Lecture 006 (March 14, 2008)

13 Homotopy fibres

Here is another consequence of the Additivity Theorem:

Theorem 13.1. *Suppose that $f : \mathbf{M} \to \mathbf{N}$ is an exact functor. Then the square*

$$\begin{array}{ccc} s_\bullet(S_\bullet(\mathbf{M}) \times_{S_\bullet(\mathbf{N})} ES_\bullet(\mathbf{N})) & \longrightarrow & s_\bullet ES_\bullet(\mathbf{N}) \\ \downarrow & & \downarrow d_* \\ s_\bullet S_\bullet(\mathbf{M}) & \xrightarrow[f_*]{} & s_\bullet S_\bullet(\mathbf{N}) \end{array}$$

is homotopy cartesian.

Recall from Section 8 (Lecture 004) that if D is a simplicial object in some category, then ED is a simplicial object in that same category with $ED_n = D_{n+1}$, and with simplicial structure maps $\tilde{\theta}^*$ induced by the ordinal number maps

$$\tilde{\theta} = \mathbf{0} * \theta : \mathbf{m+1} \cong \mathbf{0} * \mathbf{m} \to \mathbf{0} * \mathbf{n} \cong \mathbf{n+1}.$$

There is a natural map $d : ED \to D$ which is defined by the simplicial structure maps $d_0 : D_{n+1} \to D_n$.

Proof of Theorem 13.1. By the Bousfield-Friedlander Theorem [1, IV.4.9], it suffices to show that all diagrams

$$\begin{array}{ccc} s_\bullet(S_n(\mathbf{M}) \times_{S_n(\mathbf{N})} ES_n(\mathbf{N})) & \longrightarrow & s_\bullet ES_n(\mathbf{N}) \\ \downarrow & & \downarrow{\scriptstyle d_*} \\ s_\bullet S_n(\mathbf{M}) & \xrightarrow[f_*]{} & s_\bullet S_n(\mathbf{N}) \end{array}$$

are homotopy cartesian. On account of the natural exact equivalences

$$S_n(\mathbf{P}) \xrightarrow{\simeq} \mathrm{Mon}_n(\mathbf{P})$$

(for all exact categories $\mathbf{P}$) and the exact equivalence

$$S_n(\mathbf{M}) \times_{S_n(\mathbf{N})} S_{n+1}(\mathbf{N}) \xrightarrow{\simeq} \mathrm{Mon}_n(\mathbf{M}) \times_{\mathrm{Mon}_n(\mathbf{N})} \mathrm{Mon}_{n+1}(\mathbf{N})$$

it suffices to show that the diagrams

$$\begin{array}{ccc} s_\bullet(\mathrm{Mon}_n(\mathbf{M}) \times_{\mathrm{Mon}_n(\mathbf{N})} \mathrm{Mon}_{n+1}(\mathbf{N})) & \longrightarrow & s_\bullet \mathrm{Mon}_{n+1}(\mathbf{N}) \\ \downarrow & & \downarrow{\scriptstyle d_*} \\ s_\bullet \mathrm{Mon}_n(\mathbf{M}) & \xrightarrow[f_*]{} & s_\bullet \mathrm{Mon}_n(\mathbf{N}) \end{array}$$

are homotopy cartesian, where d_* takes the string of admissible monics

$$P_1 \rightarrowtail P_2 \rightarrowtail \cdots \rightarrowtail P_{n+1}$$

to the string

$$P_2/P_1 \rightarrowtail P_3/P_1 \rightarrowtail \cdots \rightarrowtail P_{n+1}/P_1.$$

Fix zero objects 0 in $\mathbf{M}$ and $\mathbf{N}$ such that $f(0) = 0$. To more properly define d_*, we make a fixed choice of quotient Q/P for all admissible monics $Q \rightarrowtail P$ of $\mathbf{N}$. In particular, we can suppose that $P/0 = P$ and $Q/Q = 0$ for all objects $P, Q \in \mathbf{N}$.

The objects of the exact category

$$\mathrm{Mon}_n(\mathbf{M}) \times_{\mathrm{Mon}_n(\mathbf{N})} \mathrm{Mon}_{n+1}(\mathbf{N})$$

are pairs (P, Q) where P is a string

$$P : P_1 \rightarrowtail \cdots \rightarrowtail P_n$$

of admissible monics of length n in $\mathbf{M}$, Q is a string

$$Q : Q_1 \rightarrowtail \cdots \rightarrowtail Q_{n+1}$$

of admissible monics of length $n+1$ in $\mathbf{N}$, and $d(Q) = f_*(P)$. In particular, the assignment $P \mapsto (P, 0 \rightarrowtail f_*(P))$ defines a functor

$$i : \mathrm{Mon}_n(\mathbf{M}) \to \mathrm{Mon}_n(\mathbf{M}) \times_{\mathrm{Mon}_n(\mathbf{N})} \mathrm{Mon}_{n+1}(\mathbf{N})$$

which is a section of the projection functor

$$pr : \mathrm{Mon}_n(\mathbf{M}) \times_{\mathrm{Mon}_n(\mathbf{N})} \mathrm{Mon}_{n+1}(\mathbf{N}) \to \mathrm{Mon}_n(\mathbf{M})$$

