Lecture 004 (October 8, 2014)

8 The K-theory spectrum

Recall that there is a poset isomorphism

$$\mathbf{0} * \mathbf{n} \cong \mathbf{n} + \mathbf{1},$$

and write

$$\tilde{\theta} = \mathbf{0} * \theta : \mathbf{m} + \mathbf{1} \cong \mathbf{0} * \mathbf{m} \to \mathbf{0} * \mathbf{n} \cong \mathbf{n} + \mathbf{1}$$

for each ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$.

Let X be a simplicial set, and write EX for the simplicial set with

$$EX_n = X_{n+1} = X(\mathbf{0} * \mathbf{n}),$$

and with structure maps $\tilde{\theta}^* : EX_n \to EX_m$ for ordinal number maps $\theta : \mathbf{m} \to \mathbf{n}$.

Example: Suppose that C is a small category. Then EBC has m-simplices given by strings of arrows

$$\sigma : \ a \to b_0 \to \cdots \to b_n,$$

and $\theta^*(\sigma)$ is the string

$$a \to b_{\theta(0)} \to \cdots \to b_{\theta(m)}.$$

It follows that there is an isomorphism

$$\bigsqcup_{a \in \mathrm{Ob}(C)} B(a/C) \xrightarrow{\cong} EBC.$$

There is a natural simplicial set map

$$p : EX \to X_0$$

which is defined by sending an n-simplex $\sigma : \Delta^{n+1} \to X$ to $\sigma(0)$. Every vertex x of X defines a vertex $s_0(x) : \Delta^1 \to X$ of EX: in this way, there is a natural simplicial set map $q : X_0 \to EX$.

Lemma 8.1. *The map $p : EX \to X_0$ is a natural strong deformation retraction, with section and homotopy inverse $q : X_0 \to EX$.*

Proof. The composite $p \cdot q$ is the identity on X_0, so it suffices to find a natural homotopy

$$h : EX \times \Delta^1 \to EX$$

from the identity on EX to $q \cdot p$ which is constant on X_0.

There is an isomorphism

$$\varinjlim_{\Delta^n \to X} E\Delta^n \xrightarrow{\cong} EX,$$

so it suffices to find the homotopy

$$h : E\Delta^n \times \Delta^1 \to E\Delta^n$$

and show that it is natural in simplices. The contracting homotopies

$$B(v/\mathbf{n}) \times \Delta^1 \to B(v/\mathbf{n})$$

onto initial objects induce homotopies

$$E\Delta^n \times \Delta^1 \cong \bigsqcup_{v \in \mathbf{n}} B(v/\mathbf{n}) \times \Delta^1 \to \bigsqcup_{v \in \mathbf{n}} B(v/\mathbf{n}) \cong E\Delta^n$$

which do the job. □

Example: Suppose that D is a simplicial (small) category. Then there is a simplicial category ED with $ED_n = D_{n+1}$, with structure functors $\tilde{\theta}^* : D_{n+1} \to D_{m+1}$. The corresponding bisimplicial set BED has vertical simplicial sets $(BED)_n$ (corresponding to strings of arrows of length n) with strong deformation retractions

$$(BED)_n \cong E(BD_n) \xrightarrow{\simeq} B(D_0)_n$$

which respect simplicial structure maps It follows that the canonical simplicial category morphism

$$ED \to D_0$$

induces a strong deformation retraction

$$BED \xrightarrow{\simeq} BD_0$$

of bisimplicial sets. This strong deformation retraction is natural in simplicial categories D.

The following result is a consequence of this observation:

Example 8.2. There is a homotopy equivalence

$$BE\,\mathrm{Iso}(S_\bullet(\mathbf{M})) \simeq *$$

for each exact category $\mathbf{M}$. This equivalence is natural in exact functors in $\mathbf{M}$.

There is a natural homotopy equivalence

$$BE\,\mathrm{Iso}(S_\bullet(\mathbf{M})) \simeq B\,\mathrm{Iso}(S_0(\mathbf{M})), \tag{1}$$

and $\mathrm{Iso}(S_0(\mathbf{M}))$ is the groupoid of zero objects of $\mathbf{M}$ and the isomorphisms between them. This is a trivial groupoid because all zero objects are initial in $\mathbf{M}$, so there is a homotopy equivalence

$$B\,\mathrm{Iso}(S_0(\mathbf{M})) \simeq *. \tag{2}$$

The desired homotopy equivalence is the composite of the equivalences (1) and (2).

The ordinal number maps $d^0 : \mathbf{n} \to \mathbf{0} * \mathbf{n}$ induce a natural simplicial set map

$$d_0 : EX \to X.$$

Suppose that $\sigma : \Delta^{n+1} \to X$ is an n-simplex of

EX. Then there is a commutative diagram

$$\begin{array}{ccc} E\Delta^{n+1} & \xrightarrow{\sigma_*} & EX \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ \Delta^{n+1} & \xrightarrow[\sigma]{} & X \end{array}$$

and the map $d_0 : E\Delta^{n+1} \to \Delta^{n+1}$ can be identified with the map

$$\bigsqcup_{v \in \mathbf{n+1}} B(v/(\mathbf{n}+\mathbf{1})) \to B(\mathbf{n}+\mathbf{1})$$

which is induced by the canonical forgetful functors $v/(\mathbf{n}+\mathbf{1}) \to \mathbf{n}+\mathbf{1}$. Note that the k-simplex

$$v \to i_0 \to \cdots \to i_k$$

of $B(v/(\mathbf{n}+\mathbf{1}))_k$ is identified with a $(k+1)$-simplex in $\Delta^{n+1}_{k+1} = E\Delta^{n+1}_k$ in the obvious way under the identification

$$\bigsqcup_{v \in \mathbf{n+1}} B(v/(\mathbf{n}+\mathbf{1})) = E\Delta^{n+1}.$$

In particular, the n-simplex

$$0 \to 1 \to 2 \to \cdots \to n+1$$

of $B(0/(\mathbf{n}+\mathbf{1}))$ maps to $\sigma \in EX_n$ under the composite

$$B(0/(\mathbf{n}+\mathbf{1})) \to E\Delta^{n+1} \xrightarrow{\sigma_*} EX.$$

The contracting homotopy

$$h : B(0/(\mathbf{n}+\mathbf{1})) \times \Delta^1 \to B(0/(\mathbf{n}+\mathbf{1})$$

is defined by functions

$$h_\tau : B(0/(\mathbf{n}+\mathbf{1}))_k \to B(0/(\mathbf{n}+\mathbf{1}))_k,$$

one for each $\tau : \mathbf{k} \to \mathbf{1}$ (satisfying compatibility conditions). Explicitly, if $\tau : \mathbf{n} \to \mathbf{1}$ is the string

$$0 \to \cdots \to 0 \to \overset{i}{1} \to \cdots \to 1 \tag{3}$$

$(0 \leq i \leq n+1)$ and

$$\gamma : 0 \to j_0 \to \cdots \to j_n$$

is an n-simplex of $B(0/(\mathbf{n}+\mathbf{1}))$, then $h_\tau(\gamma)$ is the simplex

$$0 \to 0 \to \cdots \to 0 \to j_i \to \cdots \to j_n.$$

The simplex $h_\tau(\gamma)$, interpreted as an element of Δ^{n+1}_{n+1}, has the form

$$h_\tau(\gamma) = s_0^i d_1^i(\gamma).$$

It follows that, for $\sigma \in X_{n+1} = EX_n$, the homotopy $h : EX \times \Delta^1 \to EX$ is defined by functions

