Lecture 005 (October 8, 2014)

10 Group completion

I say that a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ Z \arrow[r] & W \end{tikzcd}$$

of simplicial set maps is *homology cartesian* (for some theory h_*) if there is a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & V \arrow[d, "p"] \\ & W \end{tikzcd}$$

with i a trivial cofibration and p a fibration such that the induced map

$$X \to Z \times_W V$$

induces an isomorphism

$$h_*(X) \xrightarrow{\cong} h_*(Z \times_W V).$$

Lemma 10.1. *Suppose that $f : X \to Y$ is a map of bisimplicial sets such that all vertical maps $X_n \to Y_n$ are Kan fibrations. Suppose that every vertex $v \in Y_n$ and every ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ induce a weak equivalence* *(resp. homology isomorphism) of fibres*

$$F_{\theta^*(v)} \to F_v.$$

Then all pullback diagrams

$$\begin{tikzcd} f^{-1}(\sigma) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^{r,s} \arrow[r, "\sigma"'] & Y \end{tikzcd}$$

are homotopy (resp. homology) cartesian.

Proof. The idea is to show that all maps

$$(\gamma, \zeta)_* : f^{-1}(\tau) \to f^{-1}(\sigma)$$

induced by all bisimplex morphisms

$$\begin{tikzcd} \Delta^{k,l} \arrow[dr, "\tau"] \arrow[dd, "(\zeta,\gamma)"'] & \\ & X \\ \Delta^{r,s} \arrow[ur, "\sigma"'] & \end{tikzcd}$$

are weak equivalences (resp. homology isomorphisms)of bisimplicial sets. Then the desired result follows from Quillen's Theorem B (meaning Theorem 1.11 of Lecture 001) — see also [3, IV.5.7], or its homology analog [3, IV.5.11].

Pick a vertex $v \in \Delta^s$ and let $(1, v) : \Delta^{r,0} \to \Delta^{r,s}$ be the corresponding bisimplicial set map. Form the pullback diagrams

$$\begin{tikzcd}
\bigsqcup_{\theta:\mathbf{n}\to\mathbf{r}} f^{-1}(\theta^*(v)) \arrow[r, "v_*"] \arrow[d] & \bigsqcup_{\theta:\mathbf{n}\to\mathbf{r}} f^{-1}(\theta^*(\sigma)) \arrow[r] \arrow[d] & X_n \arrow[d, "f"] \\
\bigsqcup_{\theta:\mathbf{n}\to\mathbf{r}} \Delta^0 \arrow[r, "v"'] & \bigsqcup_{\theta:\mathbf{n}\to\mathbf{r}} \Delta^s \arrow[r] & Y_n
\end{tikzcd}$$

Then the induced map v_* is a weak equivalence (on each summand) since the simplicial set map $f : X_n \to Y_n$ is a fibration.

It therefore suffices to show that all bisimplex maps

$$\begin{tikzcd}
\Delta^{k,0} \arrow[d, "{(\zeta,1)}"'] \arrow[dr, "v"] & \\
\Delta^{r,0} \arrow[r, "w"'] & X
\end{tikzcd}$$

induce weak equivalences (resp. homology isomorphisms) $f^{-1}(v) \to f^{-1}(w)$. But there is a commutative diagram

$$\begin{tikzcd}
\bigsqcup_{\gamma:\mathbf{n}\to\mathbf{k}} f^{-1}(\gamma^*(v)) \arrow[r, "\zeta_*"] \arrow[d] & \bigsqcup_{\theta:\mathbf{n}\to\mathbf{r}} f^{-1}(\theta^*(w)) \arrow[d] \\
\bigsqcup_{\gamma:\mathbf{n}\to\mathbf{k}} f^{-1}(v) \arrow[r, "\zeta_*"'] & \bigsqcup_{\theta:\mathbf{n}\to\mathbf{r}} f^{-1}(w)
\end{tikzcd}$$

in which the vertical maps are weak equivalences (resp. homology isomorphisms) and the bisimplicial set map along the bottom diagonalizes to the weak equivalence (resp. homology isomorphism)

$$\Delta^k \times f^{-1}(v) \to \Delta^r \times f^{-1}(w)$$

and the result follows. $\square$

Corollary 10.2. *Suppose that $f : X \to Y$ is a map of bisimplicial sets such that every vertex $v \in Y_n$ and every ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ induce a weak equivalence (resp. homology isomorphism) of homotopy fibres*

$$F_{\theta^*(v)} \to F_v.$$

Then all pullback diagrams

$$\begin{tikzcd} f^{-1}(\sigma) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^{r,s} \arrow[r, "\sigma"'] & Y \end{tikzcd}$$

are homotopy (resp. homology) cartesian.

Proof. Replace the map f up to levelwise weak equivalence by a fibration p in the projective model structure for bisimplicial sets. Then p satisfies the assumptions of Lemma 10.1. $\square$

Corollary 10.3. *Suppose that $M \times X \to X$ an action of a simplicial monoid on a simplicial set X such that multiplication by the vertices $v \in M$ induce homology isomorphisms $v_* : X \to X$.*

Then the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & EM \times_M X \arrow[d] \\ * \arrow[r, "v"'] & BM \end{tikzcd}$$

is homology cartesian for each vertex $v \in M$.