The assignment of the pair $(0, Q)$ consisting of the string of identities

$$Q : Q \overset{1}{\rightarrowtail} \ldots \overset{1}{\rightarrowtail} Q$$

of length $n+1$ in $\mathbf{N}$ and the zero string

$$0 : 0 \rightarrowtail \cdots \rightarrowtail 0$$

of length n in $\mathbf{M}$ defines an exact functor

$$j : \mathbf{N} \to \mathrm{Mon}_n(\mathbf{M}) \times_{\mathrm{Mon}_n(\mathbf{N})} \mathrm{Mon}_{n+1}(\mathbf{N})$$

which is a section of the functor

$$\sigma : \mathrm{Mon}_n(\mathbf{M}) \times_{\mathrm{Mon}_n(\mathbf{N})} \mathrm{Mon}_{n+1}(\mathbf{N}) \to \mathbf{N}$$

which takes the object (P, Q) to Q_1.

The idea is, finally, to show that the map

$$s_\bullet(\mathrm{Mon}_n(\mathbf{M}) \times_{\mathrm{Mon}_n(\mathbf{N})} \mathrm{Mon}_{n+1}(\mathbf{N})) \xrightarrow{(pr_*, \sigma_*)} s_\bullet \mathrm{Mon}_n(\mathbf{M}) \times s_\bullet(\mathbf{N})$$

is a weak equivalence.

There is a natural exact sequence

$$0 \to (0, Q_1) \to (P, Q) \to (P, 0 \rightarrowtail f_*(P)) \to 0$$

The Additivity Theorem (Theorem 6.1, Lecture 003) therefore implies that the composite

$$\begin{array}{ccc} s_\bullet(\mathrm{Mon}_n(\mathbf{M}) \times_{\mathrm{Mon}_n(\mathbf{N})} \mathrm{Mon}_{n+1}(\mathbf{N})) & & \\ & \searrow^{(pr_*, \sigma_*)} & \\ \downarrow & & s_\bullet(\mathrm{Mon}_n(\mathbf{M})) \times s_\bullet(\mathbf{N}) \\ & \swarrow_{i_* + j_*} & \\ s_\bullet(\mathrm{Mon}_n(\mathbf{M}) \times_{\mathrm{Mon}_n(\mathbf{N})} \mathrm{Mon}_{n+1}(\mathbf{N})) & & \end{array}$$

is homotopic to the identity. The composite

$$(pr_*, \sigma_*)(i_* + j_*)$$

is easily seen to be homotopic to the identity. $\square$

Recall that there is a weak equivalence

$$s_\bullet ES_\bullet(\mathbf{M}) \simeq s_\bullet(\mathbf{0}),$$

and the space $s_\bullet(\mathbf{0})$ is contractible. Theorem 13.1 therefore gives an identification of the space

$$s_\bullet(S_\bullet(\mathbf{M}) \times_{S_\bullet(\mathbf{N})} ES_\bullet(\mathbf{N}))$$

with the homotopy fibre of the map

$$f_* : s^2_\bullet(\mathbf{M}) \to s^2_\bullet(\mathbf{N}).$$

A different point of view is possible: the exact equivalences

$$ES_n(\mathbf{M}) = S_{n+1}(\mathbf{M}) \xrightarrow{\simeq} \mathrm{Mon}_{n+1}(\mathbf{M})$$

identify the simplicial exact category $ES_\bullet(\mathbf{M})$ with a simplicial exact category $\mathcal{B}\,\mathrm{Mon}(\mathbf{M})$ whose category $\mathcal{B}\,\mathrm{Mon}(\mathbf{M})_n$ of n-simplices has objects given by all strings

$$P: \ P_0 \rightarrowtail P_1 \rightarrowtail \cdots \rightarrowtail P_n.$$

The morphisms of this category are natural transformations, and the exact sequences are the pointwise exact sequences. Suppose that $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map. Then one can show that the functor

$$\theta^* : \mathcal{B}\,\mathrm{Mon}(\mathbf{M})_n \to \mathcal{B}\,\mathrm{Mon}(\mathbf{M})_m$$

which takes the string P to the string

$$\theta^*(P) : \ P_{\theta(0)} \rightarrowtail P_{\theta(1)} \rightarrowtail \cdots \rightarrowtail P_{\theta(m)}$$

commutes with the exact equivalences

$$ES_\bullet(\mathbf{M})_n \simeq \mathcal{B}\,\mathrm{Mon}(\mathbf{M})_n$$

in the sense that diagram

$$\begin{array}{ccc} ES_\bullet(\mathbf{M})_n & \xrightarrow{\simeq} & \mathcal{B}\,\mathrm{Mon}(\mathbf{M})_n \\ {\scriptstyle \theta^*}\downarrow & & \downarrow{\scriptstyle \theta^*} \\ ES_\bullet(\mathbf{M})_m & \xrightarrow[\simeq]{} & \mathcal{B}\,\mathrm{Mon}(\mathbf{M})_m \end{array}$$