$$h_\tau : X_{n+1} \to X_{n+1},$$

indexed by ordinal number morphisms $\tau : \mathbf{n} \to \mathbf{1}$ written as in (3), where

$$h_\tau(\sigma) = s_0^i d_1^i(\sigma). \tag{4}$$

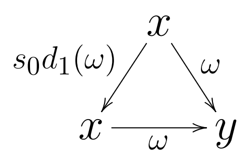
Example 8.3. Suppose that $\omega \in X_1$, so that ω represents a vertex of EX. The homotopy $h : EX \times \Delta^1 \to EX$ defines a 1-simplex $h_1(s_0(\omega)) \in EX$, where 1 denotes the identity morphism $\mathbf{1} \to \mathbf{1}$. The simplex $s_0(\omega)$ of EX_1 is the simplex $s_1(\omega) \in X_2$, and

$$h_1(s_1(\omega)) = s_0 d_1(s_1(\omega)) = s_0(\omega)$$

according to the description we have just seen. In EX, this simplex is a path

$$s_0 d_1(\omega) = d_2 s_0(\omega) \to d_1 s_0(\omega) = \omega$$

which can be represented by the picture



It follows as well that the composite

$$\Delta^1 \xrightarrow{(\omega,1)} EX \times \Delta^1 \xrightarrow{h} EX \xrightarrow{d_0} X$$

is the simplex ω.

Example 8.4. Suppose that $\mathbf{M}$ is an exact category, and that the exact functor $P : \mathrm{Ar}(\mathbf{n}+\mathbf{1}) \to \mathbf{M}$ is an $(n+1)$-simplex of $s_\bullet(\mathbf{M})$, or equivalently an n-simplex of $Es_\bullet(\mathbf{M})$. Then the homotopy

$$h : Es_\bullet(\mathbf{M}) \times \Delta^1 \to Es_\bullet(\mathbf{M})$$

of Lemma 8.1 is given by functions

$$h_\tau : s_{n+1}(\mathbf{M}) \to s_{n+1}(\mathbf{M})$$

indexed by ordinal numbers maps $\tau : \mathbf{n} \to \mathbf{1}$. If τ is the map (3) then

$$h_\tau(P) = s_0^i d_1^i(P)$$

in $s_\bullet(\mathbf{M})_{n+1}$. In terms of strings of admissible monics, h_τ takes the string

$$P(0,1) \rightarrowtail \cdots \rightarrowtail P(0,n+1)$$

to the string

$$P(0,0) \rightarrowtail \cdots \rightarrowtail P(0,0) \rightarrowtail P(0,i{+}1) \rightarrowtail \cdots \rightarrowtail P(0,n{+}1) \tag{5}$$

for $0 \leq i \leq n+1$.

Now consider the pullback

$$\begin{array}{ccc} f(\mathbf{M}) & \xrightarrow{i_*} & Es_\bullet(\mathbf{M}) \\ \downarrow & & \downarrow{\scriptstyle d_0} \\ s_\bullet(0) & \xrightarrow[i]{} & s_\bullet(\mathbf{M}) \end{array}$$

where the map i is the inclusion of the subset of all exact functors $P : \mathrm{Ar}(\mathbf{n}) \to \mathbf{M}$ such that

$$P(i,j) \cong 0$$

for all $0 < i \leq j$. Then $f(\mathbf{M})_n$ is the subset of all exact functors $P : \mathrm{Ar}(\mathbf{n}+\mathbf{1}) \to \mathbf{M}$ such that the string of admissible monics

$$P(0,1) \rightarrowtail P(0,2) \rightarrowtail \cdots \rightarrowtail P(0,n+1)$$

consists of isomorphisms.

If 0 is a distinguished zero object of $\mathbf{M}$ then there is a function

$$i_0 : \mathrm{Ob}(\mathbf{M}) \to f(\mathbf{M})_0 = Es_\bullet(\mathbf{M})_0 = s_\bullet(\mathbf{M})_1$$

which takes P to the simplex

$$\begin{array}{ccc} 0 & \rightarrowtail & P \\ & & \downarrow \\ & & 0 \end{array}$$

This function determines a simplicial set map

$$i_0 : \mathrm{Ob}(\mathbf{M}) \times \Delta^1 \to s_\bullet(\mathbf{M}). \qquad (6)$$

The description of the homotopy h of (5) implies that this map i_0 coincides with the composite

$$\mathrm{Ob}(\mathbf{M}) \times \Delta^1 \xrightarrow{i_0 \times 1} Es_\bullet(\mathbf{M}) \times \Delta^1 \xrightarrow{h} Es_\bullet(\mathbf{M}) \xrightarrow{d_0} s_\bullet(\mathbf{M}).$$

See also Remark 8.5 below.

The map i_0 of (6) induces a pointed simplicial set map

$$\sigma : (\mathrm{Ob}(\mathbf{M}) / \mathrm{Ob}(0)) \wedge S^1 \to s_\bullet(\mathbf{M}) / s_\bullet(0),$$

where $\mathrm{Ob}(0)$ is the set of zero objects of $\mathbf{M}$ and $s_\bullet(0)$ is the subcomplex of $s_\bullet(\mathbf{M})$ which consists of all exact functors $P : \mathrm{Ar}(\mathbf{n}) \to \mathbf{M}$ which take values in zero objects.

The subcomplex $s_\bullet(0)$ is contractible, by Lemma 5.1, since every zero object is uniquely isomorphic to the fixed object 0.

Remark 8.5. One can alternatively describe the composite

$$\mathrm{Ob}(\mathbf{M}) \times \Delta^1 \xrightarrow{i_0 \times 1} Es_\bullet(\mathbf{M}) \times \Delta^1 \xrightarrow{h} Es_\bullet(\mathbf{M})$$

as the map which associates the path represented by the exact functor

$$\begin{array}{ccccc} 0 & \rightarrowtail & 0 & \rightarrowtail & P \\ & & \downarrow & & \downarrow 1 \\ & & 0 & \rightarrowtail & P \\ & & & & \downarrow \\ & & & & 0 \end{array}$$

(aka. $s_1(i_0(P)) \in s_\bullet(\mathbf{M})_2$) to the object P of $\mathbf{M}$.
It follows that the composite

$$\mathrm{Ob}(\mathbf{M}) \times \Delta^1 \xrightarrow{i_0 \times 1} Es_\bullet(\mathbf{M}) \times \Delta^1 \xrightarrow{h} Es_\bullet(\mathbf{M}) \xrightarrow{d_0} s_\bullet(\mathbf{M})$$

of (6) is defined by taking the pair

$$(P, \tau : \mathbf{n} \to \mathbf{1})$$

to the composite exact functor

$$\mathrm{Ar}(\mathbf{n}) \xrightarrow{\tau_*} \mathrm{Ar}(\mathbf{1}) \xrightarrow{i_0(P)} \mathbf{M}.$$

One can also see this directly from Example 8.3.

Say that an exact category $\mathbf{M}$ together with a fixed choice of zero object 0 is a *pointed exact category*. A pointed exact functor $f : \mathbf{M} \to \mathbf{N}$ of pointed exact categories is the obvious thing: it's an exact functor which preserves the choices of zero objects.