Proof. Corollary 10.2 implies that it suffices to show, for each diagram

$$\begin{tikzcd} M^{\times n} \times X \arrow[r, "\theta^*"] \arrow[d, "pr"'] & M^{\times m} \times X \arrow[d, "pr"] \\ M^{\times n} \arrow[r, "\theta^*"'] & M^{\times m} \end{tikzcd}$$

and each vertex $v = (v_1, \dots, v_n) \in M^{\times n}$, that the induced map

$$(\theta, v) : X \to X$$

in homotopy fibres is a homology isomorphism. But this map is multiplication by the vertex

$$v_{\theta(0)-1} \cdot \dots \cdot v_2 \cdot v_1$$

and is therefore a homology isomorphism by assumption. $\square$

Example: Suppose that R is a ring and let

$$M = \bigsqcup_{n \geq 0} BGl_n(R)$$

be the simplicial monoid with monoid structure defined by block addition of matrices. Let I be the 1×1 identity matrix, and form the system of left M-modules

$$\bigsqcup_{n \geq 0} BGl_n(R) \xrightarrow{\oplus I} \bigsqcup_{n \geq 0} BGl_n(R) \xrightarrow{\oplus I} \dots$$

where $\oplus I$ denotes multiplication on the right by I. Then the colimit of this system is the simplicial set (and M-module)

$$N = \bigsqcup_{\mathbb{Z}} BGl(R).$$

and

$$EM \times_M N = \varinjlim_{n \geq 0} EM \times_M M$$

is contractible. The group homomorphisms

$$I\oplus, \oplus I : Gl_n(R) \to Gl_{n+1}(R)$$

are conjugate by a shuffle permutation, and therefore determine homotopic maps

$$BGl_n(R) \to BGl_{n+1}(R).$$

It follows that left multiplication by I induces a shift operator

$$\bigoplus_{\mathbb{Z}} h_*(BGl(R)) \to \bigoplus_{\mathbb{Z}} h_*(BGl(R)),$$

which is an isomorphism. In particular, multiplication by the vertices of $\bigsqcup_{n\geq 0} BGl_n(R)$ on the space N induces homology isomorphisms $N \to N$, and so the diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d] & EM \times_M N \arrow[d] \\ * \arrow[r, "I"'] & BM \end{tikzcd}$$

is homology cartesian. In particular, the induced map

$$\bigsqcup_{\mathbb{Z}} BGl(R) \to \Omega(B(\bigsqcup_{n\geq 0} BGl_n(R))$$

is a homology isomorphism.

Example: A completely analogous argument for the action of the monoid

$$M = \bigsqcup_{n\geq 0} B\Sigma_n$$

on the simplicial set

$$N = \bigsqcup_{\mathbb{Z}} B\Sigma_\infty$$

shows that there is a homology isomorphism

$$\bigsqcup_{\mathbb{Z}} B\Sigma_\infty \to \Omega(\bigsqcup_{n\geq 0} B\Sigma_n).$$

There is a special Γ-space model $X \mapsto \Gamma^+(X)$ for the sphere spectrum S which was introduced by Barratt, which has a weak equivalence

$$\Gamma^+(S^1) \simeq BM$$

(see [7], for example). It follows that there is a homology isomorphism

$$\bigsqcup_{\mathbb{Z}} B\Sigma_\infty \to QS^0$$

where QS^0 is the space at level 0 for a stably fibrant replacement QS for the sphere spectrum.

11 $Q = +$ theorem

Suppose that R is a unitary ring, and recall that $\mathcal{P}(R)$ denotes the exact category of finitely generated projective R-modules.

Write $K(R)$ for a stably fibrant model for the K-theory (symmetric) spectrum $K(\mathcal{P}(R))$. The space

$$K(R)^0 \simeq \Omega K(R)^1 \simeq \Omega BQ\mathcal{P}(R)$$

is an H-space with path components isomorphic to the classical group $K_0(R)$. Since $K(R)^0$ is an H-group (path components form a group), its path components are weakly equivalent, and there is a weak equivalence

$$K(R)^0 \simeq \bigsqcup_{K_0(R)} K(R)^0_0,$$

where $K(R)^0_0$ is the path component of $0 \in K_0(R)$. The path component $K(R)^0_0$ is itself an H-space, and the $Q = +$ theorem asserts the following:

Theorem 11.1. *There is a natural integral homology isomorphism*

$$BGl(R) \to K(R)^0_0.$$

In other words, the space $K(R)^0_0$ is an H-space with the integral homology of $BGl(R)$, and as such it is model for the classical plus construction $BGl(R)^+$ on $BGl(R)$. Thus, one usually sees the $Q = +$ theorem written as the assertion that there is a weak equivalence

$$\Omega BQ\mathcal{P}(R) \simeq \bigsqcup_{K_0(R)} BGl(R)^+.$$

Remark 11.2. We shall make a general study of integral homology isomorphisms $BGl(R) \to Y$ taking values in H-spaces Y in the next section of these notes, in lieu of fussing with the plus construction. It's relatively easy to see that all integral homology H-space models for a given space are weakly equivalent. The point, in applications, is to produce such a gadget $BGl(R) \to Y$ which is acyclic in the sense that its homotopy fibre F has trivial integral homology. The plus construction $BGl(R)^+$ is the traditional candidate for Y, but we'll see that the integral homology localization $L_{\mathbb{Z}}BGl(R)$ works just as well.

There are several proofs of the $Q = +$ theorem in the literature: the first appeared in [4], and more recent proofs appeared in [2] and [6]. These presentations each have particular advantages: Grayson's original proof in [4] is relatively easy to understand, the Gillet-Grayson proof in [2] makes direct use of Waldhausen's constructions, while the proof given in [6] is quite conceptual.

The proof from [6] will be presented here, in outline. There is a cost, in that one has to become comfortable with pseudo-simplicial groupoids arising from actions of symmetric monoidal categories, but these objects define homotopy types through a standard Grothendieck construction, and the arguments presented here proceed in a way that one would expect from manipulations of abelian group actions.

