Lecture 003 (September 26, 2014)

5 Waldhausen's $s_\bullet$-construction

The basic definitions and results of this section first appeared in Waldhausen's seminal paper [3]. Many of the tricks in the proofs which are given here appear in [1] and [2].

Suppose that C is some category. Write $\mathrm{Ar}(C)$ for the category whose objects are the morphisms $\alpha : a \to b$ of C. A morphism $\alpha \to \beta$ of $\mathrm{Ar}(C)$ is a commutative diagram

$$\begin{array}{ccc} a & \longrightarrow & c \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ b & \longrightarrow & d \end{array}$$

in C.

Example: In the ordinal number $\mathbf{n}$, there is a morphism $i \to j$ if and only if $i \leq j$. Thus, the objects of $\mathrm{Ar}(\mathbf{n})$ can be identified with pairs (i, j) such that $i \leq j$, and there is a morphism $(i, j) \to (k, l)$ in $\mathrm{Ar}(\mathbf{n})$ if and only if $i \leq k$ and $j \leq l$.

Observe that any functor $C \to D$ induces a functor $\mathrm{Ar}(C) \to \mathrm{Ar}(D)$.

Suppose that $\mathbf{M}$ is an exact category. The set $s_n(\mathbf{M})$ consists of all functors $P : \mathrm{Ar}(\mathbf{n}) \to \mathbf{M}$ such that the following two properties are satisfied:

1) $P(i,i) \cong 0$ for all i, and

2) if $i \leq j \leq k$ then the sequence

$$0 \to P(i,j) \rightarrowtail P(i,k) \twoheadrightarrow P(j,k) \to 0$$

is exact in $\mathbf{M}$ (ie. is in the distinguished class $\mathbf{E}$).

Say that such a functor P is *exact*.

If $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map and $P : \mathrm{Ar}(\mathbf{n}) \to \mathbf{M}$ is exact, then the composite

$$\mathrm{Ar}(\mathbf{m}) \xrightarrow{\theta_*} \mathrm{Ar}(\mathbf{n}) \xrightarrow{P} \mathbf{M}$$

is exact.

Write $s_\bullet(\mathbf{M})$ for the resulting simplicial set. This simplicial set is the *Waldhausen $s_\bullet$-construction* for an exact category $\mathbf{M}$. We'll see (Theorem 5.4 below) that $s_\bullet(\mathbf{M})$ is naturally weakly equivalent to $BQ\mathbf{M}$.

Note that if $i \leq j \leq k \leq l$ in $\mathbf{n}$ and $P : \mathrm{Ar}(\mathbf{n}) \to$

$\mathbf{M}$ is exact, then the diagram

$$\begin{array}{ccc} P(i,k) & \rightarrowtail & P(i,l) \\ \downarrow & & \downarrow \\ P(j,k) & \rightarrowtail & P(j,l) \end{array}$$

consists of admissible monics and epis as indicated, and is bicartesian since both vertical arrows have the same kernel, namely $P(i,j)$.

Example: Suppose that $P : \mathrm{Ar}(\mathbf{3}) \to \mathbf{M}$ is exact. Then P is specified by a diagram

$$\begin{array}{ccccccc} 0 & \rightarrowtail & P(0,1) & \rightarrowtail & P(0,2) & \rightarrowtail & P(0,3) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \rightarrowtail & P(1,2) & \rightarrowtail & P(1,3) \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & \rightarrowtail & P(2,3) \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

of admissible epis and monics such that all square are pullbacks (really, bicartesian). There are two ways to view this:

1) P is obtained from the string of admissible monics

$$P(0,1) \rightarrowtail P(0,2) \rightarrowtail P(0,3)$$

by putting in all possible cokernels. Dually, P is obtained from the string of admissible epis

$$P(0,3) \twoheadrightarrow P(1,3) \twoheadrightarrow P(2,3)$$

by putting in all possible kernels.

2) P is obtained from the picture

$$P(1,2) \twoheadleftarrow P(0,2) \rightarrowtail P(0,3)$$

by putting in all possible kernels and cokernels.

This last thing gives a comparison of $s_3(\mathbf{M})$ with morphisms in $Q\mathbf{M}$. Overall, the relation between $s_\bullet(\mathbf{M})$ and $BQ\mathbf{M}$ is a little complicated.

Write $S_n(\mathbf{M})$ for the category of exact functors $\mathrm{Ar}(\mathbf{n}) \to \mathbf{M}$ and their natural transformations. Then $S_n(\mathbf{M})$ is the category of n-simplices of a simplicial category $S_\bullet(\mathbf{M})$. Say that a sequence of morphisms

$$0 \to P_1 \to P_2 \to P_3 \to 0$$

in $S_n(\mathbf{M})$ is exact if all sequences

$$0 \to P_1(i,j) \rightarrowtail P_2(i,j) \twoheadrightarrow P_3(i,j) \to 0$$

are members of the distinguished class $\mathbf{E}$ in $\mathbf{M}$.

Lemma 2.3 implies that if

$$0 \to P \to Q \to R \to 0$$

is an exact sequence of functors $\mathrm{Ar}(\mathbf{n}) \to \mathbf{M}$ such that all consitutent sequences

$$0 \to P(i,j) \rightarrowtail Q(i,j) \twoheadrightarrow R(i,j) \to 0$$

are in $\mathbf{E}$, then if any two of P, Q and R are exact then so is the third. It follows that $S_n(\mathbf{M})$ and its exact sequences satisfy the axioms for an exact category. Further, it's easy to show that all simplicial structure functors $\theta^* : S_n(\mathbf{M}) \to S_m(\mathbf{M})$ exact, so that $S_\bullet(\mathbf{M})$ is a simplicial exact category.