The map

$$\sigma : (\mathrm{Ob}(\mathbf{M})/\,\mathrm{Ob}(0)) \wedge S^1 \to s_\bullet(\mathbf{M})/s_\bullet(0) \qquad (7)$$

is natural in pointed exact functors $\mathbf{M} \to \mathbf{N}$. If 0 is a distinguished zero for $\mathbf{M}$, then the exact functors $0 : \mathrm{Ar}(\mathbf{n}) \to \mathbf{M}$ which are constant at 0 are distinguished zeros for the exact categories $S_n(\mathbf{M})$, so that $S_\bullet(\mathbf{M})$ is a pointed simplicial exact category, and there is an induced bisimplicial set map

$$\sigma : (s_\bullet(\mathbf{M})/s_\bullet(0)) \wedge S^1 \to s_\bullet(S_\bullet(\mathbf{M}))/s_\bullet(S_\bullet(0)). \qquad (8)$$

Here, $s_\bullet(S_\bullet(\mathbf{M}))$ is the bisimplicial set of objects of a bisimplicial exact category

$$S^2_\bullet(\mathbf{M}) = S_\bullet(S_\bullet(\mathbf{M})),$$

and one alternatively writes

$$s^2_\bullet(\mathbf{M}) = s_\bullet(S_\bullet(\mathbf{M})).$$

The bisimplices of $s^2_\bullet(\mathbf{M})$ can be identified with functors

$$P : \mathrm{Ar}(\mathbf{m}) \times \mathrm{Ar}(\mathbf{n}) \to \mathbf{M}$$

which are exact in each variable. The bisimplicial set $s_\bullet(S_\bullet(0))$ is the subcomplex $s^2_\bullet(0)$ of $s^2_\bullet(\mathbf{M})$ which consists of all functors P as above which take values in zero objects: this is a contractible subcomplex of $s^2_\bullet(\mathbf{M})$.

Then the map σ can be rewritten as the map

$$\sigma : (s_\bullet(\mathbf{M})/s_\bullet(0)) \wedge S^1 \to s^2_\bullet(\mathbf{M})/s^2_\bullet(0)$$

of pointed bisimplicial sets.

The construction can be further iterated. Write $S^k_\bullet(\mathbf{M})$ for the k-fold simplicial exact category whose objects are the functors

$$P : \mathrm{Ar}(\mathbf{n}_1) \times \cdots \times \mathrm{Ar}(\mathbf{n}_k) \to \mathbf{M}$$

which are exact in each variable, and write $s^k_\bullet(\mathbf{M})$ for its k-fold simplicial set of objects. Write $s^k_\bullet(0)$ for the objects P of $s^k_\bullet(\mathbf{M})$ which take values in zero objects. Then $s^k_\bullet(0)$ is contractible, and there are k-fold pointed simplicial set maps

$$\sigma : (s^k_\bullet(\mathbf{M})/s^k_\bullet(0)) \wedge S^1 \to s^{k+1}_\bullet(\mathbf{M})/s^{k+1}_\bullet(0). \quad (9)$$

These maps are called *bonding maps.*

The pointed spaces

$$\operatorname{Ob}(\mathbf{M})/\operatorname{Ob}(0), s_\bullet(\mathbf{M})/s_\bullet(0), s_\bullet^2(\mathbf{M})/s_\bullet^2(0), \dots$$

and the bonding maps σ of (9) determine a spectrum $K(\mathbf{M})$, which is called the *K-theory spectrum* of the exact category $\mathbf{M}$.

The assignment

$$\mathbf{M} \mapsto K(\mathbf{M})$$

is natural in pointed exact categories $\mathbf{M}$.

Lemma 8.6. *Suppose that $f : \mathbf{M} \to \mathbf{N}$ is an exact equivalence. Then the induced map $f_* : s_\bullet^k(\mathbf{M}) \to s_\bullet^k(\mathbf{N})$ is a weak equivalence.*

When I say that a map of k-fold simplicial sets is a weak equivalence, I mean that it induces a weak equivalence of associated diagonal simplicial sets.

Proof. The induced functor

$$S_\bullet^n(\mathbf{M}) \to S_\bullet^n(\mathbf{N})$$

of n-fold simplicial exact categories is an exact equivalence of exact categories in all multi-simplicial degrees, and for all n. It follows that the induced map

$$s_\bullet S_\bullet^{k-1}(\mathbf{M}) \to s_\bullet S_\bullet^{k-1}(\mathbf{N})$$

is a weak equivalence. □

According to the description given of the map h in (5), the composite homotopy

$$Es_\bullet(\mathbf{M}) \times \Delta^1 \xrightarrow{h} Es_\bullet(\mathbf{M}) \xrightarrow{d_0} s_\bullet(\mathbf{M})$$

maps the subobjects

$$Es_\bullet(0) \times \Delta^1 \text{ and } Es_\bullet(\mathbf{M}) \times \{0\}$$

into $s_\bullet(0)$, and therefore induces a pointed map

$$h_* : (Es_\bullet(\mathbf{M})/Es_\bullet(0)) \wedge \Delta^1_* \to s_\bullet(\mathbf{M})/s_\bullet(0),$$

naturally in $\mathbf{M}$. Here Δ^1_* is the 1-simplex Δ^1 pointed by the vertex 0.

The adjoint of h_* is a map

$$\tilde{h} : Es_\bullet(\mathbf{M})/Es_\bullet(0) \to P(s_\bullet(\mathbf{M})/s_\bullet(0))$$

taking values in the path space which fits into a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ob}(\mathbf{M})/\mathrm{Ob}(0) & & & & \\
\downarrow{\scriptstyle i_{0*}} & & & & \\
f(\mathbf{M})/f(0) & \longrightarrow & Es_\bullet(\mathbf{M})/Es_\bullet(0) & \xrightarrow{d_{0*}} & s_\bullet(\mathbf{M})/s_\bullet(0) \\
\downarrow & & \simeq\downarrow{\scriptstyle \tilde{h}} & & \downarrow{\scriptstyle 1} \\
\Omega(s_\bullet(\mathbf{M})/s_\bullet(0)) & \longrightarrow & P(s_\bullet(\mathbf{M})/s_\bullet(0)) & \longrightarrow & s_\bullet(\mathbf{M})/s_\bullet(0)
\end{array} \tag{10}$$

The sequence along the bottom is the "path-loop fibration" (PX is paths in X starting at the base point — to make this actually work you have to replace $s_\bullet(\mathbf{M})/s_\bullet(0)$ by a fibrant model). The map $\tilde{h}$ is a weak equivalence on account of Lemma 8.1, since $s_\bullet(\mathbf{M})$ and $s_\bullet(0)$ have the same sets of vertices. The composite map

$$\mathrm{Ob}(\mathbf{M})/\,\mathrm{Ob}(0) \xrightarrow{i_{0*}} f(\mathbf{M})/f(0) \to \Omega(s_\bullet(\mathbf{M})/s_\bullet(0))$$

is the adjoint of the map σ of (7).

The diagram (10) is natural in exact categories with distinguished zero object. Thus, by applying this construction to the simplicial exact categories $S^k_\bullet(M)$ we obtain a list of commutative diagrams of multisimplicial sets

$$\begin{array}{ccccc}
s^k_\bullet(\mathbf{M})/s^k_\bullet(0) & & & & \\
\downarrow{\scriptstyle i_{0*}} & & & & \\
f^{k+1}(\mathbf{M})/f^{k+1}(0) & \longrightarrow & Es^{k+1}_\bullet(\mathbf{M})/Es^{k+1}_\bullet(0) & \xrightarrow{d_{0*}} & s^{k+1}_\bullet(\mathbf{M})/s^{k+1}_\bullet(0) \\
\downarrow & & \simeq\downarrow{\scriptstyle \tilde{h}} & & \downarrow{\scriptstyle 1} \\
\Omega(s^{k+1}_\bullet(\mathbf{M})/s^{k+1}_\bullet(0)) & \to & P(s^{k+1}_\bullet(\mathbf{M})/s^{k+1}_\bullet(0)) & \to & s^{k+1}_\bullet(\mathbf{M})/s^{k+1}_\bullet(0)
\end{array} \tag{11}$$

where, for notational convenience, $f^{k+1}(\mathbf{M}) = f(S^k_\bullet(\mathbf{M}))$ and $Es^{k+1}_\bullet(\mathbf{M}) = Es_\bullet(S^k_\bullet(\mathbf{M}))$.