Here are the steps:

1) Recall (Remark 5.7 of Lecture 003) that there is a natural weak equivalence

$$B \operatorname{Iso} S_\bullet \mathcal{P}(R) \simeq s_\bullet \mathcal{P}(R) = K(R)^1, \tag{1}$$

and that there are weak equivalences of groupoids

$$\operatorname{Iso} S_n \mathcal{P}(R) \xrightarrow{\simeq} \operatorname{Iso} \operatorname{Mon}_n \mathcal{P}(R) \tag{2}$$

which are defined by taking the exact functors $P : \operatorname{Ar}(\mathbf{n}) \to \mathcal{P}(R)$ to the strings

$$P(0,1) \rightarrowtail P(0,2) \rightarrowtail \cdots \rightarrowtail P(0,n).$$

The object $\operatorname{Iso} \operatorname{Mon}_n \mathcal{P}(R)$ is the groupoid of isomorphisms of strings of admissible (aka. split) monomorphisms of length $n-1$, and these groupoids assemble into a pseudo-simplicial groupoid

$$\operatorname{Iso} \operatorname{Mon}_\bullet \mathcal{P}(R).$$

Further, the equivalences (2) define a pseudo-natural equivalence

$$\operatorname{Iso} S_\bullet \mathcal{P}(R) \xrightarrow{\simeq} \operatorname{Iso} \operatorname{Mon}_\bullet \mathcal{P}(R). \tag{3}$$

There is an exact functor

$$\mathcal{P}(R)^{\times n} \to \operatorname{Mon}_n \mathcal{P}(R)$$

which sends an n-tuple $(P_1, P_2, \dots, P_n)$ of projective modules to the string

$$P_1 \rightarrowtail P_1 \oplus P_2 \rightarrowtail \cdots \rightarrowtail P_1 \oplus \cdots \oplus P_n$$

which specializes to an equivalence of groupoids

$$(\operatorname{Iso}\mathcal{P}(R))^{\times n} \to \operatorname{Iso}\operatorname{Mon}_n \mathcal{P}(R).$$

These equivalences of groupoids together determine a pseudo-natural equivalence

$$B_{\oplus}(\operatorname{Iso}\mathcal{P}(R)) \simeq \operatorname{Iso}\operatorname{Mon}_n \mathcal{P}(R), \tag{4}$$

where $B_{\oplus}(\operatorname{Iso}\mathcal{P}(R))$ is the pseudo-simplicial category which is canonically determined by the symmetric monoidal category $\operatorname{Iso}(\mathcal{P}(R))$ with direct sum. The category in simplicial degree n is the groupoid $\operatorname{Iso}\mathcal{P}(R)^{\times n}$ and the faces and degeneracies of $B_{\oplus}\operatorname{Iso}(\mathcal{P}(R))$ are determined by the direct sum $\oplus$ functor and zero object 0 in exact analogy with the way that the faces and degeneracies of the classifying simplicial set BA are constructed from an abelian group by using the group addition and identity.

The equivalences (1), (3) and (4) together amount to a "delooping" of the $Q = +$ theorem.

2) The object $B_{\oplus}\operatorname{Iso}(\mathcal{P}(R))$ is a special case of a simplicial category construction BS that is available for any symmetric monoidal category S: BS_n is the category $S^{\times n}$, and the sum $\oplus : S \times S \to S$ and 0-object of S define the face and degeneracy functors.

Suppose that the functor

$$\oplus : S \times N \to S$$

defines a coherent action of a small symmetric monoidal category on a small category N. This means, in part, that there is a natural isomorphism

$$s_1 \oplus (s_2 \oplus x) \cong (s_1 \oplus s_2) \oplus x$$

which is suitably compatible with all associativity and symmetry isomorphisms of S. Such an action is (has to be) good enough to define a pseudo-simplicial object $ES \times_S N$ which can be defined analogously with the Borel construction, together with a pseudo-natural transformation

$$ES \times_S N \to BS$$

which is defined by the projection

$$S \times \cdots \times S \times N \to S \times \cdots \times S$$

(n factors) in simplicial degree n.

The symmetric monoidal category S has a coherent action on the pseudo-functor $ES \times_S N$, which is given in simplicial degree n by the assignment

$$(s, (s_1, \ldots, s_n, x)) \mapsto (s_1, \ldots, s_n), x \oplus s).$$

If A is an abelian group acting on a set X, then A acts on the translation category $E_A X$ by

$$(s, x \xrightarrow{t} tx) \mapsto sx \xrightarrow{t} tsx.$$

There are commutative diagrams

$$\begin{tikzcd} x \arrow[r, "s"] \arrow[d, "t"'] & sx \arrow[d, "t"] \\ tx \arrow[r, "s"'] & tsx \end{tikzcd}$$

so that the action by $s \in A$ on the simplicial set $EA \times_A X$ is homotopic to the identity.

In the same way (ie. via the same simplicial homotopy formulas) the objects of S act by the identity on $ES \times_S N$ in the homotopy category.