Lemma 5.1. *Suppose that the exact functors $f, g : \mathbf{M} \to \mathbf{N}$ are naturally isomorphic. Then the induced maps*

$$f_*, g_* : s_\bullet(\mathbf{M}) \to s_\bullet(\mathbf{N})$$

are homotopic.

Proof. A simplicial set map $h : X \times \Delta^1 \to Y$ consists of functions $h_\tau : X_n \to Y_n$, one for each $\tau : \mathbf{n} \to \mathbf{1}$, $n \geq 0$, such that the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{h_\tau} & Y_n \\ {\scriptstyle \theta^*}\downarrow & & \downarrow{\scriptstyle \theta^*} \\ X_m & \xrightarrow[h_{\tau\theta}]{} & Y_m \end{array}$$

commutes for each ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$. In effect,

$$h_\tau(\sigma) = h(\sigma, \tau)$$

for all $\sigma : \Delta^n \to X$ and $\tau : \Delta^n \to \Delta^1$.

Suppose that $P : \mathrm{Ar}(\mathbf{n}) \to \mathbf{M}$ is exact, and let $h_\tau(P)$ be the composite

$$\mathrm{Ar}(\mathbf{n}) \xrightarrow{(P,\tau_*)} \mathbf{M} \times \mathrm{Ar}(\mathbf{1}) \xrightarrow{1\times s} \mathbf{M} \times \mathbf{1} \xrightarrow{h} \mathbf{N}.$$

Here, h is the natural isomorphism

$$f(N) = h(N,0) \cong h(N,1) = g(N),$$

and $s : \mathrm{Ar}(\mathbf{1}) \to \mathbf{1}$ is the source map $(i,j) \mapsto i$. More explicitly, the functor $h_\tau P$ is specified by the assignment

$$(i,j) \mapsto \begin{cases} fP(i,j) & \text{if } \tau(i) = 0, \\ gP(i,j) & \text{if } \tau(i) = 1. \end{cases}$$

In all cases, there are natural isomorphisms

$$h_\tau(P) \cong fP \cong gP,$$

so that $h_\tau(P)$ is exact. □

Here's some fun with ordinal numbers: suppose that $\mathbf{n}$ is an ordinal number, let $\mathbf{n}^o$ denote the opposite poset

$$n^o \to \cdots \to 1^o \to 0^o,$$

and let $\mathbf{n}^o * \mathbf{n}$ be the poset join

$$n^o \to \cdots \to 1^o \to 0^o \to 0 \to 1 \to \cdots \to n.$$

or, more suggestively

$$\begin{array}{ccccccc} 0^o & \leftarrow & 1^o & \leftarrow & \dots & \leftarrow & n^o \\ \downarrow & & \downarrow & & & & \downarrow \\ 0 & \rightarrow & 1 & \rightarrow & \dots & \rightarrow & n \end{array}$$

Example: $\mathbf{1}^o * \mathbf{1} \cong \mathbf{3}$.

Generally, $\mathbf{n}^o * \mathbf{n} \cong \mathbf{2n} + \mathbf{1}$.

Every ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ induces a functor (aka. ordinal number map)

$$\theta^o * \theta : \mathbf{m}^o * \mathbf{m} \to \mathbf{n}^o * \mathbf{n},$$

and the assignment $\mathbf{n} \mapsto \mathbf{n}^o * \mathbf{n}$ defines a functor

$$e : \mathbf{\Delta} \to \mathbf{\Delta}.$$

Let X be a simplicial set, and write

$$X^e = X \cdot e^{op},$$

so that

$$X^e_n = X(\mathbf{n}^o * \mathbf{n}).$$

The simplicial set X^e is the *edgewise subdivision* of X.

The canonical ordinal number inclusions

$$\omega_n : \mathbf{n} \to \mathbf{n}^o * \mathbf{n}$$

are natural in $\mathbf{n}$, and hence define a natural map

$$\omega : X^e \to X$$

of simplicial sets.

Lemma 5.2. *The natural map $\omega : X^e \to X$ is a weak equivalence.*

Proof. The simplicial set X is a homotopy colimit of its simplices in the sense that the collection of all simplices $\sigma : \Delta^n \to X$ defines a weak equivalence

$$\underrightarrow{\text{holim}}_{\,\Delta^n \to X}\, \Delta^n \xrightarrow{\simeq} X.$$

This is proved by showing that each map

$$\underrightarrow{\text{holim}}_{\,\Delta^n \to X}\, \Delta^n_m \xrightarrow{\simeq} X_m$$

is a weak equivalence. It follows that all maps

$$\underrightarrow{\text{holim}}_{\,\Delta^n \to X}\, (\Delta^n)^e_n \xrightarrow{\simeq} X^e_n$$

are weak equivalences, and so the map

$$\underrightarrow{\text{holim}}_{\,\Delta^n \to X}\, (\Delta^n)^e \xrightarrow{\simeq} X^e$$

is a weak equivalence.

There is a commutative diagram

$$\begin{array}{ccc} \underrightarrow{\text{holim}}_{\,\Delta^n \to X}\, (\Delta^n)^e & \xrightarrow{\simeq} & X^e \\ {\scriptstyle \omega_*}\downarrow & & \downarrow{\scriptstyle \omega} \\ \underrightarrow{\text{holim}}_{\,\Delta^n \to X}\, \Delta^n & \xrightarrow[\simeq]{} & X \end{array}$$

so it suffices to show that all simplicial sets $(\Delta^n)^e$ are contractible.