commute, so there is a simplicial exact equivalence

$$ES_\bullet(\mathbf{M}) \simeq \mathcal{B}\,\mathrm{Mon}(\mathbf{M}).$$

Finally, for a fixed choice of zero object 0, there is a simplicial contracting homotopy

$$\begin{array}{ccccccc} 0 & \rightarrowtail & 0 & \rightarrowtail & \cdots & \rightarrowtail & 0 \\ \downarrow & & \downarrow & & & & \downarrow \\ P_0 & \rightarrowtail & P_1 & \rightarrowtail & \cdots & \rightarrowtail & P_n \end{array}$$

for the simplicial exact category $\mathcal{B}\,\mathrm{Mon}(\mathbf{M})$. It follows that the space

$$s_\bullet ES_\bullet(\mathbf{M})$$

is contractible.

There is a "dual" construction. Let X be a simplicial object, and write $E'X$ for the simplicial object with

$$E'X_n = X_{n+1},$$

and with simplicial structure maps induced by the ordinal number morphisms

$$\mathbf{m}+\mathbf{1} \cong \mathbf{m} * \mathbf{0} \xrightarrow{\theta * \mathbf{0}} \mathbf{n} * \mathbf{0} \cong \mathbf{n}+\mathbf{1}.$$

The ordinal number morphisms

$$d^{n+1} : \mathbf{n} \to \mathbf{n} * \mathbf{0} \cong \mathbf{n}+\mathbf{1}$$

induce a natural morphism of simplicial objects

$$d : E'X \to X.$$

There is an exact equivalence

$$S_{n+1}(\mathbf{M}) \to \mathrm{Epi}_{n+1}(\mathbf{M}),$$

which is defined by taking an exact functor $P : \mathrm{Ar}(\mathbf{n}+\mathbf{1}) \to \mathbf{M}$ to the string of admissible epimorphisms

$$P(0, n+1) \twoheadrightarrow P(1, n+1) \twoheadrightarrow \cdots \twoheadrightarrow P(n, n+1).$$

Write $\mathcal{B}\,\mathrm{Epi}(\mathbf{M})_n = \mathrm{Epi}_{n+1}(\mathbf{M})$ for the exact category whose objects are all strings

$$P : \; P_0 \twoheadrightarrow P_1 \twoheadrightarrow \cdots \twoheadrightarrow P_n.$$

If $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, write $\theta^*(P)$ for the string

$$P_{\theta(0)} \twoheadrightarrow P_{\theta(1)} \twoheadrightarrow \cdots \twoheadrightarrow P_{\theta(m)}.$$

Then the assignment $P \mapsto \theta^*(P)$ defines an exact functor, and the diagram

$$\begin{array}{ccc} S_{n+1}(\mathbf{M}) & \xrightarrow{\simeq} & \mathcal{B}\,\mathrm{Epi}(\mathbf{M})_n \\ {\scriptstyle (\theta * \mathbf{0})^*}\downarrow & & \downarrow{\scriptstyle \theta^*} \\ S_{m+1}(\mathbf{M}) & \xrightarrow[\simeq]{} & \mathcal{B}\,\mathrm{Epi}(\mathbf{M})_m \end{array}$$

commutes.

It follows that there is a simplicial exact equivalence

$$E'S_\bullet(\mathbf{M}) \simeq \mathcal{B}\,\mathrm{Epi}(\mathbf{M}).$$

Pick a zero object 0 of $\mathbf{M}$, and observe that there is a contracting homotopy

$$\begin{array}{ccccccc} P_0 & \twoheadrightarrow & P_1 & \twoheadrightarrow & \cdots & \twoheadrightarrow & P_n \\ \downarrow & & \downarrow & & & & \downarrow \\ 0 & \twoheadrightarrow & 0 & \twoheadrightarrow & \cdots & \twoheadrightarrow & 0 \end{array}$$

for the simplicial exact category $\mathcal{B}\,\mathrm{Epi}(\mathbf{M})$. The space

$$s_\bullet E'S_\bullet(\mathbf{M})$$

is therefore contractible.

Theorem 13.2. *Suppose that $f : \mathbf{M} \to \mathbf{N}$ is an exact functor. Then the square*

$$\begin{array}{ccc} s_\bullet(S_\bullet(\mathbf{M}) \times_{S_\bullet(\mathbf{N})} E'S_\bullet(\mathbf{N})) & \longrightarrow & s_\bullet E'S_\bullet(\mathbf{N}) \\ \downarrow & & \downarrow d \\ s_\bullet S_\bullet(\mathbf{M}) & \xrightarrow[f_*]{} & s_\bullet S_\bullet(\mathbf{N}) \end{array}$$

is homotopy cartesian.