The composite

$$s^k_\bullet(\mathbf{M})/s^k_\bullet(0) \xrightarrow{i_{0*}} f^{k+1}(\mathbf{M})/f^{k+1}(0) \to \Omega(s^{k+1}_\bullet(\mathbf{M})/s^{k+1}_\bullet(0))$$

is the adjoint σ_* of the map σ of (9).

Lemma 8.7. *The map*

$$i_{0*} : s^k_\bullet(\mathbf{M})/s^k_\bullet(0) \to f^{k+1}(\mathbf{M})/f^{k+1}(0)$$

is a weak equivalence for $k \geq 1$*.*

Proof. It is enough to show that the map

$$i_{0*} : s^k_\bullet(\mathbf{M}) \to f^{k+1}(\mathbf{M})$$

is a weak equivalence. The map $i_0 : \mathrm{Ob}(\mathbf{M}) \to f(\mathbf{M})$ is the object level map of a map

$$i_0 : \mathbf{M} \to F(\mathbf{M})$$

of simplicial exact categories, where $F(\mathbf{M})_n$ is the exact category whose objects are the exact functors $P : \mathrm{Ar}(\mathbf{n+1}) \to \mathbf{M}$ such that all morphisms in the string

$$P(0,1) \rightarrowtail P(0,2) \rightarrowtail \cdots \rightarrowtail P(0,n+1)$$

are isomorphisms. The composite

$$\mathbf{M} \xrightarrow{i_0} S_1(\mathbf{M}) = F(\mathbf{M})_0 \xrightarrow{s^*} F(\mathbf{M})_n$$

is an exact equivalence for each $n \geq 0$, so that the map $i_0 : \mathbf{M} \to F(\mathbf{M})$ is a simplicial exact equivalence. Applying the functor $s^k_\bullet$ to this exact equivalence gives the map $i_{0*} : s^k_\bullet(\mathbf{M}) \to f^{k+1}(\mathbf{M})$, which is therefore a weak equivalence by Lemma 8.6. □

Theorem 8.8. *The maps*

$$s^k_\bullet(\mathbf{M})/s^k_\bullet(0) \to Es^{k+1}_\bullet(\mathbf{M})/Es^{k+1}_\bullet(0) \xrightarrow{d_{0*}} s^{k+1}_\bullet(\mathbf{M})/s^{k+1}_\bullet(0)$$

form a fibre homotopy sequence if $k \geq 1$.

Corollary 8.9. *The adjoint map*

$$\sigma_* : s^k_\bullet(\mathbf{M})/s^k_\bullet(0) \to \Omega(s^{k+1}_\bullet(\mathbf{M})/s^{k+1}_\bullet(0))$$

is a weak equivalence for $k \geq 1$.

Corollary 8.9 follows from Theorem 8.8 by a comparison of fibre sequences. The Corollary says that the K-theory spectrum $K(\mathbf{M})$ is an Ω-spectrum in levels 1 and above.

Proof of Theorem 8.8. It's enough to show that the maps

$$s^k_\bullet(\mathbf{M}) \to Es^{k+1}_\bullet(\mathbf{M}) \xrightarrow{d_0} s^{k+1}_\bullet(\mathbf{M})$$

form a fibre sequence. But this sequence may be identified with the effect of applying the functor $s^k_\bullet$ to the sequence

$$\mathbf{M} \to ES_\bullet(\mathbf{M}) \xrightarrow{d_0} S_\bullet(\mathbf{M})$$

of maps of simplicial exact categories.

The additivity theorem (Theorem 6.1) holds for the functors $\mathbf{M} \mapsto s^k_\bullet \operatorname{Ex}(\mathbf{M})$ for $k > 1$. To see this, look at the diagram

$$\begin{array}{ccc} s^k_\bullet \operatorname{Ex}(\mathbf{M}) & \xrightarrow{(s^k_\bullet f, s^k_\bullet g)} & s^k_\bullet(\mathbf{M}) \times s^k(\mathbf{M}) \\ \cong\downarrow & & \downarrow\cong \\ s_\bullet \operatorname{Ex} S^{k-1}_\bullet(\mathbf{M}) & \xrightarrow[(s_\bullet f, s_\bullet g)]{} & s_\bullet S^{k-1}(\mathbf{M}) \times s_\bullet S^{k-1}(\mathbf{M}) \end{array}$$

The map on the bottom is a weak equivalence by Theorem 6.1, and so the map on top is a weak equivalence too. The symmetric monoidal structure of $\mathbf{M}$ induces an H-space structure on $s^k_\bullet(\mathbf{M})$, and it follows, just as in the proof of Corollary 7.2, that any exact sequence

$$0 \to f \rightarrowtail g \twoheadrightarrow h \to 0$$

of exact functors $\mathbf{M} \to \mathbf{N}$ determines maps

$$f_*, g_*, h_* : s^k_\bullet(\mathbf{M}) \to s^k_\bullet(\mathbf{N})$$

which satisfy the relation

$$g_* = f_* + h_*.$$

Now consider the sequence of exact functors

$$\mathbf{M} \xrightarrow{i} S_{n+1}(\mathbf{M}) \xrightarrow{d_0} S_n(\mathbf{M}) \qquad (12)$$

where the functor i associates to an object P the exact functor $i(P) : \operatorname{Ar}(\mathbf{n+1}) \to \mathbf{M}$ which is determined (by a fixed choice of zero object 0) by letting the string

$$i(P)(0,1) \rightarrowtail i(P)(0,2) \rightarrowtail \cdots \rightarrowtail i(P)(0,n+1)$$

be the string of identities

$$P \overset{1}{\rightarrowtail} P \overset{1}{\rightarrowtail} \ldots \overset{1}{\rightarrowtail} P \tag{13}$$

and so $i(P)(i,j) = 0$ for all other pairs $i \leq j$. The sequence of functors (12) can be replaced up to exact equivalence by the sequence

$$\mathbf{M} \xrightarrow{i} \mathrm{Mon}_{n+1}(\mathbf{M}) \xrightarrow{p} \mathrm{Mon}_n(\mathbf{M}),$$

where $i(P)$ is the sequence of identities (13), and p sends the string

$$P_1 \rightarrowtail P_2 \rightarrowtail \cdots \rightarrowtail P_{n+1} \tag{14}$$

to the string

$$P_2/P_1 \rightarrowtail P_3/P_1 \rightarrowtail \cdots \rightarrowtail P_{n+1}/P_1.$$

The functor p has a section

$$\sigma : \mathrm{Mon}_n(\mathbf{M}) \to \mathrm{Mon}_{n+1}(\mathbf{M})$$

(up to isomorphism) which sends the string

$$Q_1 \rightarrowtail \cdots \rightarrowtail Q_n$$

to the string

$$0 \rightarrowtail Q_1 \rightarrowtail \cdots \rightarrowtail Q_n.$$

The functor $i : \mathbf{M} \to \mathrm{Mon}_{n+1}(\mathbf{M})$ has a left inverse q which sends the string (14) to the object P_1.