The simplicial set $(\Delta^n)^e$ is the nerve of a poset whose objects are the relations $(i,j) : i \leq j$ of $\mathbf{n}$. There is a morphism $(i,j) \leq (k,l)$ in this poset if and only if $k \leq i$ and $j \leq l$. The object $(0,n)$ is terminal in this poset, so that $(\Delta^n)^e$ is contractible. $\square$

Suppose once again that $\mathbf{M}$ is an exact category, and let the exact functor $P : \mathrm{Ar}(\mathbf{n}^o * \mathbf{n}) \to \mathbf{M}$ define an n-simplex of $s_\bullet(\mathbf{M})^e$. Then for $i \leq j$ the pictures

$$\begin{array}{ccccc} & & P(j^o,i) & & \\ & \swarrow & & \searrow & \\ P(i^o,i) & & & & P(j^o,j) \end{array}$$

define morphisms

$$\alpha^P_{i,j} : P(i^o,i) \to P(j^o,j)$$

of $Q\mathbf{M}$. Further, if $i \leq j \leq k$ then the square in the diagram

$$\begin{array}{ccccccccc} & & & & P(k^0,i) & & & & \\ & & & \swarrow & & \searrow & & & \\ & & P(j^o,i) & & & & P(k^o,j) & & \\ & \swarrow & & \searrow & & \swarrow & & \searrow & \\ P(i^o,i) & & & & P(j^o,j) & & & & P(k^o,k) \end{array}$$

is bicartesian, so that assigning the morphism $\alpha^P_{i,j}$ to the relation $i \leq j$ defines a functor $\pi_n(P) : \mathbf{n} \to Q\mathbf{M}$.

We have therefore defined a function

$$\pi_n : (s_\bullet(\mathbf{M})^e)_n \to BQ\mathbf{M}_n$$

The functions π_n are natural in $\mathbf{n}$ and therefore define a simplicial set map

$$\pi : s_\bullet(\mathbf{M})^e \to BQ\mathbf{M}.$$

The exact functors $P : \mathrm{Ar}(\mathbf{n}^o * \mathbf{n}) \to \mathbf{M}$ and the natural isomorphisms between them define a groupoid $\mathrm{Iso}(S_\bullet(\mathbf{M}))^e_n$, which is the n-simplex groupoid of a simplicial groupoid $\mathrm{Iso}(S_\bullet(\mathbf{M}))^e$. There is a groupoid $\mathrm{Iso}(BQ\mathbf{M})_n$ with objects given by the functors $\mathbf{n} \to Q\mathbf{M}$ and whose morphisms are the natural isomorphisms of such functors. The groupoid $\mathrm{Iso}(BQ\mathbf{M})_n$ is the n-simplex groupoid of a simplicial groupoid $\mathrm{Iso}(BQ\mathbf{M})$, and it is easy to see that the simplicial set map π above is the object level part of a map

$$\pi : \mathrm{Iso}(S_\bullet(\mathbf{M}))^e \to \mathrm{Iso}(BQ\mathbf{M})$$

of simplicial groupoids.

Lemma 5.3. *The morphism of groupoids*

$$\pi_n : \mathrm{Iso}(S_\bullet(\mathbf{M}))^e_n \to \mathrm{Iso}(BQ\mathbf{M})_n.$$

induces a weak equivalence

$$B\,\mathrm{Iso}(S_\bullet(\mathbf{M}))^e_n \to B\,\mathrm{Iso}(BQ\mathbf{M})_n.$$

Proof. Suppose that $P, Q : \mathrm{Ar}(\mathbf{n}^0 * \mathbf{n}) \to \mathbf{M}$ are exact functors. A natural isomorphism $\theta : \pi_n(P) \to \pi_n(Q)$ consists of isomorphisms

$$\theta_i = \theta(i^o, i) : P(i^o, i) \xrightarrow{\cong} Q(i^o, i)$$

such that the diagrams

$$\begin{array}{ccc}
P(i^o,i) & \xrightarrow{\alpha^P_{i,j}} & P(j^o,j) \\
{\scriptstyle \theta_i}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \theta_j} \\
Q(i^o,i) & \xrightarrow[\alpha^Q_{i,j}]{} & Q(j^o,j)
\end{array}$$

commute in $Q\mathbf{M}$. It follows that there is a uniquely determined natural isomorphism

$$\theta(j^o, i) : P(j^o, i) \xrightarrow{\cong} Q(j^o, i)$$

such that the diagrams

$$\begin{array}{ccccc}
P(i^o,i) & \twoheadleftarrow & P(j^o,i) & \rightarrowtail & P(j^o,j) \\
{\scriptstyle \theta_i}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \theta(j^o,i)} & & {\scriptstyle \cong}\downarrow{\scriptstyle \theta_j} \\
Q(i^o,i) & \twoheadleftarrow & Q(j^o,i) & \rightarrowtail & Q(j^o,j)
\end{array}$$

commute. The comparison of exact sequences

$$\begin{array}{ccccccccc}
0 & \rightarrow & P(n^o,i^o) & \rightarrowtail & P(n^o,0) & \twoheadrightarrow & P(i^o,0) & \rightarrow & 0 \\
 & & \vdots & & {\scriptstyle \cong}\downarrow{\scriptstyle \theta(n^o,0)} & & {\scriptstyle \cong}\downarrow{\scriptstyle \theta(i^o,0)} & & \\
0 & \rightarrow & Q(n^o,i^o) & \rightarrowtail & Q(n^o,0) & \twoheadrightarrow & Q(i^o,0) & \rightarrow & 0
\end{array}$$

uniquely determines isomorphisms $\theta(n^o, i^o)$ for $i < n$. It follows that the natural isomorphism θ uniquely determines a natural isomorphism

$$\begin{array}{ccccccc} P(n^o,(n-1)^o) & \rightarrowtail & \dots & \rightarrowtail & P(n^o,n-1) & \rightarrowtail & P(n^o,n) \\ {\scriptstyle \theta(n^o,(n-1)^o)}\downarrow\cong & & & & \cong\downarrow{\scriptstyle \theta(n^o,n-1)} & & \cong\downarrow{\scriptstyle \theta(n^o,n)} \\ Q(n^o,(n-1)^o) & \rightarrowtail & \dots & \rightarrowtail & Q(n^o,n-1) & \rightarrowtail & Q(n^o,n) \end{array}$$