It follows that the space

$$s_\bullet(S_\bullet(\mathbf{M}) \times_{S_\bullet(\mathbf{N})} E'S_\bullet(\mathbf{N}))$$

has the homotopy type of the homotopy fibre of the map

$$f_* : s^2_\bullet(\mathbf{M}) \to s^2_\bullet(\mathbf{N}).$$

Proof of Theorem 13.2. The proof is effectively the same as that of Theorem 13.1. It suffices to show that all diagrams

$$\begin{array}{ccc} s_\bullet(\mathrm{Epi}_n(\mathbf{M}) \times_{\mathrm{Epi}_n(\mathbf{N})} \mathrm{Epi}_{n+1}(\mathbf{N})) & \longrightarrow & s_\bullet \mathrm{Epi}_{n+1}(\mathbf{N}) \\ pr* \downarrow & & \downarrow d_* \\ s_\bullet \mathrm{Epi}_n(\mathbf{M}) & \xrightarrow[f_*]{} & s_\bullet \mathrm{Epi}_n(\mathbf{N}) \end{array}$$

are homotopy cartesian. The functor

$$d : \mathrm{Epi}_{n+1}(\mathbf{N}) \to \mathrm{Epi}_n(\mathbf{N})$$

takes the string of admissible epis

$$P_0 \twoheadrightarrow P_1 \twoheadrightarrow \cdots \twoheadrightarrow P_n$$

to the string

$$K: \; K_0 \twoheadrightarrow K_1 \twoheadrightarrow \cdots \twoheadrightarrow K_{n-1},$$

where

$$K_i = \ker(P_i \twoheadrightarrow P_n).$$

Choose zero objects of $\mathbf{M}$ and $\mathbf{N}$ such that $f(0) = 0$, and choose kernels K for admissible epis $P \twoheadrightarrow Q$ such that P is the kernel of $P \to 0$ and 0 is the kernel of the identity $1 : P \to P$.

Objects of the exact category

$$\mathrm{Epi}_n(\mathbf{M}) \times_{\mathrm{Epi}_n(\mathbf{N})} \mathrm{Epi}_{n+1}(\mathbf{N})$$

are pairs (P, Q) such that $f(P)$ is the kernel of the morphism

$$\begin{array}{ccccccc} Q_0 & \twoheadrightarrow & Q_1 & \twoheadrightarrow & \cdots & \twoheadrightarrow & Q_{n-1} \\ \downarrow & & \downarrow & & & & \downarrow \\ Q_n & \xrightarrow[1]{} & Q_n & \xrightarrow[1]{} & \cdots & \xrightarrow[1]{} & Q_n \end{array}$$

Then there is a natural short exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f(P_0) & \rightarrowtail & Q_0 & \twoheadrightarrow & Q_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & f(P_{n-1}) & \rightarrowtail & Q_{n-1} & \twoheadrightarrow & Q_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \rightarrowtail & Q_n & \underset{1}{\twoheadrightarrow} & Q_n & \longrightarrow & 0
\end{array}$$

and it follows from that Additivity Theorem that there is a weak equivalence

$$\begin{aligned} s_\bullet(\mathrm{Epi}_n(\mathbf{M}) \times_{\mathrm{Epi}_n(\mathbf{N})} \mathrm{Epi}_{n+1}(\mathbf{N})) \\ \xrightarrow{\simeq} s_\bullet(\mathrm{Epi}_n(\mathbf{M})) \times s_\bullet(\mathbf{N}) \end{aligned}$$

which is induced by the functor which takes the pair (P, Q) to the pair (P, Q_n). □

14 The Resolution Theorem

Theorem 14.1. *Suppose that* $\mathbf{P}$ *is a subcategory of an exact category* $\mathbf{M}$ *such that* $\mathbf{P}$ *is full and closed under extensions, and such that*

- *every admissible subobject of an object of* $\mathbf{P}$ *is in* $\mathbf{P}$*,*
- *every object* Q *of* $\mathbf{M}$ *has a cover* $P \twoheadrightarrow Q$ *by an object of* $\mathbf{P}$*.*

Then the inclusion $i : \mathbf{P} \to \mathbf{M}$ induces a stable equivalence $K(\mathbf{P}) \to K(\mathbf{M})$.

Under the assumption that $\mathbf{P}$ is full and closed under extensions in $\mathbf{M}$, the category $\mathbf{P}$ is an exact category in which a sequence of $\mathbf{P}$ is exact if and only if it is exact in $\mathbf{M}$.

Proof. We show that the model for the homotopy fibre of the map

$$i_* : s^2_\bullet(\mathbf{P}) \to s^2_\bullet(\mathbf{M})$$

which is specified by Theorem 13.2 is contractible.