3) Generally, if the functor $M \times N \to N$ defines a coherent action by a small monoidal category M on a category N, then we are entitled to a Borel construction $EM \times_M N$ and a pseudo-natural transformation

$$\pi : EM \times_M N \to BM,$$

by the same formulas as above. Here's a consequence of Corollary 10.2:

Corollary 11.3. *Suppose given a coherent action $M \times N \to N$ by a small monoidal category*

M on a small category N. Suppose also that action by all objects of N induces weak equivalences (resp. homology isomorphisms) $N \to N$. Then BN is the homotopy (resp. homology) fibre of the map

$$\pi_* : B(EM \times_M N) \to B(BM).$$

4) Suppose that S is a small symmetric monoiodal category, and form the Borel construction

$$ES \times_S (S \times S)$$

for the diagonal action of S on $S \times S$. Then S acts on $ES \times_S (S \times S)$ via

$$(s, (s_1, \dots, s_n, (t_1, t_2))) \mapsto (s_1, \dots, s_n, (s \oplus t_1, t_2)).$$

This action is invertible in the homotopy category, with inverse

$$(s, (s_1, \dots, s_n, (t_1, t_2))) \mapsto (s_1, \dots, s_n, (t_1, s \oplus t_2))$$

by 3) above, and it follows from Corollary 11.3 that the maps

$$\begin{tikzcd} ES \times_S (S \times S) \arrow[r] & ES \times_S (S \times S) \times_l ES \arrow[d] \\ & BS \end{tikzcd}$$

induce a fibre sequence of spaces.

At the same time there are weak equivalences

$$\begin{aligned} ES \times_S (S \times S) \times_l ES &\cong ES \times_S ((ES \times_S S) \times S) \\ &\simeq ES \times_S S \\ &\simeq * \end{aligned}$$

The object $ES \times_S S$ is contractible in the same way that $EA \times_A A \cong EA$ is contractible for abelian groups A.

We have proved the following:

Theorem 11.4. *There is a weak equivalence*

$$B(ES \times_S (S \times S)) \simeq \Omega B(BS),$$

for each small symmetric monoidal category S.

By comparison with [4], the object $ES \times_S (S \times S)$ is a "generalized $S^{-1}S$-construction". It is also a homotopy theoretic group completion: it is a model for the space at level 0 in the stably fibrant spectrum associated to the symmetric monoidal category S by Γ-space methods.

5) Suppose that S acts coherently on a category N, as in 2) above (in particular, everthing is small).

There is a spectral sequence for

$$H_*(B(ES \times_S (N \times S)))$$

arising from the map

$$ES \times_S (N \times S) \to ES \times_S S.$$

This spectral sequence has

$$\begin{aligned} E_2^{p,q} &= H_p(ES \times_S S, H_q(BN)) \\ &\Rightarrow H_{p+q}(B(ES \times_S (N \times S))). \end{aligned}$$

We can localize at the action by the abelian monoid $\pi_0(S)$, which monoid acts invertibly on

$$H_*(B(ES \times_S (N \times S)))$$

via

$$(s, ((s_1, \ldots s_n, (n, t))) \mapsto (s_1, \ldots, s_n)(s \oplus n, t))$$

by construction. It follows that there is a spectral sequence

$$\begin{aligned} E_2^{p,q} &= H_p(ES \times_S S, \pi_0(S)^{-1} H_q(BN)) \\ &\Rightarrow H_{p+q}(B(ES \times_S (N \times S))). \end{aligned} \tag{5}$$

The action by all elements of $ES \times_S S$ on

$$\pi_0(S)^{-1} H_*(BN)$$

is invertible and $B(ES \times_S S)$ is contractible. This means that the spectral sequence (5) collapses, and so there is an isomorphism

$$H_*(B(ES \times_S (N \times S))) \cong \pi_0(S)^{-1} H_*(BN). \tag{6}$$

This isomorphism is induced by the functor

$$N \to N \times S$$

which is defined by $n \mapsto (n, 0)$.

6) What about $S = \operatorname{Iso}\mathcal{P}(R)$?

The map

$$S \to ES \times_S (S \times S)$$

defined by $P \mapsto (P, 0)$ induces the isomorphism

$$\pi_0(S)^{-1}H_*(BS) \cong H_*(B(ES \times_S (S \times S))).$$

The monoid $\pi_0(S)$ is isomorphism classes of projective modules, and the localization

$$\pi_0(S)^{-1}H_*(BS)$$

may be computed as the filtered colimit

$$H_*(BS) \xrightarrow{\oplus R} H_*(BS) \xrightarrow{\oplus R} \dots$$

since every projective module is a split summand of a free module.

The projective modules P, Q represent the same path component of

$$\pi_0(ES \times_S (S \times S)) = (\pi_0(S) \times \pi_0(S))/\pi_0(S) = K_0(R)$$

if and only if $P \oplus R^n \cong Q \oplus R^n$, meaning that P and Q are in the same stable equivalence class.

Write S_P for the groupoid of isomorphisms of projective modules stably equivalent to P. It follows that the homology of the component of 0 is computed by the filtered colimit

$$H_*BS_0 \xrightarrow{\oplus R} H_*BS_R \xrightarrow{\oplus R} H_*BS_{R\oplus R} \to \dots.$$

The comparison

$$\begin{array}{ccccc} BGl_1(R) & \longrightarrow & BGl_2(R) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \\ BS_R & \longrightarrow & BS_{R\oplus R} & \longrightarrow & \dots \end{array}$$

induces a homology isomorphism on colimits since every object of BS_{R^n} is stably free.

It follows that there is a homology isomorphism

$$BGl(R) \to B(ES \times_S (S \times S))_0,$$

which gives the $Q = +$ result.

12 H-spaces

Throughout this section, $H_*(X)$ denotes the integral homology of a space X.

An H-space will be a pointed simplicial set X, equipped with a pointed map $m : X \times X \to X$ such that the composite

$$X \vee X \overset{i}{\subset} X \times X \xrightarrow{m} X$$

is pointed homotopic to the fold map

$$\nabla : X \vee X \to X$$

which is the identity on each summand. The base point of an H-space X will be denoted by e.

Examples include the loop space

$$\Omega Y = \mathbf{hom}_*(S^1, Y)$$

of a pointed Kan complex Y and all simplicial groups.