of strings of admissible monics. But this means that there is a unique natural isomorphism $\theta : P \xrightarrow{\cong} Q$ which specializes to the θ_i, since the comparison of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & P(n^o,r) & \rightarrowtail & P(n^o,s) & \twoheadrightarrow & P(r,s) & \to & 0 \\ & & {\scriptstyle \theta(n^o,r)}\downarrow\cong & & \cong\downarrow{\scriptstyle \theta(n^o,s)} & & \vdots & & \\ 0 & \to & Q(n^o,r) & \rightarrowtail & Q(n^o,s) & \twoheadrightarrow & Q(r,s) & \to & 0 \end{array}$$

determines an isomorphism $\theta(r,s) : P(r,s) \xrightarrow{\cong} Q(r,s)$ uniquely, for any $n^o \leq r \leq s$ in $\mathbf{n}^o * \mathbf{n}$. The functor π_n is therefore fully faithful.

Suppose that $\alpha : \mathbf{n} \to Q\mathbf{M}$ is a functor, and choose representatives

$$\alpha(i) = P(i^o, i) \twoheadleftarrow P(j^o, i) \rightarrowtail P(j^o, j) = \alpha(j)$$

for all morphisms $\alpha(i) \to \alpha(j)$ in $Q\mathbf{M}$. Then for all relations $i \leq j$ there is a uniquely determined diagram

$$
\begin{array}{ccccc}
 & & P(n^o, i) & & \\
 & \swarrow\!\!\!\swarrow & & \searrow & \\
 & P(j^o, i) & & P(n^o, j) & \\
 \swarrow\!\!\!\swarrow & & \searrow \quad \swarrow\!\!\!\swarrow & & \searrow \\
P(i^o, i) & & P(j^o, j) & & P(n^o, n)
\end{array}
$$

for which the square is bicartesian and the composites

$$P(n^o, i) \rightarrowtail P(n^o, j) \rightarrowtail P(n^o, n)$$

and

$$P(n^o, i) \twoheadrightarrow P(j^o, i) \twoheadrightarrow P(i^o, i)$$

coincide with the chosen representatives for the map $\alpha(i) \to \alpha(n)$ in $Q\mathbf{M}$. It follows that there is a uniquely determined string of admissible monics

$$P(n^o, 0) \rightarrowtail P(n^o, 1) \rightarrowtail \cdots \rightarrowtail P(n^o, n)$$

such that all composites $P(n^o, i) \rightarrowtail P(n^o, n)$ are the original choices of representatives. Similarly, there is a uniquely determined string of admissible epis

$$P(n^o, 0) \twoheadrightarrow \cdots \twoheadrightarrow P(1^o, 0) \twoheadrightarrow P(0^o, 0)$$

such that all composites $P(i^o, 0) \twoheadrightarrow P(0^o, 0)$ are original choices.

For $0 \leq i < n$, define $P(n^o, i^o)$ by the exact sequence

$$0 \to P(n^o, i^o) \rightarrowtail P(n^o, 0) \twoheadrightarrow P(i^o, 0) \to 0.$$

Then if $i \leq j < n$ there is a unique admissible monic $P(n^o, j^o) \rightarrowtail P(n^o, i^o)$ such that the diagram

$$\begin{array}{ccc} P(n^o, j^o) & \rightarrowtail & P(n^o, 0) \\ \downarrow & \nearrow & \\ P(n^o, i^o) & & \end{array}$$

commutes. Then the string of admissible monics

$$P(n^o, (n-1)^o) \rightarrowtail \ldots P(n^o, 0^o) \rightarrowtail P(n^o, 0) \rightarrowtail \cdots \rightarrowtail P(n^o, n)$$

determines an exact functor $P : \mathrm{Ar}(\mathbf{n}^o * \mathbf{n}) \to \mathbf{M}$ with $P(r, s)$ defined by the exact sequence

$$0 \to P(n^o, r) \rightarrowtail P(n^o, s) \twoheadrightarrow P(r, s) \to 0$$

for $n^o \leq r \leq s \leq n$ such that $\pi_n(P) = \alpha$ in $\mathrm{Iso}(BQ\mathbf{M})_n$.

The functor π_n is a functor between groupoids which is fully faithful and is surjective on objects. It is therefore an equivalence. □

Theorem 5.4. *There are weak equivalences*

$$s_\bullet(\mathbf{M}) \underset{\simeq}{\overset{\omega}{\leftarrow}} s_\bullet(\mathbf{M})^e \underset{\simeq}{\overset{\pi}{\to}} BQ\mathbf{M}$$

for each exact category $\mathbf{M}$*. These maps are natural in exact functors.*

Proof. The fact that ω is a weak equivalence is consequence of a general phenomenon for simplicial sets, which is given by Lemma 5.2.

It follows from Lemma 5.3 that the map

$$\pi : \mathrm{Iso}(S_\bullet(\mathbf{M}))^e \to \mathrm{Iso}(BQ\mathbf{M})$$

is a weak equivalence of simplicial groupoids.