There is an exact sequence

$$0 \to i \cdot q \rightarrowtail 1 \twoheadrightarrow \sigma \cdot p \to 0$$

of exact functors $\mathrm{Mon}_{n+1}(\mathbf{M}) \to \mathrm{Mon}_{n+1}(\mathbf{M})$, so that the is a relation

$$1 = i_* q_* + \sigma_* p_*$$

of maps $s^k_\bullet \mathrm{Mon}_{n+1}(\mathbf{M}) \to s^k_\bullet \mathrm{Mon}_{n+1}(\mathbf{M})$ in the homotopy category. It follows that the map

$$(q_*, p_*) : s^k_\bullet \mathrm{Mon}_{n+1}(\mathbf{M}) \to s^k_\bullet(\mathbf{M}) \times s^k_\bullet \mathrm{Mon}_n(\mathbf{M})$$

is a weak equivalence with homotopy inverse induced by the composite functor

$$\mathbf{M} \times \mathrm{Mon}_n(\mathbf{M}) \xrightarrow{(i,\sigma)} \mathrm{Mon}_{n+1}(\mathbf{M}) \times \mathrm{Mon}_{n+1}(\mathbf{M}) \xrightarrow{\oplus} \mathrm{Mon}_{n+1}(\mathbf{M}).$$

In particular the sequences

$$s^k_\bullet(\mathbf{M}) \xrightarrow{i_*} s^k_\bullet \mathrm{Mon}_{n+1}(\mathbf{M}) \xrightarrow{p_*} s^k_\bullet \mathrm{Mon}_n(\mathbf{M})$$

and

$$s^k_\bullet(\mathbf{M}) \xrightarrow{i_*} s^k_\bullet S_{n+1}(\mathbf{M}) \xrightarrow{d_{0*}} s^k_\bullet S_n(\mathbf{M})$$

are homotopy fibre sequences. All spaces $s^k_\bullet(\mathbf{N})$ are connected (see below), so it follows from a theorem of Bousfield and Friedlander [1], [2, IV.4.9], that the maps

$$s^k_\bullet(\mathbf{M}) \to s^k_\bullet ES_\bullet(\mathbf{M}) \to s^k_\bullet S_\bullet(\mathbf{M})$$

form a homotopy fibre sequence. □

Remark 8.10. The space

$$s^k_\bullet(\mathbf{N}) = s_\bullet S^{k-1}_\bullet(\mathbf{N})$$

is connected for each exact category $\mathbf{N}$, since all of the simplicial sets

$$s_\bullet S^{k-1}_{i_1,\dots,i_{k-1}}(\mathbf{N})$$

are connected. One can show that if X is a k-fold simplicial object in simplicial sets such that all of the constituent spaces $X_{i_1,\dots,i_k}$ are connected, then the associated diagonal simplicial set $d(X)$ is connected (exercise).

Since all spaces $s^k_\bullet(\mathbf{M})$ are connected for $k \geq 1$, then all spaces $K(\mathbf{M})^k$ are connected for $k \geq 1$. It follows from Theorem 8.8 that the space $K(\mathbf{M})^k$ is $(k-1)$-connected for $k \geq 1$, and then one computes explicitly to show that the K-theory spectrum is connective, and that the stable homotopy groups $\pi_i K(\mathbf{M})$ are given by

$$\pi_i K(\mathbf{M}) \cong \pi_i \Omega K(\mathbf{M})^1 = \pi_{i+1} BQ(\mathbf{M})$$

for $i \geq 0$. In other words the stable homotopy groups of $K(\mathbf{M})$ coincide up to isomorphism with the classical K-groups of $\mathbf{M}$ in positive degrees, and are 0 in negative degrees.

9 Symmetries and products

Suppose that X is a set, and recall that the n-fold product $X^{\times n}$ can be identified with the set of functions $x : \underline{n} \to X$, where

$$\underline{n} = \{1, 2, \ldots, n\}.$$

Any symmetric group element $\sigma \in \Sigma_n$ determines a bijection

$$\sigma^* : X^{\times n} \xrightarrow{\cong} X^{\times n}, \tag{15}$$

which is defined by sending an element $x : \underline{n} \to X$ to the composite

$$\underline{n} \xrightarrow{\sigma} \underline{n} \xrightarrow{x} X.$$

This assignment is covariant in X and contravariant in σ, and it follows that there is a (standard) natural left action

$$\Sigma_n \times X^{\times n} \to X^{\times n} \tag{16}$$

of the group Σ_n on $X^{\times n}$ which sends the pair (σ, x) to the element $(\sigma^{-1})^*(x)$. Alternatively, in terms of n-tuples,

$$\sigma(x_1, \dots, x_n) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}).$$

If X is a pointed space, this action preserves the base point $(*, \dots, *)$ of $X^{\times n}$, and it also preserves the subspace of $X^{\times n}$ which consists of all (n-tuples) $x : \underline{n} \to X$ such that $x(i) = *$ for some i. It follows that the action (16) induces an action

$$\Sigma_n \times X^{\wedge n} \to X^{\wedge n} \tag{17}$$

of Σ_n on the n-fold smash product $X^{\wedge n}$ which is natural in pointed X. This action specializes (for example) to the usual left action

$$\Sigma_n \times (S^1)^{\wedge n} \to (S^1)^{\wedge n}. \tag{18}$$

Suppose that $\mathbf{M}$ is an exact category, pointed by a zero object 0. Recall that the K-theory spectrum $K(\mathbf{M})$ consists of the pointed "spaces"

$$\operatorname{Ob}(\mathbf{M})/\operatorname{Ob}(0), s_\bullet(\mathbf{M})/s_\bullet(0), s_\bullet^2(\mathbf{M})/s_\bullet^2(0), \dots$$

Well, this is kind of a lie, because the objects in this list are multi-simplicial sets. What we mean to write is

$$K(\mathbf{M})^k = d(s_\bullet^k(\mathbf{M})/s_\bullet^k(0)),$$

where $d : s^k\mathbf{Set} \to s\mathbf{Set}$ is the multisimplicial diagonal functor. Thus, for example, $d(s^k_\bullet(\mathbf{M}))$ is the simplicial set with n-simplices given by all functors

$$P : \mathrm{Ar}(\mathbf{n})^{\times k} \to \mathbf{M}$$

which are exact in all variables. If $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, then the m-simplex $\theta^*(P)$ is the composite functor

$$\mathrm{Ar}(\mathbf{m})^{\times k} \xrightarrow{\theta_*^{\times k}} \mathrm{Ar}(\mathbf{n})^{\times k} \xrightarrow{P} \mathbf{M}.$$