I need a concept with a little less structure: a *multiplicative space* is a pointed simplicial set X with a pointed map $m : X \times X \to X$, and a multiplicative map is a pointed map $f : X \to Y$ such that the diagram

$$\begin{tikzcd} X \times X \arrow[r, "m_X"] \arrow[d, "f \times f"'] & X \arrow[d, "f"] \\ Y \times Y \arrow[r, "m_Y"'] & Y \end{tikzcd} \tag{7}$$

commutes.

Note that if X is a multiplicative space and the trivial cofibration $j : X \to Y$ is a fibrant model of X, then Y acquires a multiplicative structure in such a way that j is a multiplicative map. In effect, there is an extension in the diagram

$$\begin{tikzcd} X \times X \arrow[r, "m_X"] \arrow[d, "j\times j"'] & X \arrow[d, "j"] \\ Y \times Y \arrow[r, dotted, "m_Y"'] & Y \end{tikzcd}$$

since $j \times j$ is a trivial cofibration and Y is fibrant.

Note that if X is an H-space then the fibrant model Y has the structure of an H-space, since the commutative diagram

$$\begin{tikzcd} X \vee X \arrow[r, "\nabla"] \arrow[d, "j\vee j"', "\simeq"] & X \arrow[d, "j", "\simeq"'] \\ Y \vee Y \arrow[r, "m_Y \cdot i"'] & Y \end{tikzcd}$$

forces the map $m_Y \cdot i$ to be ∇ in the pointed homotopy category.

Lemma 12.1. *Suppose that $f : X \to Y$ is a multiplicative map of connected multiplicative spaces, where Y is an H-space. Suppose also that f is an integral homology isomorphism and that the induced homomorphism $\pi_1(X) \to \pi_1(Y)$ is surjective. Then f is acyclic in the sense that its homotopy fibre has the integral homology of a point.*

There is a rather extensive study of acyclic maps in [5].

Proof. We can assume that the multiplicative space Y is fibrant, by the remarks above.

We can also assume that the map f is a fibration. To see this, find a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

such that p is a fibration and i is a trivial cofibration, and observe that the lifting m_Z exists in the diagram

$$\begin{tikzcd} X \times X \arrow[r, "i \cdot m_X"] \arrow[d, "i \times i"'] & Z \arrow[d, "p"] \\ Z \times Z \arrow[r, "m_Y(p \times p)"'] \arrow[ur, dotted, "m_Z"] & Y \end{tikzcd}$$

since p is a fibration and $i \times i$ is a trivial fibration. It follows that the fibration p is a multiplicative map which is weakly equivalent to the homology isomorphism f.

Suppose henceforth that f is a fibration with fibre F over the base point $e \in Y$, and that Y is fibrant. It follows that X is fibrant.

Suppose that the loop $\alpha : \Delta^1 \to Y$ represents an element $[\alpha]$ of $\pi_1(Y, e)$, and let the space $f^{-1}(\alpha)$

be defined by the pullback diagram

$$\begin{tikzcd}
f^{-1}(\alpha) \arrow[r, "\alpha_*"] \arrow[d, "p_*"'] & X \arrow[d, "f"] \\
\Delta^1 \arrow[r, "\alpha"'] & Y
\end{tikzcd}$$

I claim that the induced weak equivalences

$$F \xrightarrow{d^1_\alpha} f^{-1}(\alpha) \xleftarrow{d^0_\alpha} F$$

induce the identity morphism

$$H_*(F) \to H_*(F).$$

This would imply that the fundamental groupoid of Y acts trivially on $H_*(F)$.

To repeat a standard fact, the question of whether or not $(d^0_\alpha)^{-1}d^1_\alpha$ is the identity on $H_*(F)$ is independent of the choice of representative α for the homotopy element $[\alpha] \in \pi_1(Y, e)$. If $\alpha' : \Delta^1 \to Y$ is a second representative for $[\alpha]$ then there is a 2-simplex $\sigma : \Delta^2 \to Y$ such that

$$\partial(\sigma) = (d_0\sigma, d_1\sigma, d_2\sigma) = (e, \alpha, \alpha'),$$

and computing relative to the pullback $f^{-1}(\sigma)$ shows that

$$(d^0_e)^{-1}d^1_e(d^0_\alpha)^{-1}d^1_\alpha = (d^0_{\alpha'})^{-1}d^1_{\alpha'}$$

as morphisms $F \to F$ in the homotopy category, while $(d^0_e)^{-1}d^1_e = 1$.

The map $\pi_1(X) \to \pi_1(Y)$ is surjective by assumption. Thus, if the loop $\alpha : \Delta^1 \to Y$ represents an element $[\alpha] \in \pi_1(Y, e)$, there is a lifting

$$\begin{tikzcd} & X \arrow[d, "f"] \\ \Delta^1 \arrow[ur, dotted, "\alpha'"] \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

of α to a loop α' of X.

Let α'_* be the composite

$$F \times \Delta^1 \xrightarrow{i \times \alpha'} X \times X \xrightarrow{m_X} X,$$

where $i : F \to X$ is the inclusion of the fibre. Then the square

$$\begin{tikzcd} F \times \Delta^1 \arrow[r, "\alpha'_*"] \arrow[d, "pr"'] & X \arrow[d, "f"] \\ \Delta^1 \arrow[r, "m_Y(e,\alpha)"'] & Y \end{tikzcd}$$

commutes, so there is a unique map

$$\theta : F \times \Delta^1 \to f^{-1}(\tilde{\alpha}),$$

where $\tilde{\alpha} = m(e, \alpha))$ and so $[\alpha] = [\tilde{\alpha}]$ in $\pi_1(Y, e)$ since Y is an H-space.