The natural weak equivalence $\omega : s_\bullet(\mathbf{M})^e \simeq s_\bullet(\mathbf{M})$ and Lemma 5.1 together imply that the functor $\mathbf{M} \mapsto s_\bullet(\mathbf{M})^e$ takes exact equivalences to weak equivalences. It follows that the exact equivalences $\mathbf{M} \to \mathrm{Iso}_n(\mathbf{M})$ of Exercise 3.7 induce a weak equivalence of bisimplicial sets

$$\eta : s_\bullet(\mathbf{M})^e \to B(\mathrm{Iso}(S_\bullet(\mathbf{M}))^e$$

which induced by the inclusions of objects into the corresponding groupoid of isomorphisms in each simplicial degree. There is a corresponding map

$$\eta' : BQ\mathbf{M} \to B\,\mathrm{Iso}(BQ\mathbf{M})$$

and a commutative diagram

$$\begin{array}{ccc} s_\bullet(\mathbf{M})^e & \xrightarrow[\simeq]{\eta} & B(\mathrm{Iso}(S_\bullet(\mathbf{M}))^e \\ {\scriptstyle\pi}\downarrow & & \simeq\downarrow{\scriptstyle\pi_*} \\ BQ\mathbf{M} & \xrightarrow[\eta']{} & B\,\mathrm{Iso}(BQ\mathbf{M}) \end{array}$$

and it remains to show that the map η' is a weak equivalence.

The bisimplicial set $B\,\mathrm{Iso}(BQ\mathbf{M})$ is the bisimplicial nerve of a bicategory whose 2-cells are commutative diagrams

$$\begin{array}{ccc} P_1 & \longrightarrow & Q_1 \\ \cong\downarrow & & \downarrow\cong \\ P_2 & \longrightarrow & Q_2 \end{array}$$

in the category $Q\mathbf{M}$. Write $\mathrm{Iso}_n(BQ\mathbf{M})$ for the category whose objects are all strings of isomorphisms

$$P:\ P_0 \xrightarrow{\cong} P_1 \xrightarrow{\cong} \dots \xrightarrow{\cong} P_n$$

and whose morphisms are all natural transformations of such. Then (you've seen this before) there are functors

$$Q\mathbf{M} \xrightarrow{f} \mathrm{Iso}_n\, Q\mathbf{M} \xrightarrow{g} Q\mathbf{M},$$

where $f(Q)$ is the string of identities

$$Q \xrightarrow{1} \dots \xrightarrow{1} Q$$

and $g(P) = P_0$. Then the functors f and g determine an equivalence of categories $Q\mathbf{M} \simeq \operatorname{Iso}_n Q\mathbf{M}$ for each $n \geq 0$, and it follows that η' is a weak equivalence. □

Exercise 5.5. Show that there is an isomorphism of simplicial categories

$$\operatorname{Iso}_n Q\mathbf{M} \cong Q(\operatorname{Iso}_n \mathbf{M})$$

where $\operatorname{Iso}_n \mathbf{M}$ is the exact category described at the end of Section 2.

Remark 5.6. The argument in the last paragraph of the proof of Theorem 5.4 is part of a very general phenomenon. Suppose that C is a small category, and let $\operatorname{Iso}(\operatorname{Ar}(C))$ be the bicategory whose 2-cells are all commutative diagrams

$$\begin{array}{ccc} a & \longrightarrow & b \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ a' & \longrightarrow & b' \end{array}$$

in C. Then there is a weak equivalence

$$BC \simeq B\operatorname{Iso}(\operatorname{Ar}(C))$$

which is induced by the inclusion of objects in all vertical groupoids.

Remark 5.7. Write $\mathrm{Mon}_n(\mathbf{M})$ for the category of strings of admissible monics

$$P_1 \rightarrowtail P_2 \rightarrowtail \cdots \rightarrowtail P_n$$

and their natural transformations. Let

$$\mathrm{Iso}(\mathrm{Mon}_n(\mathbf{M}))$$

be the groupoid of isomorphisms in this category. Suppose that $P : \mathrm{Ar}(\mathbf{n}) \to \mathbf{M}$ is an exact functor. Then we have already seen that P is completely determined up to isomorphism by the string of admissible monics

$$P(0,1) \rightarrowtail P(0,2) \rightarrowtail \cdots \rightarrowtail P(0,n). \quad (1)$$

It follows that the functor

$$m : S_n(\mathbf{M}) \to \mathrm{Mon}_n(\mathbf{M})$$

which sends P to the string (1) induces an equivalence of groupoids

$$m : \mathrm{Iso}(S_n(\mathbf{M})) \xrightarrow{\simeq} \mathrm{Iso}(\mathrm{Mon}_n(\mathbf{M})).$$

For a sequence of admissible monics

$$P :\ 0 \rightarrowtail P_1 \rightarrowtail \cdots \rightarrowtail P_n,$$

write $P_0 = 0$. Then, if $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map there is an induced string of admissible monics

$$\theta^*(P) :\ 0 \rightarrowtail P_{\theta(1)}/P_{\theta(0)} \rightarrowtail \cdots \rightarrowtail P_{\theta(m)}/P_{\theta(0)},$$

subject to making some choices of quotients. The assignment $P \mapsto \theta^*(P)$ defines a functor

$$\theta^* : \mathrm{Iso}(\mathrm{Mon}_n(\mathbf{M})) \to \mathrm{Iso}(\mathrm{Mon}_m(\mathbf{M}))$$

for each ordinal number map θ. The quotients in question are only defined up to canonical isomorphism, and so there are canonical isomorphisms

$$1_{\mathbf{n}}^* \cong 1, \quad (\gamma\theta)^* \cong \theta^*\gamma^*$$

which are coherent in the sense that they satisfy certain cocycle conditions. This means that the assigment $\mathbf{n} \mapsto \mathrm{Iso}(\mathrm{Mon}_n(\mathbf{M}))$ defines a *pseudo-functor*

$$\mathrm{Iso}(\mathrm{Mon}_\bullet(\mathbf{M})) : \mathbf{\Delta}^{op} \rightsquigarrow \mathbf{Gpd}$$

taking values in small groupoids. That's okay, because any such pseudo-functor can be *rectified* (by using the "Grothendieck construction") to produce a simplicial object in groupoids up to pointwise equivalence.