The symmetric group Σ_k in k-letters acts (on the right) on the category $\mathrm{Ar}(\mathbf{n})^{\times k}$ by permuting factors, and therefore acts on the left on the sets $d(s^k_\bullet(\mathbf{M})_n$: for $\sigma \in \Sigma_k$ and P as above, $\sigma(P)$ is the composite

$$\mathrm{Ar}(\mathbf{n})^{\times k} \xrightarrow{\sigma^*} \mathrm{Ar}(\mathbf{n})^{\times k} \xrightarrow{P} \mathbf{M}.$$

where σ^* is the map defined in (15). The functions

$$\sigma : d(s^k_\bullet(\mathbf{M}))_n \to d(s^k_\bullet(\mathbf{M}))_n$$

plainly respect the simplicial structure, so that there is a natural induced left action

$$\Sigma_k \times d(s^k_\bullet(\mathbf{M})) \to d(s^k_\bullet(\mathbf{M})),$$

and hence a natural pointed action

$$\Sigma_k \times K(\mathbf{M})^k \to K(\mathbf{M})^k.$$

The bonding map

$$\sigma_K : S^1 \wedge K(\mathbf{M})^k \to K(\mathbf{M})^{k+1}$$

is induced by a $(k+1)$-fold simplicial set map

$$h : \Delta^1 \times s^k_\bullet(\mathbf{M}) \to s^{k+1}_\bullet(\mathbf{M})$$

which takes the pair (τ, P) consisting of the n-simplex $\tau : \mathbf{n} \to \mathbf{1}$ and the multisimplex

$$P : \mathrm{Ar}(\mathbf{n}_1) \times \cdots \times \mathrm{Ar}(\mathbf{n}_k) \to \mathbf{M}$$

to the composite

$$\begin{array}{c} \mathrm{Ar}(\mathbf{n}) \times \mathrm{Ar}(\mathbf{n}_1) \times \cdots \times \mathrm{Ar}(\mathbf{n}_k) \\ \downarrow {\scriptstyle \tau_* \times 1} \\ \mathrm{Ar}(\mathbf{1}) \times \mathrm{Ar}(\mathbf{n}_1) \times \cdots \times \mathrm{Ar}(\mathbf{n}_k) \xrightarrow[P_*]{} \mathbf{M} \end{array}$$

where P_* is the unique functor such that

$$P_*((0,0), \epsilon_1, \ldots, \epsilon_k) = P_*((1,1), \epsilon_1, \ldots, \epsilon_k) = 0$$

and

$$P_*((0,1), \epsilon_1, \ldots, \epsilon_k) = P(\epsilon_1, \ldots, \epsilon_k).$$

Similarly, the iterated bonding map

$$\sigma_K : S^r \wedge K(\mathbf{M})^k \to K(\mathbf{M})^{r+k}$$

(S^r denotes the r-fold smash $(S^1)^{\wedge r}$) is induced on diagonals by the $(r+k)$-fold simplicial set map

$$h : \Delta^1 \times \cdots \times \Delta^1 \times s^k_\bullet(\mathbf{M}) \to s^{r+k}_\bullet(\mathbf{M})$$

which takes the $(r+1)$-tuple

$$(\mathbf{m}_1 \xrightarrow{\tau_1} \mathbf{1}, \ldots, \mathbf{m}_r \xrightarrow{\tau_r} \mathbf{1}, P)$$

to the composite

$$\begin{array}{c} \mathrm{Ar}(\mathbf{m}_1) \times \cdots \times \mathrm{Ar}(\mathbf{m}_r) \times \mathrm{Ar}(\mathbf{n}_1) \times \cdots \times \mathrm{Ar}(\mathbf{n}_k) \\ \Big\downarrow {\scriptstyle \tau_{1*} \times \cdots \times \tau_{r*} \times 1} \\ \mathrm{Ar}(\mathbf{1}) \times \cdots \times \mathrm{Ar}(\mathbf{1}) \times \mathrm{Ar}(\mathbf{n}_1) \times \cdots \times \mathrm{Ar}(\mathbf{n}_k) \\ \Big\downarrow {\scriptstyle P_*} \\ \mathbf{M} \end{array}$$

where P_* is the uniquely determined functor such that

$$P_*((0,1), \ldots, (0,1), \epsilon_1, \ldots, \epsilon_k) = P(\epsilon_1, \ldots, \epsilon_k)$$

and

$$P_*(\gamma_1, \ldots, \gamma_r, \epsilon_1, \ldots, \epsilon_k) = 0$$

if some $\gamma_j \neq (0,1)$.

In particular, the induced map

$$h : (\Delta^1)^{\times r} \times d(s^k_\bullet(\mathbf{M})) \to d(s^{r+k}_\bullet(\mathbf{M}))$$

is defined on n-simplices by taking the $(r+1)$-tuple

$$(\mathbf{n} \xrightarrow{\tau_1} \mathbf{1}, \ldots, \mathbf{n} \xrightarrow{\tau_r} \mathbf{1}, P)$$

to the composite

$$\mathrm{Ar}(\mathbf{n})^{\times r} \times \mathrm{Ar}(\mathbf{n})^{\times k} \xrightarrow{\tau_{1*} \times \cdots \times \tau_{r*} \times 1} \mathrm{Ar}(\mathbf{1})^{\times r} \times \mathrm{Ar}(\mathbf{n})^{\times k} \xrightarrow{P_*} \mathbf{M}.$$

If $\gamma \in \Sigma_r$, then

$$\gamma(\tau_1, \ldots, \tau_r) = (\tau_{\gamma^{-1}(1)}, \ldots, \tau_{\gamma^{-1}(r)})$$

in $(\Delta^1)^{\times r}_n$, and there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Ar}(\mathbf{n})^{\times r} & \xrightarrow{\gamma^*} & \mathrm{Ar}(\mathbf{n})^{\times r} \\ {\scriptstyle \tau_{\gamma^{-1}(1)*}\times\cdots\times\tau_{\gamma^{-1}(r)*}} \downarrow & & \downarrow {\scriptstyle \tau_{1*}\times\cdots\times\tau_{r*}} \\ \mathrm{Ar}(\mathbf{1})^{\times r} & \xrightarrow[\gamma^*]{} & \mathrm{Ar}(\mathbf{1})^{\times r} \end{array}$$

It follows that the iterated bonding map

$$\sigma_K : S^r \wedge K(\mathbf{M})^k \to K(\mathbf{M})^{r+k}$$

is $(\Sigma_r \times \Sigma_k)$-equivariant, and so we have proved

Proposition 9.1. *The K-theory spectrum $K(\mathbf{M})$ has the structure of a symmetric spectrum. This symmetric spectrum structure is natural in pointed exact categories $\mathbf{M}$.*

Here's a first observation:

Lemma 9.2. *Suppose that the exact functors $f, g : \mathbf{M} \to \mathbf{N}$ are naturally isomorphic. Then the induced maps $f_* : K(\mathbf{M}) \to K(\mathbf{N})$ of (symmetric) spectra represent the same map in the stable category.*

Proof. The source and target maps

$$s, t : \mathrm{Iso}(\mathbf{N}) \to \mathbf{N}$$

are exact equivalences, and therefore induce stable equivalences

$$s_*, t_* : K(\mathrm{Iso}(\mathbf{N})) \to K(\mathbf{N})$$

by Lemma 5.1 (Lecture 003) and Lemma 8.6 above. There is a commutative diagram of exact equivalences

$$\begin{array}{ccc} & & \mathrm{Iso}(\mathbf{N}) \\ & \overset{\sigma}{\nearrow} & \downarrow {\scriptstyle (s,t)} \\ \mathbf{N} & \xrightarrow[\Delta]{} & \mathbf{N} \times \mathbf{N} \end{array} \tag{19}$$

where Δ is the diagonal functor and σ assigns the identity map $1_P : P \to P$ to each object P of $\mathbf{M}$. It follows that the maps s_* and t_* coincide in the stable category.

Finally, a natural isomorphism $h : f \cong g$ is an exact functor $h : \mathbf{M} \to \mathrm{Iso}(\mathbf{N})$ such that the diagram

$$\begin{array}{ccc} & & \mathrm{Iso}(\mathbf{N}) \\ & \overset{h}{\nearrow} & \downarrow {\scriptstyle (s,t)} \\ \mathbf{M} & \xrightarrow[(f,g)]{} & \mathbf{N} \times \mathbf{N} \end{array}$$

commutes. There is an induced commutative diagram

$$\begin{array}{ccc} & & K(\mathrm{Iso}(\mathbf{N})) \\ & \overset{h_*}{\nearrow} & \downarrow {\scriptstyle (s_*,t_*)} \\ K(\mathbf{M}) & \xrightarrow[(f_*,g_*)]{} & K(\mathbf{N}) \times K(\mathbf{N}) \end{array}$$

so that $f_* = s_* h_* = t_* h_* = g_*$ in the stable category, as claimed. $\square$

The moral of this last proof is that the diagram (19) is a path object for the category of exact categories.