There is a commutative diagram of weak equivalences

$$\begin{tikzcd}
F \arrow[r, "d^1"] \arrow[d, "\theta_1"'] & F \times \Delta^1 \arrow[d, "\theta"] & F \arrow[l, "d^0"'] \arrow[d, "\theta_0"] \\
F \arrow[r, "d^1_{\tilde{\alpha}}"'] & f^{-1}(\tilde{\alpha}) & F \arrow[l, "d^0_{\tilde{\alpha}}"]
\end{tikzcd}$$

But finally, the maps θ_0 and θ_1 are induced by the composites

$$F \xrightarrow{d^i} F \times \Delta^1 \xrightarrow{1\times\alpha'} X \times X \xrightarrow{m_*} X,$$

and these composites coincide, so that $\theta_0 = \theta_1$. It follows that $(d^0_{\tilde{\alpha}})^{-1} d^1_{\tilde{\alpha}}$ is the identity in the homotopy category, and hence induces the identity in homology, as required.

The space F is connected, by the connectivity assumption on X and the surjectivity of the homomorphism $\pi_1(X) \to \pi_1(Y)$.

Since the fundamental groupoid $\pi(Y)$ acts trivially on $H_*(F)$, the Serre spectral sequence for the fibration $f : X \to Y$ has the standard form:

$$E_2^{p,q} = H_p(Y, H_q(F)) \Rightarrow H_{p+q}(X).$$

The assumption that $f : X \to Y$ is a homology isomorphism implies that the quotient map $H_n(X) \twoheadrightarrow E_\infty^{n,0}$ is an isomorphism and that $E_2^{n,0} = E_r^{n,0} = E_\infty^{n,0}$ for all $n \geq 0$. In particular, all differentials defined on the $(n,0)$ line are trivial.

The sequences

$$E_2^{2,0} \xrightarrow{d_2=0} E_2^{0,1} \to E_\infty^{0,1} \to 0$$

and

$$0 \to E_\infty^{0,1} \to H_1(X) \xrightarrow{\cong} E_\infty^{1,0} \to 0$$

are exact, so that

$$H_1(F) = E_2^{0,1} \cong E_\infty^{0,1} \cong 0$$

Similarly, one can show that all higher homology groups of F vanish. □

There is a model structure on the category $s\mathbf{Set}$ (hence also for pointed simplicial sets), for which the weak equivalences $X \to Y$ are those maps which induce integral homology isomorphisms

$$H_*(X) \cong H_*(Y),$$

and the cofibrations are the monomorphisms. This model structure was originally introduced by Bousfield in [1], but has since been written up in many places, for example in [3]. The model structure is cofibrantly generated so that fibrant replacement is functorial: there is a natural cofibration $j : X \to L_{\mathbb{Z}}(X)$ such that j is an integral homology isomorphism and $L_{\mathbb{Z}}(X)$ is fibrant for this model structure.

The space $L_{\mathbb{Z}}(X)$ is called the integral homology localization of X, and spaces X for which the fibrant replacement j_X is a weak equivalence are said to be *integral homology local.* Here's an observation:

Corollary 12.2. *Suppose that X is an H-space. Then X is integral homology local.*

Proof. The map

$$j_X \times j_X : X \times X \to L_{\mathbb{Z}}(X) \times L_{\mathbb{Z}}(X)$$

is a cofibration and an H_*-isomorphism, so that the extension exists in the diagram

$$\begin{tikzcd} X \times X \arrow[r, "m_X"] \arrow[d, "j_X \times j_X"'] & X \arrow[d, "j_X"] \\ L_{\mathbb{Z}}(X) \times L_{\mathbb{Z}}(X) \arrow[r, dotted, "m_*"'] & L_{\mathbb{Z}}(X) \end{tikzcd}$$

It follows that there is an H-space structure on $L_{\mathbb{Z}}(X)$ for which the homology isomorphism j_X is multiplicative.

Then the Corollary follows from Lemma 12.1. In effect, the map

$$\pi_1(X) \to \pi_1(L_{\mathbb{Z}}(X))$$

is an isomorphism so that the map j_X is acyclic. The fundamental group $\pi_1(F)$ of the homotopy fibre F is abelian by a long exact sequence argument, so that

$$\pi_1(F) \cong H_1(F) = 0.$$

The space F is therefore simply connected with $\tilde{H}_*(F) = 0$, so that F is contractible by the Hurewicz Theorem. □

Corollary 12.3. *Suppose that $f : X \to Y$ is an integral homology isomorphism between connected H-spaces. Then f is a weak equivalence.*

Proof. There is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j_X"', "\simeq"] & Y \arrow[d, "\simeq"', "j_Y"] \\ L_{\mathbb{Z}}(X) \arrow[r, "f_*"'] & L_{\mathbb{Z}}(Y) \end{tikzcd}$$

in which the maps j_X and j_Y are weak equivalences by Corollary 12.2, and f_* is a weak equivalence by assumption. □

Now suppose that R is a ring, and define a group homomorphism

$$\oplus : Gl(R) \times Gl(R) \to Gl(R)$$

by sending the pair of matrices (A, B) to the matrix $A \oplus B$, where

$$(A \oplus B)_{i,j} = \begin{cases} A_{k,l} & \text{if } i = 2k-1,\ j = 2l-1, \\ B_{k,l} & \text{if } i = 2k,\ j = 2l, \text{ and} \\ \delta_{i,j} & \text{otherwise.} \end{cases}$$

Then there is a commutative diagram of ring homomorphisms

$$\begin{tikzcd} Gl(R) \arrow[r, "in_L"] \arrow[dr, "u_{odd}"'] & Gl(R) \times Gl(R) \arrow[d, "\oplus"] & Gl(R) \arrow[l, "in_R"'] \arrow[dl, "u_{ev}"] \\ & Gl(R) & \end{tikzcd}$$

Here, $in_L(A) = (A, e)$ and $in_R(B) = (e, B)$ where e is the identity of $Gl(R)$, and u_{odd} and u_{ev} are names for the displayed composites.

