & \\ & Q & & & & K & \\ {\scriptstyle p_Q}\swarrow & & \nwarrow{\scriptstyle i_Q} & & {\scriptstyle p_K}\swarrow & & \nwarrow{\scriptstyle i_K} \\ 0 & & & 0 & & & 0 \end{array}$$

in $\mathbf{M}$ in which the indicated square is a pullback. It follows that

$$i^{1}_{K!}p^{!}_{K}i^{-1}_{Q!}p^{!}_{Q} = i^{-1}_{K!}i^{-1}_{!}p^{!}p^{!}_{Q} = i^{-1}_{P!}p^{!}_{P}$$

in $GQ\mathbf{M}$.

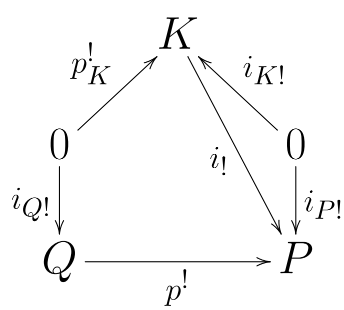
The composite

$$Q\mathbf{M} \xrightarrow{\phi} K_0(\mathbf{M}) \xrightarrow{\psi} GQ\mathbf{M}$$

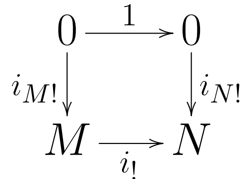
is naturally isomorphic to the canonical functor $\eta : Q\mathbf{M} \to GQ\mathbf{M}$: the isomorphism $\epsilon : \psi\phi \to \eta$ is defined on object $P$ by

$$\epsilon_P = i_{P!} : 0 \rightarrowtail P.$$

In effect, there are commutative diagrams in $Q\mathbf{M}$




for an admissible epi $p : P \twoheadrightarrow Q$ (with kernel $i : K \to P$), and



for an admissible monic $i : M \rightarrowtail N$.

It follows that the composite

$$GQ\mathbf{M} \xrightarrow{\phi_*} K_0(\mathbf{M}) \xrightarrow{\psi} GQ\mathbf{M}$$

is naturally isomorphic to the identity functor on $GQ\mathbf{M}$. One also shows that the composite

$$K_0(\mathbf{M}) \xrightarrow{\psi} GQ\mathbf{M} \xrightarrow{\phi_*} K_0(\mathbf{M})$$

is the identity on $K_0(\mathbf{M})$: $\psi([P])$ is the composite $i_{P!}^{-1} p_P^!$, while $\phi_*$ takes $p_P^!$ to $[P]$ and takes $i_{P!}$ to $[0]$. □

## References

[1] Daniel Quillen. Higher algebraic $K$-theory. I. In *Algebraic $K$-theory, I: Higher $K$-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.