In the pullback diagram

$$\begin{array}{ccc} \mathrm{Epi}_n(\mathbf{P}) \times_{\mathrm{Epi}_n(\mathbf{M})} \mathrm{Epi}_{n+1}(\mathbf{M}) & \longrightarrow & \mathrm{Epi}_{n+1}(\mathbf{M}) \\ \downarrow & & \downarrow d \\ \mathrm{Epi}_n(\mathbf{P}) & \xrightarrow[i]{} & \mathrm{Epi}_n(\mathbf{M}) \end{array}$$

the pullback object is exactly equivalent to the exact category $\mathcal{B}\,\mathrm{Epi}^{\mathbf{P}}(\mathbf{M})_n$ whose objects consist of strings of admissible epimorphisms

$$Q_0 \twoheadrightarrow Q_1 \twoheadrightarrow \cdots \twoheadrightarrow Q_n$$

with kernels in the subcategory $\mathbf{P}$. It follows that there is a simplicial exact equivalence

$$S_\bullet(\mathbf{P}) \times_{S_\bullet(\mathbf{M})} E'S_\bullet(\mathbf{M}) \xrightarrow{\simeq} \mathcal{B}\,\mathrm{Epi}^{\mathbf{P}}(\mathbf{M}).$$

There is an obvious simplicial exact functor

$$j : \mathcal{B}\,\mathrm{Epi}(\mathbf{P}) \to \mathcal{B}\,\mathrm{Epi}^{\mathbf{P}}(\mathbf{M}).$$

The idea is to show that this functor j induces a weak equivalence

$$j_* : s_\bullet(\mathcal{B}\,\mathrm{Epi}(\mathbf{P})) \xrightarrow{\cong} s_\bullet(\mathcal{B}\,\mathrm{Epi}^{\mathbf{P}}(\mathbf{M})),$$

and then use the fact that $s_\bullet(\mathcal{B}\,\mathrm{Epi}(\mathbf{P}))$ is contractible.

The simplicial set map

$$s_n(\mathcal{B}\,\mathrm{Epi}(\mathbf{P})) \to s_n(\mathcal{B}\,\mathrm{Epi}^{\mathbf{P}}(\mathbf{M}))$$

can be identified with the map induced on nerves by the functor

$$\mathrm{Epi}(S_n(\mathbf{P})) \xrightarrow{j} \mathrm{Epi}^{S_n(\mathbf{P})}(S_n(\mathbf{M}))$$

where $\mathrm{Epi}(S_n(\mathbf{P}))$ is the category of admissible epimorphisms in $S_n(\mathbf{P})$ and $\mathrm{Epi}^{S_n(\mathbf{P})}(S_n(\mathbf{M}))$ is the category of admissible epimorphisms in $S_n(\mathbf{M})$ with kernels in $S_n(\mathbf{P})$.

The inclusion $i : S_n(\mathbf{P}) \to S_n(\mathbf{M})$ of exact categories satisfies the assumptions of the Theorem. It therefore suffices to show that the functor

$$j : \mathrm{Epi}(\mathbf{P}) \to \mathrm{Epi}^{\mathbf{P}}(\mathbf{M})$$

induces a weak equivalence

$$j : B\,\mathrm{Epi}(\mathbf{P}) \to B\,\mathrm{Epi}^{\mathbf{P}}(\mathbf{M}).$$

For $Q \in \mathbf{M}$, the objects of the slice category j/Q are covers $q : P \twoheadrightarrow Q$ by objects $P \in \mathbf{P}$ with kernel in $\mathbf{P}$, and the morphisms are commutative diagrams

$$\begin{array}{ccc} P & \overset{q}{\searrow} & \\ {\scriptstyle \pi}\big\downarrow & & Q \\ P' & \underset{q'}{\nearrow} & \end{array}$$

of admissible epimorphisms (with $ker(\pi)$ in $\mathbf{P}$). The category j/Q is non-empty by assumption.

Take a fixed object $p : P_0 \twoheadrightarrow Q$ and form the pullback diagrams

$$\begin{array}{ccc} P_0 \times_Q P & \longrightarrow & P \\ \big\downarrow & \overset{q_*}{\searrow} & \big\downarrow {\scriptstyle q} \\ P_0 & \underset{p}{\longrightarrow} & Q \end{array}$$

for all other objects q of j/Q. Then $P_0 \times_Q P$ is in $\mathbf{P}$ and the dotted arrow is an admissible epi with kernel in $\mathbf{P}$ since $\mathbf{P}$ is closed under extensions. The pullback diagram is natural in q, and so there are natural transformations

$$p \leftarrow q_* \to q.$$

In particular j/Q is contractible. $\square$

Let $\mathbf{P}_n$ be the full subcategory of $\mathbf{M}$ whose objects are those M having $\mathbf{P}$-resolutions

$$0 \to P_m \rightarrowtail P_{m-1} \to \cdots \to P_0 \twoheadrightarrow M \to 0$$

(composed of exact sequences of $\mathbf{M}$), with $m \leq n$. Let $\mathbf{P}_\infty$ be the full subcategory on those objects M which have $\mathbf{P}$-resolutions of bounded length.