Further, the equivalences

$$m : \mathrm{Iso}(S_n(\mathbf{M})) \xrightarrow{\simeq} \mathrm{Iso}(\mathrm{Mon}_n(\mathbf{M})).$$

are not natural in simplicial structure maps, but they are natural up to canonical isomorphism, and therefore form a pseudo-natural transformation

$$m : \mathrm{Iso}(S_\bullet(\mathbf{M})) \to \mathrm{Iso}(\mathrm{Mon}_\bullet(\mathbf{M})),$$

which induces a weak equivalence of the respective Grothendieck constructions.

It follows that the Grothendieck construction for the pseudo-functor $\mathrm{Iso}(\mathrm{Mon}_\bullet(\mathbf{M}))$ is another model for the K-theory space $BQ\mathbf{M}$ of an exact category $\mathbf{M}$.

A detailed account of these constructions can be found in [2].

6 Additivity

Suppose that $\mathbf{M}$ is an exact category, and recall (from Section 2) that $\mathrm{Ex}(\mathbf{M})$ is the exact category whose objects are the exact sequences

$$0 \to P' \rightarrowtail P \twoheadrightarrow P'' \to 0, \tag{2}$$

whose morphisms are the comparisons of exact sequences, and whose exact sequences are the termwise exact sequences.

Observe that there is an exact equivalence

$$S_2(\mathbf{M}) \xrightarrow{\simeq} \mathrm{Ex}(\mathbf{M})$$

which sends an exact functor $P : \mathrm{Ar}(\mathbf{2}) \to \mathbf{M}$ to the exact sequence

$$0 \to P(0,1) \rightarrowtail P(0,2) \twoheadrightarrow P(1,2) \to 0.$$

There are exact functors $s, t : \mathrm{Ex}(\mathbf{M}) \to \mathbf{M}$ which are defined, respectively, sending the sequence (2) to the objects P' and P''.

The "Additivity Theorem" is the following:

Theorem 6.1. *The simplicial set map*

$$(t, s) : s_\bullet \mathrm{Ex}(\mathbf{M}) \to s_\bullet\mathbf{M} \times s_\bullet\mathbf{M}$$

is a weak equivalence.

The proof of this result will occupy the remainder of this section.

The idea is to show that the square

$$\begin{array}{ccc} s_\bullet\mathbf{M} & \xrightarrow{c} & s_\bullet \mathrm{Ex}(\mathbf{M}) \\ \downarrow & & \downarrow{\scriptstyle s} \\ * & \xrightarrow[0]{} & s_\bullet\mathbf{M} \end{array} \tag{3}$$

is homotopy cartesian, where c is induced by the exact functor $c : \mathbf{M} \to \mathrm{Ex}(\mathbf{M})$ which sends an object P to the sequence

$$0 \to 0 \rightarrowtail P \overset{1}{\twoheadrightarrow} P \to 0.$$

If this can be done, then the comparison of homotopy fibre sequences (of connected spaces)

$$\begin{array}{ccccc} & & s_\bullet \operatorname{Ex}(\mathbf{M}) & & \\ & \overset{c}{\nearrow} & \Big\downarrow{\scriptstyle (t,s)} & \overset{s}{\searrow} & \\ s_\bullet\mathbf{M} & & & & s_\bullet\mathbf{M} \\ & \underset{in_L}{\searrow} & & \underset{pr_R}{\nearrow} & \\ & & s_\bullet\mathbf{M} \times s_\bullet\mathbf{M} & & \end{array}$$

implies that (t, s) is a weak equivalence. Here, $in_L(P) = (P, 0)$ and pr_R is projection onto the right factor.

Recall that every simplex $P : \Delta^n \to s_\bullet\mathbf{M}$ defines a pullback diagram

$$\begin{array}{ccc} s^{-1}(P) & \longrightarrow & s_\bullet \operatorname{Ex}(\mathbf{M}) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle s} \\ \Delta^n & \xrightarrow[P]{} & s_\bullet\mathbf{M} \end{array}$$

The assertion that the square (3) is homotopy cartesian is a consequence of the following:

Lemma 6.2. *The functor* $\mathbf{\Delta}/s_\bullet\mathbf{M} \to s\mathbf{Set}$ *defined by* $P \mapsto s^{-1}(P)$ *is a diagram of equivalences in simplicial sets.*

In effect, if Lemma 6.2 holds, then the diagram

$$\begin{array}{ccc} s^{-1}(0) & \longrightarrow & \underrightarrow{\mathrm{holim}}_P\ s^{-1}(P) \\ \Big\downarrow & & \Big\downarrow \\ * & \xrightarrow[0]{} & s_\bullet\mathbf{M} \end{array}$$

is homotopy cartesian by Theorem 1.11 (aka. Quillen's Theorem B), and there are weak equivalences

$$s^{-1}(0) \xrightarrow{\simeq} s_\bullet \mathbf{M}$$

(see the proof of Lemma 6.2) and

$$\underrightarrow{\text{holim}}_{\,P:\Delta^n \to s_\bullet \mathbf{M}}\ s^{-1}(P) \xrightarrow{\simeq} s_\bullet \operatorname{Ex}(\mathbf{M}).$$

Proof of Lemma 6.2. It is enough to show that the composite

$$s^{-1}(P) \to s_\bullet \operatorname{Ex}(\mathbf{M}) \xrightarrow{t} s_\bullet \mathbf{M}$$

is a weak equivalence for each simplex P.