Any injective function $v : \mathbb{N} \to \mathbb{N}$ defines a homomorphism

$$v_* : Gl(R) \to Gl(R)$$

with

$$v_*(A)_{i,j} = \begin{cases} A_{k,l} & \text{if } i = v(k)\ j = v(l), \text{ and} \\ \delta_{i,j} & \text{otherwise.} \end{cases}$$

From this point of view, u_{ev} is the homomorphism corresponding to the usual injection $\mathbb{N} \to \mathbb{N}$ which picks off the even numbers, and u_{odd} arises from the injection corresponding to the odd numbers.

Lemma 12.4. *The group homomorphism $v_* : Gl(R) \to Gl(R)$ induces a homology isomorphism $BGl(R) \to BGl(R)$.*

Proof. Suppose that m is an upper bound for the set of numbers $v(\underline{n})$. Then $v_*(Gl_n(R)) \subset Gl_m(R)$ and there is a commutative diagram of group homomorphisms

$$\begin{array}{ccc} Gl_n(R) & \longrightarrow & Gl(R) \\ {\scriptstyle v_*}\downarrow & & \downarrow{\scriptstyle v_*} \\ Gl_m(R) & \longrightarrow & Gl(R) \end{array}$$

in which the horizontal maps are canonical inclusions. The map $v_* : Gl_n(R) \to Gl_m(R)$ is conjugate via some permutation matrix to the canonical inclusion $Gl_n(R) \subset Gl_m(R)$.

Thus if $\alpha \in H_*(BGl_n(R))$ is in the kernel of the map

$$H_*(BGl_n(R)) \to H_*(BGl(R) \xrightarrow{v_*} H_*(BGl(R))$$

then $v_*(\alpha) = 0$ in $H_*(BGl_m(R))$ for some upper bound m on the set $v(\underline{n})$, and so $\alpha \mapsto 0$ in $H_*(BGl_m(R)$ under the canonical map $Gl_n(R) \to Gl_m(R)$. Thus, α represents 0 in $H_*(BGl(R))$.

Given $\beta \in H_*(BGl_n(R)$, choose an upper bound m on the set $v(\underline{n})$, and observe that the image of

β under the canonical map

$$H_*(BGl_n(R)) \to H_*(BGl_m(R))$$

is also in the image of

$$v_* : H_*(BGl_n(R)) \to H_*(BGl_m(R)),$$

because the maps are the same. It follows that the image of β in $H_*(BGl(R))$ is in the image of v_*. $\square$

The injective function $v : \mathbb{N} \to \mathbb{N}$ therefore induces a homotopy category isomorphism

$$v_* : L_{\mathbb{Z}}Bl(R) \to L_{\mathbb{Z}}BGl(R).$$

Lemma 12.5. *The group completion of the monoid* $\mathrm{Mon}(\mathbb{N})$ *of monomorphisms* $v : \mathbb{N} \to \mathbb{N}$ *is trivial.*

Corollary 12.6. *The direct sum homomorphism* $\oplus : Gl(R) \times Gl(R) \to Gl(R)$ *gives the space* $L_{\mathbb{Z}}BGl(R)$ *the structure of an* H*-space.*

Proof. Define a map

$$\oplus : L_{\mathbb{Z}}BGl(R) \times L_{\mathbb{Z}}BGl(R) \to L_{\mathbb{Z}}BGl(R)$$

by solving the extension problem

$$\begin{tikzcd} BGl(R) \times BGl(R) \arrow[r, "\oplus"] \arrow[d, "j \times j"'] & BGl(R) \arrow[d, "j"] \\ L_{\mathbb{Z}}BGl(R) \times L_{\mathbb{Z}}BGl(R) \arrow[r, dotted, "\oplus"'] & L_{\mathbb{Z}}BGl(R) \end{tikzcd}$$

Then the diagram

$$\begin{tikzcd}
BGl(R) \arrow[r, "u_{odd}"] \arrow[d, "j"'] & BGl(R) \arrow[d, "j"] \\
L_{\mathbb{Z}}BGl(R) \arrow[r, "\oplus\cdot in_L"'] & L_{\mathbb{Z}}BGl(R)
\end{tikzcd}$$

commutes, so that $\oplus \cdot in_L$ and u_{odd*} coincide in the pointed homotopy category. But then Lemma 12.5 implies that $\oplus \cdot in_L$ is pointed homotopic to the identity. Similarly

$$\oplus \cdot in_R \simeq u_{ev*} \simeq 1,$$

and so $L_{\mathbb{Z}}BGl(R)$ has the desired H-space structure $\square$

Proof of Lemma 12.5. If $u : \mathbb{N} \to \mathbb{N}$ has infinitely many fixed points then enumerating fixed points gives a morphism $i : \mathbb{N} \to \mathbb{N}$ such that $u \cdot i = i$. It follows that $u = e$ in the associated group.

For general case, given a morphism $v : \mathbb{N} \to \mathbb{N}$, find a morphism $u : \mathbb{N} \to \mathbb{N}$ such that $v \cdot u$ and u have infinitely many fixed points. Then if follows that $v = e$ in the associated group.