Suppose that $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{N}$ are exact categories. A *biexact functor* (or *biexact pairing*) is functor

$$\otimes : \mathbf{M}_1 \times \mathbf{M}_2 \to \mathbf{N}$$

which is exact in each variable in the sense that all functors

$$\otimes(P,\ ) : \mathbf{M}_2 \to \mathbf{N} \text{ and } \otimes(\ ,Q) : \mathbf{M}_1 \to \mathbf{N}$$

are exact.

Example: The tensor product functors

$$\otimes : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$$

and

$$\otimes : \mathcal{P}(X) \times \mathbf{M}(X) \to \mathbf{M}(X)$$

for vector bundles $\mathcal{P}(X)$ and coherent sheaves $\mathbf{M}(X)$ on a Noetherian scheme X are standard examples of biexact functors.

Every biexact pairing $\otimes : \mathbf{M}_1 \times \mathbf{M}_2 \to \mathbf{N}$ induces a simplicial set map

$$\otimes : d(s^r_\bullet(\mathbf{M}_1)) \times d(s^s_\bullet(\mathbf{M}_2)) \to d(s^{r+s}_\bullet(\mathbf{N})).$$

Explictly, given exact functors $P : \mathrm{Ar}(\mathbf{n})^{\times r} \to \mathbf{M}_1$ and $Q : \mathrm{Ar}(\mathbf{n})^{\times s} \to \mathbf{M}_2$, the n-simplex $\otimes(P,Q)$

is the composite

$$\mathrm{Ar}(\mathbf{n})^{\times r} \times \mathrm{Ar}(\mathbf{n})^{\times s} \xrightarrow{P\times Q} \mathbf{M}_1 \times \mathbf{M}_2 \xrightarrow{\otimes} \mathbf{N}.$$

To go further, we need to replace the biexact pairing $\otimes$ up to equivalence by a biexact pairing

$$\otimes_0 : \mathbf{M}_1 \times \mathbf{M}_2 \to \mathbf{N}_0$$

which takes values in an exact category $\mathbf{N}_0$ having a unique zero object 0. This is done by letting $\otimes_0$ be the composite

$$\mathbf{M}_1 \times \mathbf{M}_2 \xrightarrow{\otimes} \mathbf{N} \xrightarrow{r} \mathbf{N}_0,$$

where $r : \mathbf{N} \to \mathbf{N}_0$ is the retraction map onto an exact subcategory $\mathbf{N}_0 \subset \mathbf{N}$ having 0 as the only zero object — see the Appendix. This is harmless, since the retraction map is natural in exact categories and is an exact equivalence. We shall henceforth assume that the target $\mathbf{N}$ of the biexact pairing

$$\otimes : \mathbf{M}_1 \times \mathbf{M}_2 \to \mathbf{N}$$

has a unique zero object 0.

The map

$$\otimes : d(s^r_\bullet(\mathbf{M}_1)) \times d(s^s_\bullet(\mathbf{M}_2)) \to d(s^{r+s}_\bullet(\mathbf{N})).$$

is plainly $(\Sigma_r \times \Sigma_s)$-equivariant, and $P \otimes Q = 0$ if either P or Q is a zero object. It follows that the pairing $\otimes$ induces a pointed $(\Sigma_r \times \Sigma_s)$-equivariant map

$$\otimes_* : K(\mathbf{M}_1)^r \wedge K(\mathbf{M}_2)^s \to K(\mathbf{N})^{r+s}. \qquad (20)$$

It is an exercise (which uses the assumption that $\mathbf{N}$ has only one zero object) to show that the diagrams

$$\begin{array}{ccc}
(\Delta^1)^{\times k} \times s^r_\bullet(\mathbf{M}_1) \times s^s_\bullet(\mathbf{M}_2) & \xrightarrow{h\times 1} & s^{k+r}_\bullet(\mathbf{M}_1) \times s^s_\bullet(\mathbf{M}_2) \\
{\scriptstyle 1\times\otimes}\downarrow & & \downarrow{\scriptstyle \otimes} \\
(\Delta^1)^{\times k} \times s^{r+s}_\bullet(\mathbf{N}) & \xrightarrow[h]{} & s^{k+r+s}_\bullet(\mathbf{N})
\end{array}$$

and

$$\begin{array}{ccccc}
(\Delta^1)^{\times k} \times s^r_\bullet(\mathbf{M}_1) \times s^s_\bullet(\mathbf{M}_2) & \xrightarrow{h\times 1} & s^{k+r}_\bullet(\mathbf{M}_1) \times s^s_\bullet(\mathbf{M}_2) & \xrightarrow{\otimes} & s^{k+r+s}_\bullet(\mathbf{N}) \\
{\scriptstyle t\times 1}\downarrow & & & & \downarrow{\scriptstyle c(k,r)\oplus 1} \\
s^r_\bullet(\mathbf{M}_1) \times (\Delta^1)^{\times k} \times s^s_\bullet(\mathbf{M}_2) & \xrightarrow[1\times h]{} & s^r(\mathbf{M}_1) \times s^{k+s}_\bullet(\mathbf{M}_2) & \xrightarrow[\otimes]{} & s^{r+k+s}_\bullet(\mathbf{N})
\end{array}$$

commute. Here,

$$t : (\Delta^1)^{\times k} \times s^r_\bullet(\mathbf{M}_1) \to s^r_\bullet(\mathbf{M}_1) \times (\Delta^1)^{\times k}$$

interchanges factors, and the element $c(k,r) \in \Sigma_{k+r}$ shuffles the first k entries of $\underline{k+r}$ past the last r entries.

It follows that the diagrams

$$\begin{array}{ccc}
S^k \wedge K(\mathbf{M}_1)^r \wedge K(\mathbf{M}_2)^s & \xrightarrow[\sigma_K \wedge 1]{} & K(\mathbf{M}_1)^{k+r} \wedge K(\mathbf{M}_2)^s \\
{\scriptstyle 1\wedge\otimes_*}\downarrow & & \downarrow{\scriptstyle \otimes_*} \\
S^k \wedge K(\mathbf{N})^{r+s} & \xrightarrow[\sigma_K]{} & K(\mathbf{N})^{k+r+s}
\end{array} \tag{21}$$

and

$$\begin{array}{ccccc}
S^k \wedge K(\mathbf{M}_1)^r \wedge K(\mathbf{M}_2)^s & \xrightarrow{\sigma_K \wedge 1} & K(\mathbf{M}_1)^{k+r} \wedge K(\mathbf{M}_2)^s & \xrightarrow{\otimes_*} & K(\mathbf{N})^{k+r+s} \\
{\scriptstyle t\wedge 1}\downarrow & & & & \downarrow{\scriptstyle c(k,r)\oplus 1} \\
K(\mathbf{M}_1)^r \wedge S^k \wedge K(\mathbf{M}_2)^s & \xrightarrow[1\wedge\sigma_K]{} & K(\mathbf{M}_1)^r \wedge K(\mathbf{M}_2)^{k+s} & \xrightarrow[\otimes_*]{} & K(\mathbf{N})^{r+k+s}
\end{array} \tag{22}$$

commute.