If the simplex $0 : \Delta^0 \to s_\bullet \mathbf{M}$ defined by a 0-object of $\mathbf{M}$ then $s^{-1}(0) = s_\bullet \operatorname{Ex}_0(\mathbf{M})$, where $\operatorname{Ex}_0(\mathbf{M})$ is the exact category of all exact sequences

$$0 \to 0 \rightarrowtail A \twoheadrightarrow B \to 0.$$

The exact functor $t : \operatorname{Ex}(\mathbf{M}) \to \mathbf{M}$ restricts to an exact equivalence $t : \operatorname{Ex}_0(\mathbf{M}) \to \mathbf{M}$, and it follows that the composite

$$f : s^{-1}(0) \to s_\bullet \operatorname{Ex}(\mathbf{M}) \xrightarrow{t} s_\bullet \mathbf{M}$$

is a weak equivalence. This map f has a section $g : s_\bullet \mathbf{M} \to s^{-1}(0)$ which is defined by taking a simplex Q to the exact sequence

$$0 \to 0 \rightarrowtail Q \xrightarrow{1} Q \to 0.$$

Consider the diagram

$$\begin{array}{ccccccccc}
s_\bullet \mathbf{M} & \xrightarrow[\simeq]{g} & s^{-1}(0) & \xrightarrow{n_*} & s^{-1}(P) & \longrightarrow & s_\bullet \operatorname{Ex}(\mathbf{M}) & \xrightarrow{t} & s_\bullet \mathbf{M} \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle s} & & \\
 & & \Delta^0 & \xrightarrow[n]{} & \Delta^n & \xrightarrow[P]{} & s_\bullet \mathbf{M} & &
\end{array}$$

where n denotes the corresponding object of $\mathbf{n}$ and $P \cdot n$ is a zero object 0 of $\mathbf{M}$, so that the composite

$$s^{-1}(0) \xrightarrow{n_*} s^{-1}(P) \to s_\bullet \operatorname{Ex}(\mathbf{M}) \xrightarrow{t} s_\bullet \mathbf{M}$$

is the weak equivalence f. The composite of all horizontal arrows along the top of the diagram is the identity, so it suffices to show that the composite

$$\psi : s^{-1}(P) \to s_\bullet \operatorname{Ex}(\mathbf{M}) \xrightarrow{t} s_\bullet \mathbf{M} \xrightarrow{g} s^{-1}(0) \xrightarrow{n_*} s^{-1}(P)$$

is homotopic to the identity. This would mean that the map

$$s^{-1}(P) \to s_\bullet \operatorname{Ex}(\mathbf{M}) \xrightarrow{t} s_\bullet \mathbf{M}$$

is a homotopy equivalence.

An m-simplex of $s^{-1}(P)$ is a pair

$$(\mathbf{m} \xrightarrow{\theta} \mathbf{n}, \theta^*(P) \rightarrowtail A \twoheadrightarrow B),$$

where

$$0 \to \theta^* P \rightarrowtail A \twoheadrightarrow B \to 0$$

an exact sequence of $S_m\mathbf{M}$. The composite map ψ is then defined by

$$\psi(\mathbf{m} \xrightarrow{\theta} \mathbf{n}, \theta^*(P) \rightarrowtail A \twoheadrightarrow B)$$
$$= (\mathbf{m} \xrightarrow{n} \mathbf{n}, 0 \rightarrowtail B \overset{1}{\twoheadrightarrow} B),$$

where n is the constant ordinal number map which takes all i to n.

We define a homotopy $H : s^{-1}(P) \times \Delta^1 \to s^{-1}(P)$ such that the diagram

$$\begin{array}{ccc} s^{-1}(P) \times \Delta^1 & \xrightarrow{H} & s^{-1}(P) \\ {\scriptstyle pr\times 1}\downarrow & & \downarrow{\scriptstyle pr} \\ \Delta^n \times \Delta^1 & \xrightarrow[h]{} & \Delta^n \end{array}$$

commutes, where pr is the defining projection and h is the homotopy defined by the functor

$$\begin{array}{ccccccc} 0 & \to & 1 & \to & \dots & \to & n \\ \downarrow & & \downarrow & & & & \downarrow \\ n & \to & n & \to & \dots & \to & n \end{array}$$

which flows into the terminal object n of $\mathbf{n}$.

The homotopy h is given in simplicial degree m by the functions

$$h_\tau : \hom(\mathbf{m}, \mathbf{n}) \to \hom(\mathbf{m}, \mathbf{n}),$$

one for each morphism $\tau : \mathbf{m} \to \mathbf{1}$, where $h_\tau(\theta)$ is the composite

$$\mathbf{m} \xrightarrow{(\theta,\tau)} \mathbf{n} \times \mathbf{1} \xrightarrow{h} \mathbf{n}.$$

There is a (unique) natural transformation

$$\theta \to h_\tau(\theta),$$

which induces a natural transformation

$$\mathrm{Ar}(\theta) \to \mathrm{Ar}(h_\tau(\theta))$$

of induced functors $\mathrm{Ar}(\mathbf{m}) \to \mathrm{Ar}(\mathbf{n})$. This transformation, in turn, induces a map

$$\theta^*(P) \to h_\tau(\theta)^*(P)$$

in $S_m\mathbf{M}$. Form the pushout

$$\begin{array}{ccc} \theta^*(P) & \rightarrowtail & A \\ \downarrow & & \downarrow \\ h_\tau(\theta)^*(P) & \rightarrowtail & A_\tau \end{array} \tag{4}$$

Then the sequence

$$0 \to h_\tau(\theta)^*(P) \rightarrowtail A_\tau \twoheadrightarrow B \to 0$$

is exact, and we are entitled to set

$$\begin{aligned} &h_\tau(\mathbf{m} \xrightarrow{\theta} \mathbf{n}, \theta^*(P) \rightarrowtail A \twoheadrightarrow B) \\ &\quad = (h_\tau(\theta), h_\tau(\theta)^*(P) \rightarrowtail A_\tau \twoheadrightarrow B). \end{aligned}$$