Starting with $a_1 = 1$, one inductively finds a sequence of numbers n_i, a_i, b_i such that

$$n_i + 1 \leq a_{i+1}, v(a_{i+1}), b_{i+1} \leq n_i,$$

and such that $b_{i+1} \neq a_{i+1}, v(a_{i+1})$. Then one defines bijections

$$u : [n_i + 1, n_{i+1}] \to [n_i + 1, n_{i+1}]$$

on intervals such that $u(v(a_{i+1})) = a_{i+1}$ and $u(b_{i+1}) = b_{i+1}$. Piecing together these bijections gives a bijection $u : \mathbb{N} \to \mathbb{N}$ such that $v(u(v(a_i))) = v(a_i)$ and $u(b_i) = b_i$. $\square$

Proposition 12.7. *The integral homology localization map*

$$j : BGl(R) \to L_{\mathbb{Z}}BGl(R)$$

is acyclic.

Proof. The map j is multiplicative by construction, and is an integral homology isomorphism. The space $L_{\mathbb{Z}}BGl(R)$ is an H-space by Corollary 12.6, and so the homomorphism

$$Gl(R) = \pi_1(BGl(R)) \to \pi_1(L_{\mathbb{Z}}BGl(R))$$

is surjective. The map j is therefore acyclic by Lemma 12.1. $\square$

An application:

Let $E(R)$ denote the subgroup of $Gl(R)$ which is generated by elementary transformation matrices

$e_{i,j}(a)$, $i \neq j$, $a \in R$. $E(R)$ is often called the *elementary subgroup* of $Gl_n(R)$

Every elementary transformation has determinant 1, so that $E(R) \subset Sl(R)$.

Lemma 12.8. *1) The subgroup $E(R)$ is perfect.*

2) (Whitehead lemma) $E(R) = [Gl(R), Gl(R)]$.

Proof. Statement 1) follows from the identities

$$[e_{i,j}(a), e_{j,k}(b)] = e_{i,h}(ab)$$

which hold for $i \neq k$.

For statement 2), we have the matrix identities:

$$\begin{bmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{bmatrix}$$

$$\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -A^{-1} & I \end{bmatrix} \begin{bmatrix} I & A \\ 0 & A \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$$

for $A, B \in Gl_n(R)$, and where I is the $(n \times n)$ identity matrix. All matrices of the form

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \text{ and } \begin{bmatrix} I & 0 \\ B & I \end{bmatrix}$$

are in $E(R)$. $\square$

The path component $K(R)^0_0$ of $0 \in K_0(R)$ in the space

$$K(R)^0 = \Omega BQ\mathcal{P}(R)$$

is an H-space which is equipped with an integral homology isomorphism

$$BGl(R) \to K(R)^0_0,$$

by the $Q = +$ Theorem (Theorem 11.1). H-spaces are local for integral homology theory (Corollary 12.2), so there is a natural weak equivalence

$$L_{\mathbb{Z}}BGl(R) \simeq K(R)^0_0,$$

We can therefore identify the map $BGl(R) \to K(R)^0_0$ given by $Q = +$, up to weak equivalence, with the integral homology localization map

$$j : BGl(R) \to L_{\mathbb{Z}}BGl(R).$$

In particular, there are isomorphisms

$$K_i(R) \cong \pi_i L_{\mathbb{Z}}BGl(R)$$

for $i \geq 1$ which are natural in rings R.

Proposition 12.9. *There are natural isomorphisms*

$$K_1(R) \cong Gl(R)/E(R)$$

and

$$K_2(R) \cong H_2(E(R), \mathbb{Z})$$

for rings R.

Proof. The group

$$K_1(R) \cong \pi_1(L_{\mathbb{Z}} BGl(R))$$

is isomorphic to

$$Gl(R)/[Gl(R), Gl(R)] \cong Gl(R)/E(R)$$

by Lemma 12.8, and the fact that j is a homology isomorphism taking values in an H-space (Corollary 12.6).

Form the pullback diagrams

$$\begin{tikzcd} X \arrow[r, "j_*"] \arrow[d] & Y \arrow[r] \arrow[d] & E(Gl(R)/E(R)) \arrow[d, "\pi"] \\ BGl(R) \arrow[r, "j"'] & L_{\mathbb{Z}} BGl(R) \arrow[r, "f"'] & B(Gl(R)/E(R)) \end{tikzcd}$$

where f is an isomorphism on fundamental groups. Then Y is the universal cover of $L_{\mathbb{Z}} BGl(R)$ and X is the homotopy fibre of the map $BGl(R) \to B(Gl(R)/E(R))$ which is induced by the canonical homomorphism $Gl(R) \to Gl(R)/E(R)$. It follows that there is a weak equivalence

$$X \simeq BE(R).$$

The map j_* has the same homotopy fibre as does j, and is therefore acyclic by Proposition 12.7. A spectral sequence argument shows the map j_* is an integral homology isomorphism, so that j_* induces an isomorphism

$$H_2(BE(R)) \cong H_2(Y). \tag{8}$$

But

$$H_2(Y) \cong \pi_2(Y) \tag{9}$$

by the Hurewicz Theorem, and the map $Y \to L_Z BGl(R)$ induces an isomorphism

$$\pi_2(Y) \cong \pi_2(L_{\mathbb{Z}} BGl(R)) \cong K_2(R). \tag{10}$$

The equivalences (8), (9) and (10) together give the desired result. $\square$

If R is a field F, then $E(F) = Sl(F)$, and the group

$$K_2(F) = H_2(BSl(F), \mathbb{Z})$$

(also called the Schur multiplier of the infinite special linear group $Sl(F)$) has a the presentation given by Matsumoto's thesis:

$$K_2(F) \cong F^* \otimes F^* / \langle t \otimes 1 - t |\ t \neq 1 \rangle.$$

See [8] for a proof (and for a lot of other things about K-theory in low degrees).

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