Lemma 14.2. *Suppose that* $\mathbf{P}$ *is full and closed under extensions in the exact category* $\mathbf{M}$*. Suppose further that*

a) all admissible epis $P \to P'$ *between objects of* $\mathbf{P}$ *in* $\mathbf{M}$ *are admissible epis of* $\mathbf{P}$*,*

b) given any admissible epi $p : M \twoheadrightarrow P$ *with* $P \in \mathbf{P}$*, there is a diagram*

$$\begin{array}{ccc} P' & & \\ \downarrow & \overset{p'}{\searrow} & \\ M & \xrightarrow[p]{} & P \end{array}$$

with p' *an admissible epi of* $\mathbf{P}$*.*

Then we have the following:

1) $\mathbf{P}_n$ *is closed under extensions in* $\mathbf{M}$*,*

2) all admissible epis between objects of $\mathbf{P}_n$ *in* $\mathbf{M}$ *are admissible epis of* $\mathbf{P}_n$*,* *3) given an exact sequence*

$$0 \to M' \rightarrowtail M \twoheadrightarrow M'' \to 0,$$

$$\textit{if } M \in \mathbf{P}_n \textit{ and } M'' \in \mathbf{P}_{n+1} \textit{ then } M' \in \mathbf{P}_n.$$

Proof. Suppose that

$$0 \to M' \overset{i}{\rightarrowtail} M \overset{p}{\twoheadrightarrow} M'' \to 0 \qquad (1)$$

is an exact sequence of $\mathbf{M}$ with $M', M'' \in \mathbf{P}_n$, and choose exact sequences

$$0 \to K_0'' \rightarrowtail P_0'' \twoheadrightarrow M'' \to 0$$

and

$$0 \to K_0' \rightarrowtail P_0' \twoheadrightarrow M' \to 0$$

with $P_0', P_0'' \in \mathbf{P}$ and $K_0', K_0'' \in \mathbf{P}_{n-1}$. Form the pullback

$$\begin{array}{ccc} M \times_{M'} P_0'' & \overset{p_*}{\twoheadrightarrow} & P_0'' \\ \downarrow & & \downarrow \\ M & \underset{p}{\twoheadrightarrow} & M'' \end{array}$$

Then there is a commutative diagram

$$\begin{array}{ccc} P' & & \\ \downarrow & \overset{\pi}{\searrow} & \\ M \times_{M'} P_0'' & \underset{p_*}{\twoheadrightarrow} & P_0'' \end{array}$$

with $P' \in \mathbf{P}$ and π and admissible epi, by assumption a). The kernel $P'' \rightarrowtail P'$ of π is an admissible monic of $\mathbf{P}$, by assumption b). It follows that there is a diagram

$$\begin{array}{ccccc} K'_0 \oplus P'' & \rightarrowtail & K & \twoheadrightarrow & K'_0 \\ \downarrow & & \downarrow & & \downarrow \\ P'_0 \oplus P'' & \rightarrowtail & P''_0 \oplus P' & \twoheadrightarrow & P''_0 \\ \downarrow & & \downarrow & & \downarrow \\ M' & \rightarrowtail & M & \twoheadrightarrow & M'' \end{array}$$

in which all rows and columns are exact sequences. But then $K'_0 \oplus P''$ and K''_0 are in $\mathbf{P}_{n-1}$ which (inductively) is closed under extensions, so that $K \in \mathbf{P}_{n-1}$ and so $M \in \mathbf{P}_n$, and we have proved statement 1).

For 2), suppose given an exact sequence (1) with $M, M'' \in \mathbf{P}_n$, and choose an exact sequence

$$0 \to K''_0 \rightarrowtail P''_0 \twoheadrightarrow M'' \to 0$$

with $P''_0 \in \mathbf{P}$ and $K''_0 \in \mathbf{P}_{n-1}$. Then there is an exact sequence

$$0 \to M' \rightarrowtail M \times_{M''} P''_0 \twoheadrightarrow P''_0 \to 0$$

and $M \times_{M''} P''_0 \in \mathbf{P}_n$ by statement 1), from the exact sequence

$$0 \to K''_0 \rightarrowtail M \times_{M''} P_0 \twoheadrightarrow M \to 0.$$

We can therefore assume that $M'' \in \mathbf{P}$.

Choose an exact sequence

$$0 \to K_0 \rightarrowtail P_0 \twoheadrightarrow M \to 0$$

with $P_0 \in \mathbf{P}$ and $K_0 \in \mathbf{P}_{n-1}$. Then there are exact sequences

$$0 \to M' \times_M P_0 \rightarrowtail P_0 \twoheadrightarrow M'' \to 0$$

and

$$0 \to K_0 \rightarrowtail M' \times_M P_0 \twoheadrightarrow M' \to 0$$

so that $M' \times_M P_0 \in \mathbf{P}$ and $M' \in \mathbf{P}_n$.