To make the pushout construction actually work the diagrams (4) have to be constructed correctly. These diagrams are induced by pushouts

$$\begin{array}{ccc} P(i,j) & \stackrel{i}{\rightarrowtail} & A \\ {\scriptstyle \alpha_*}\downarrow & & \downarrow{\scriptstyle \gamma} \\ P(k,l) & \rightarrowtail & \overline{A} \end{array} \tag{5}$$

where

$$0 \to P(i,j) \stackrel{i}{\rightarrowtail} A \stackrel{p}{\twoheadrightarrow} B \to 0$$

is an exact sequence of $\mathbf{M}$ and $\alpha : (i,j) \to (k,l)$ is a morphism of $\mathrm{Ar}(\mathbf{n})$. The diagrams (5) can be chosen so that $\gamma = 1_A : A \to A$ if α is an identity morphism, and such that $\gamma = p$ if $k = l$. Form the pushouts (5) for the set of diagrams

$$\begin{array}{ccccccccc} 0 & \to & P(i,j) & \rightarrowtail & A & \twoheadrightarrow & B & \to & 0 \\ & & {\scriptstyle \alpha_*}\downarrow & & & & & & \\ & & P(k,l) & & & & & & \end{array}$$

in which the displayed exact sequence is in $\mathbf{E}$, subject to these constraints.

Then the induced pushout diagrams (4) define functions $h_\tau : s^{-1}(P)_m \to s^{-1}(P)_m$ such that the diagrams

$$\begin{array}{ccc} s^{-1}(P)_m & \xrightarrow{h_\tau} & s^{-1}(P)_m \\ {\scriptstyle\gamma^*}\downarrow & & \downarrow{\scriptstyle\gamma_*} \\ s^{-1}(P)_k & \xrightarrow[h_{\tau\gamma}]{} & s^{-1}(P)_k \end{array}$$

commute for all ordinal number morphisms $\gamma : \mathbf{k} \to \mathbf{m}$, and such that the functions h_τ define a homotopy from the identity to ψ. □

7 H-space structure

Suppose that $\mathbf{M}$ is an exact category. Then the direct sum functor

$$\oplus : \mathbf{M} \times \mathbf{M} \to \mathbf{M}$$

defined by $(P, Q) \mapsto P \oplus Q$ is exact, and induces a map

$$\oplus : s_\bullet(\mathbf{M}) \times s_\bullet(\mathbf{M}) \to s_\bullet(\mathbf{M}). \qquad (6)$$

Any zero object 0 of $\mathbf{M}$ defines a vertex $0 : * \to s_\bullet(\mathbf{M})$ which defines a 2-sided identity up to homotopy for the map $\oplus$: in effect there are canonical natural isomorphisms

$$0 \oplus P \cong P \cong P \oplus 0.$$

The functor $\oplus$ is asssociative up to canonical isomorphism, so that the map (6) is associative up to canonical homotopy. In particular the direct sum functor $\oplus$ and choice of zero object 0 give the space $s_\bullet(\mathbf{M})$ the structure of a homotopy associative (even homotopy commutative) H-space.

As in the previous section, let $\mathrm{Ex}(\mathbf{M})$ be the exact category of exact sequences

$$0 \to P' \rightarrowtail P \twoheadrightarrow P'' \to 0 \tag{7}$$

in $\mathbf{M}$. Recall that there are exact functors

$$s, t : \mathrm{Ex}(\mathbf{M}) \to \mathbf{M}$$

which take the exact sequence (7) to P' and P'', respectively. There is a further exact functor

$$tot : \mathrm{Ex}(\mathbf{M}) \to \mathbf{M}$$

which takes the exact sequence (7) to the object P.

The following result is a consequence of the additivity theorem Theorem 6.1:

Corollary 7.1. *The H-space structure on $s_\bullet(\mathbf{M})$ determines a relation of simplicial set maps*

$$tot_* = s_* + t_* : s_\bullet \mathrm{Ex}(\mathbf{M}) \to s_\bullet(\mathbf{M}).$$

Proof. There is a functor

$$\oplus : \mathbf{M} \times \mathbf{M} \to \mathrm{Ex}(\mathbf{M})$$

which takes a pair (P, Q) to the exact sequence

$$0 \to P \rightarrowtail P \oplus Q \twoheadrightarrow Q \to 0.$$

The composite $(s, t)\oplus$ is the identity on $\mathbf{M} \times \mathbf{M}$, so that the induced map

$$\oplus_* : s_\bullet(\mathbf{M}) \times s_\bullet(\mathbf{M}) \to s_\bullet(\mathrm{Ex}(\mathbf{M}))$$

is a homotopy inverse for (s_*, t_*), by the additivity theorem. At the same time the composite exact functor

$$\mathbf{M} \times \mathbf{M} \xrightarrow{\oplus} \mathrm{Ex}(\mathbf{M}) \xrightarrow{tot} \mathbf{M}$$

is the direct sum functor for $\mathbf{M}$, and therefore induces the H-space structure. It follows that

$$tot_* = tot_* \oplus_* (s_*, t_*) = \oplus_*(s_*, t_*) = s_* + t_*,$$

as required. □

Corollary 7.2. *Suppose that*

$$0 \to f \rightarrowtail g \twoheadrightarrow h \to 0$$

is an exact sequence of exact functors $\mathbf{M} \to \mathbf{N}$. *Then there is a relation*

$$g_* = f_* + h_* : s_\bullet(\mathbf{M}) \to s_\bullet(\mathbf{N}).$$

Proof. The exact sequence of exact functors determines an exact functor

$$\mathbf{M} \to \mathrm{Ex}(\mathbf{N}).$$

Use Corollary 7.1 to finish the proof. □

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