The data for a smash product pairing

$$\otimes_* : K(\mathbf{M}_1) \wedge_\Sigma K(\mathbf{M}_2) \to K(\mathbf{N})$$

consists of a family of maps (20) satisfying the commutativity conditions (21) and (22) — see [4, p.518], for example. We have therefore proved the following:

Proposition 9.3. *Suppose that*

$$\otimes : \mathbf{M}_1 \times \mathbf{M}_2 \to \mathbf{N}$$

is a biexact pairing, and that $\mathbf{N}$ *has a unique zero object. Then the induced maps*

$$\otimes_* : K(\mathbf{M}_1)^r \wedge K(\mathbf{M}_2)^s \to K(\mathbf{N})^{r+s}$$

form a morphism

$$\otimes_* : K(\mathbf{M}_1) \wedge_\Sigma K(\mathbf{M}_2) \to K(\mathbf{N})$$

of symmetric spectra.

There is a natural shift operator $Z \mapsto Z[1]$ for symmetric spectra Z, where

$$Z[1]^n = Z^{1+n},$$

and $\sigma \in \Sigma_n$ acts as $1 \oplus \sigma \in \Sigma_{n+1}$ on Z^{1+n}. The structure map $\sigma : S^r \wedge Z[1]^n \to Z[1]^{r+n}$ is defined to be the composite

$$S^r \wedge Z^{1+n} \xrightarrow{\sigma} Z^{r+1+n} \xrightarrow{c(r,1)\oplus 1} Z^{1+r+n},$$

and $Z[1]$ has the structure of a symmetric spectrum. The adjoints $\sigma_* : Z^n \to \Omega Z^{1+n}$ of the composite maps

$$Z^n \wedge S^1 \xrightarrow{t} S^1 \wedge Z^n \xrightarrow{\sigma} Z^{1+n}$$

together determine a natural map

$$\tilde{\sigma} : Z \to \Omega Z[1]$$

of symmetric spectra.

This map is not a stable equivalence of symmetric spectra in general: Jeff Smith has given a counterexample — see [5].

If, however, the underlying spectrum consists of Kan complexes and is an Ω-spectrum above some level, then the map of spectra underlying $\tilde{\sigma}$ is a stable equivalence, so that $\tilde{\sigma}$ is a stable equivalence of symmetric spectra [3], [5, Lem. 10]. In particular, if Z is an Ω-spectrum above level 0, then $\Omega Z[1]$ is stably fibrant and the stable equivalence $Z \to \Omega Z[1]$ is a stably fibrant model for Z.

It follows from Corollary 8.9 that, up to levelwise replacement of $K(\mathbf{M})$ by an injective fibrant model, the symmetric spectrum $\Omega K(\mathbf{M})[1]$ is a stably fibrant replacement for the symmetric spectrum $K(\mathbf{M})$. In particular, we have the following:

Lemma 9.4. *Any stably fibrant replacement*

$$K(\mathbf{M}) \to FK(\mathbf{M})$$

in symmetric spectra consists of weak equivalences

$$K(\mathbf{M})^n \to FK(\mathbf{M})^n$$

for $n \geq 1$, and $FK(\mathbf{M})^0$ is the derived loop space of $K(\mathbf{M})^1$.

Appendix: Some category theory

The contents of this section are meant to indicate that making a choice of zero object in an exact category $\mathbf{M}$ is harmless.

The moral of the following is that we can form an equivalent exact category $\mathbf{M}_0$ with the same non-zero objects as $\mathbf{M}$, and with only one zero object. Further, this construction is natural: a choice of zero object 0 in $\mathbf{M}$ gives $\mathbf{M}_0$ the structure of a strong deformation retract of $\mathbf{M}$, and any exact functor $f : \mathbf{M} \to \mathbf{N}$ determines a unique exact functor $f_* : \mathbf{M}_0 \to \mathbf{N}_0$ which commutes with retractions in the sense that the diagram of exact functors

$$\begin{array}{ccc} \mathbf{M} & \xrightarrow{r} & \mathbf{M}_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathbf{N} & \xrightarrow[r]{} & \mathbf{N}_0 \end{array}$$

commutes.

Suppose that C is a category with a full subcategory A which is a trivial groupoid. Suppose further that A is closed under isomorphisms: if there is an isomorphism $x \xrightarrow{\cong} a$ with $a \in A$, then $x \in A$. Pick an object 0 in A, and let C_0 be the full subcategory on the objects $\mathrm{Ob}(C) - \mathrm{Ob}(A) \cup \{0\}$.

Define a function $r : \mathrm{Ob}(C) \to \mathrm{Ob}(C_0)$ by setting

$$r(x) = \begin{cases} x & \text{if } x \in \mathrm{Ob}(C) - \mathrm{Ob}(A), \text{ and} \\ 0 & \text{if } x \in \mathrm{Ob}(A). \end{cases}$$

Then there are isomorphisms

$$\eta_x : x \xrightarrow{\cong} r(x)$$

such that $\eta_x = 1_x$ if $x \in \mathrm{Ob}(C) - \mathrm{Ob}(A)$ and η_x is the canonical isomorphism $x \to 0$ in the groupoid A if $x \in \mathrm{Ob}(A)$. If $\alpha : x \to y$ is a morphism of C, let $r(\alpha)$ be the unique morphism such that the diagram

$$\begin{array}{ccc} x & \xrightarrow[\cong]{\eta_x} & r(x) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle r(\alpha)} \\ y & \xrightarrow[\eta_y]{\cong} & r(y) \end{array}$$

commutes. Then if $i : C_0 \subset C$ is the inclusion of C_0 in C, we see that $r \cdot i$ is the identity on C_0, and the isomorphisms η_x determine a natural isomorphism

$$\eta : 1_C \xrightarrow{\cong} i \cdot r.$$

Note that the isomorphism η restricts to the identity isomorphism

$$i \cong i \cdot r \cdot i = i$$

on the image of the inclusion functor i, so that C_0 is a strong deformation retract of C.

Suppose that D is a category with full subcategory B which is a trivial groupoid. Suppose that B is closed under isomorphisms. Pick an object 0 of B and form the full subcategory D_0 of D in the same way that C_0 is constructed above. Then there is a functor $r : D \to D_0$ which together with the inclusion $i : D_0 \to D$ forms a strong deformation retraction of D onto D_0.

Suppose that $f : C \to D$ is a functor such that $f(A) \subset B$, and let f_* be the composite

$$C_0 \xrightarrow{i} C \xrightarrow{f} D \xrightarrow{r} D_0.$$

I claim that the diagram of functors

$$\begin{array}{ccc} C & \xrightarrow{r} & C_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*} \\ D & \xrightarrow[r]{} & D_0 \end{array} \tag{23}$$

commutes as required.

Consider the commutative diagram

$$\begin{array}{ccccc} f(x) & \xrightarrow{\eta_{f(x)}} & irf(x) & & \\ & \searrow^{f(\eta_x)} & & \searrow^{irf(\eta_x)} & \\ & & f(ir(x)) & \xrightarrow[\eta_{f(ir(x))}]{} & irf(ir(x)) \end{array}$$

If $f(x) \in B$ then $f(ir(x)) \in B$ and

$$0 = rf(x) \xrightarrow{rf(\eta_x)} r(f(ir(x)) = 0$$

is the identity morphism. If $f(x) \notin B$ then $x \notin A$ and the map $\eta_x : x \to ir(x)$ is the identity morphism, and so

$$rf(\eta_x) : rf(x) \to rf(ir(x))$$

is the identity morphism as well. It follows that the functors rf and $rfir = f_* r$ coincide, and so the diagram (23) commutes.

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