To prove statement 3), suppose given an exact sequence (1) with $M \in \mathbf{P}_n$ and $M'' \in \mathbf{P}_{n+1}$. Choose an exact sequence

$$0 \to K_0'' \rightarrowtail P_0'' \twoheadrightarrow M'' \to 0$$

with $P_0'' \in \mathbf{P}$ and $K_0'' \in \mathbf{P}_n$. Then from the exact sequence

$$0 \to K_0'' \rightarrowtail M \times_{M''} P_0'' \twoheadrightarrow M \to 0$$

and statement 1) we see that $M \times_{M''} P_0'' \in \mathbf{P}_n$. The exact sequence

$$0 \to M' \rightarrowtail M \times_{M''} P_0'' \twoheadrightarrow P_0'' \to 0$$

and statement 2) imply that $M' \in \mathbf{P}_n$. $\square$

Theorem 14.3 (Resolution Theorem)**.** *Suppose that* $\mathbf{P}$ *is full and closed under extensions in the exact category* $\mathbf{M}$*, and that* $\mathbf{P}$ *and* $\mathbf{M}$ *satisfy the conditions a) and b) of Lemma 14.2. Then the inclusions*

$$\mathbf{P} \subset \mathbf{P}_1 \subset \mathbf{P}_2 \subset \cdots \subset \mathbf{P}_\infty$$

induce stable equivalences

$$K(\mathbf{P}) \simeq K(\mathbf{P}_1) \simeq K(\mathbf{P}_2) \simeq \cdots \simeq K(\mathbf{P}_\infty).$$

Proof. The inclusion $\mathbf{P}_n \subset \mathbf{P}_{n+1}$ satisfies the conditions of Theorem 14.1: the subcategory $\mathbf{P}_n$ is closed under extensions by statement 2) of Lemma 14.2, and every object $M \in \mathbf{P}_{n+1}$ has a cover $P_0 \twoheadrightarrow M$ with $P_0 \in \mathbf{P} \subset \mathbf{P}_n$.

As categories

$$\mathbf{P}_\infty = \cup_n \mathbf{P}_n$$

so that $K(\mathbf{P}_\infty)$ is the filtered colimit of the diagram of spectra

$$K(\mathbf{P}) \to K(\mathbf{P}_1) \to K(\mathbf{P}_2) \to \dots$$

and all of these maps are stable equivalences. □

Suppose that X is a regular Noetherian scheme. Then every coherent sheaf on X has a finite resolution by vector bundles, so that Theorem 14.3 implies the following major result:

Corollary 14.4. *Suppose that X is a regular Noetherian scheme. Then the inclusion $\mathcal{P}(X) \subset \mathbf{M}(X)$ of vector bundles in coherent sheaves induces a stable equivalence*

$$K(X) = K(\mathcal{P}(X)) \simeq K(\mathbf{M}(X)) = K'(X).$$

15 The Dévissage Theorem

Theorem 15.1. *Suppose that $\mathbf{B}$ is a a non-empty subcategory of a small abelian category $\mathbf{A}$ which is closed under taking finite direct sums, subobjects and quotients in $\mathbf{A}$. Suppose that every object Q of $\mathbf{A}$ has a finite filtration*

$$0 = F_{-1} \rightarrowtail F_0 \rightarrowtail F_1 \rightarrowtail \cdots \rightarrowtail F_n = Q$$

with all filtration quotients $F_i/F_{i-1} \in \mathbf{B}$. Then the inclusion $i : \mathbf{B} \to \mathbf{A}$ induces a stable equivalence $K(\mathbf{B}) \simeq K(\mathbf{A})$.

The categories $\mathbf{B}$ and $\mathbf{A}$ are exact categories, with all monomorphisms and epimorphisms admissible, so the statement of the Theorem makes sense.

Proof. From Theorem 13.1, the homotopy fibre of the map

$$i_* : s^2_\bullet(\mathbf{B}) \to s^2_\bullet(\mathbf{A})$$

is equivalent to the space

$$s_\bullet \mathcal{B}\,\mathrm{Mon}_{\mathbf{B}}(\mathbf{A})$$

associated to the simplicial exact category $\mathcal{B}\,\mathrm{Mon}_{\mathbf{B}}(\mathbf{A})$ whose category of n-simplices has objects given by strings of monomorphisms

$$M_0 \rightarrowtail M_1 \rightarrowtail \cdots \rightarrowtail M_n$$

with all quotients M_j/M_i in the subcategory $\mathbf{B}$ for $i \leq j$.

NB: We do not know that $\mathbf{B}$ is closed under extensions, so it's not clear that the simplicial set of objects of $\mathcal{B}\,\mathrm{Mon}_{\mathbf{B}}(\mathbf{A})$ is the nerve of a category.

There is a "forgetful" morphism of simplicial exact categories

$$p : \mathcal{B}\,\mathrm{Mon}_{\mathbf{B}}(\mathbf{A}) \to \mathcal{B}\,\mathrm{Mon}(\mathbf{A}),$$

and we show that this map induces a weak equivalence

$$s_\bullet(\mathcal{B}\,\mathrm{Mon}_{\mathbf{B}}(\mathbf{A})) \simeq s_\bullet(\mathcal{B}\,\mathrm{Mon}(\mathbf{A})).$$

This would complete the proof, since the space on the right is contractible.

The simplicial set map

$$p_* : s_n(\mathcal{B}\,\mathrm{Mon}_{\mathbf{B}}(\mathbf{A})) \to s_n(\mathcal{B}\,\mathrm{Mon}(\mathbf{A}))$$

can be identified with the simplicial set map

$$\mathrm{Ob}(\mathcal{B}\,\mathrm{Mon}_{S_n(\mathbf{B})}(S_n(\mathbf{A})) \to B\,\mathrm{Mon}(S_n(\mathbf{A}))$$

which is induced by the forgetful morphism corresponding to $S_n(\mathbf{B}) \subset S_n(\mathbf{A})$. The inclusion $i : S_n(\mathbf{B}) \subset S_n(\mathbf{A})$ satisfies the assumptions of the Theorem, so it suffices to show that the simplicial set map

$$p : \mathrm{Ob}(\mathcal{B}\,\mathrm{Mon}_{\mathbf{B}}(\mathbf{A})) \to B\,\mathrm{Mon}(\mathbf{A})$$

is a weak equivalence.

The simplicial set $B\,\mathrm{Mon}(\mathbf{A})$ is the homotopy colimit of the nerves

$$B(\mathrm{Mon}(\mathbf{A})/P)$$

of its various slice categories, so that the total space $\mathrm{Ob}(\mathcal{B}\,\mathrm{Mon}_{\mathbf{B}}(\mathbf{A}))$ is the homotopy colimit of the simplicial sets $p^{-1}(P)$ which are defined by the pullback squares

$$\begin{array}{ccc} p^{-1}(P) & \longrightarrow & \mathrm{Ob}(\mathcal{B}\,\mathrm{Mon}_{\mathbf{B}}(\mathbf{A})) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p} \\ B(\mathrm{Mon}(\mathbf{A})/P) & \longrightarrow & B(\mathrm{Mon}(\mathbf{A})). \end{array}$$

It therefore suffices to show that the maps p_* are weak equivalences. We do this by showing that if $m : P \rightarrowtail P'$ is a monic with quotient P'/P in $\mathbf{B}$, then the induced map $m_*: p^{-1}(P) \to p^{-1}(P')$ is a weak equivalence.

To see that this is enough, observe that the n-simplices of $p^{-1}(P)$ are strings of monics

$$A_0 \rightarrowtail \cdots \rightarrowtail A_n \rightarrowtail P$$

of $\mathbf{A}$ with $A_j/A_i \in \mathbf{B}$ for all $i \leq j$. If P is a zero object, then $p^{-1}(P) = B(\mathbf{0}/P)$, which is contractible. By assumption, every object $P \in \mathbf{A}$ has a filtration

$$0 = F_{-1} \rightarrowtail F_0 \rightarrowtail \cdots \rightarrowtail F_n = P$$

with $F_i/F_{i-1} \in \mathbf{B}$, and so it would follow that there is a weak equivalence

$$* \simeq p^{-1}(0) \simeq p^{-1}(P).$$

Suppose that $P \rightarrowtail P'$ has $P'/P \in \mathbf{B}$, and let the string of monics

$$B_0 \rightarrowtail \cdots \rightarrowtail B_n \rightarrowtail P' \tag{2}$$

be an n-simplex of $p^{-1}(P')$. Then pulling back over $P \rightarrowtail P'$ defines a string of monomorphisms

$$B_0^* \rightarrowtail \cdots \rightarrowtail B_n^* \rightarrowtail P' \tag{3}$$

of monomorphisms, and there are monomorphisms $B_j^*/B_i^* \rightarrowtail B_j/B_i$, so that all B_j^*/B_i^* are in $\mathbf{B}$.

Assigning the string (3) to the string (2) therefore defines a simplicial set map

$$\sigma : p^{-1}(P') \to p^{-1}(P).$$

For $i \leq j$ there is a monomorphism

$$B_j/B_i^* \rightarrowtail B_j/B_i \oplus B_j/B_j^*,$$

and it follows that B_j/B_i^* is in $\mathbf{B}$. All strings

$$B_0^* \rightarrowtail \cdots \rightarrowtail B_i^* \rightarrowtail B_i \rightarrowtail \cdots \rightarrowtail B_n \rightarrowtail P'$$

are therefore elements of $p^{-1}(P')$. It follows that there is a homotopy

$$m_*\sigma \simeq 1 : p^{-1}(P') \to p^{-1}(P').$$

There is also a homotopy

$$\sigma m_* \simeq 1 : p^{-1}(P) \to p^{-1}(P),$$

so that m_* is a weak equivalence as required. $\square$

References

[1] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.