Lecture 009 (April 12, 2011)

21 Some algebra: filtered and graded rings

A ring A is said to be *filtered* if the underlying abelian group has a filtration

$$0 = F_{-1}(A) \subset F_0(A) \subset \cdots \cup_{i \geq -1} F_i(A) = A,$$

such that

$$F_p(A) \cdot F_q(A) \subset F_{p+q}(A)$$

under the ring multiplication, and $1 \in F_0(A)$. The canonical example is a polynomial ring $R[t]$, filtered by degree.

The *associated graded ring* $gr(A) = (gr_p(A))_{p \geq 0}$ of a filtered ring A has

$$gr(A)_p = F_p(A)/F_{p-1}(A),$$

with the obvious multiplication.

An A-module M is said to be *filtered* if M has a filtration

$$0 = F_{-1}(M) \subset F_0(M) \subset \cdots \cup_{i \geq -1} F_i(M) = M,$$

such that

$$F_p(A) \cdot F_q(M) \subset F_{p+q}(M)$$

The *associated graded module*

$$gr(M) = (F_p(M)/F_{p-1}(M))_{p\geq 0}$$

is a (positively) graded module over $gr(A)$.

Example: Any ideal in a polynomial ring $R[t]$ can be filtered by degree.

A given module M has multiple filtered A-module structures: given a filtration F_*M and $n \in \mathbb{Z}$ there is a filtered A-module structure $F_*^{(n)}(M)$ on M with

$$F_p^{(n)}(M) = \begin{cases} F_{p+n}(M) & \text{if } p+n \geq 0, \text{ and} \\ 0 & \text{if } p+n < 0. \end{cases}$$

Write $M^{(n)}$ for the module M together with the filtration $F_*^{(n)}(M)$, and observe that there is a natural isomorphism

$$gr(M^{(n)}) \cong gr(M)(n)$$

where $gr(M)(n)$ is the shifted (or twisted) graded $gr(A)$-module in the usual sense (at least if $n \leq 0$ — otherwise it is truncated in positive degrees).

Lemma 21.1. *Suppose that M is a filtered A-module such that $gr(M)$ is finitely generated. Then M is finitely generated.*

Proof. Pick a set of homogeneous generators

$$x_i \in F_{n_i}M/F_{n_i-1}M, \ 1 \leq i \leq m,$$

for $gr(M)$, and choose elements $z_i \in F_{n_i}M$ such that $z_i \mapsto x_i$ under the canonical surjection

$$F_{n_i}M \to F_{n_i}M/F_{n_i-1}M.$$

There is a homomorphism of filtered A-modules

$$\phi_i : F^{(-n_i)}A \to M$$

such that $\phi_i(1) = z_i$. Adding up these homomorphisms defines a map

$$\sum \phi_i : \bigoplus_{i=1}^{m} F^{(-n_i)}A \to M$$

which induces a surjective homomorphism

$$\bigoplus_{i=1}^{m} gr(A)(-n_i) \to gr(M)$$

of graded modules which picks up the generators x_i. But then the filtered module map $\sum \phi_i$ is surjective by an induction on filtration degree, and so the underlying A-module homomorphism

$$\oplus_{i=1}^{n} A \to M$$

is surjective. □

Corollary 21.2. *Suppose that A is a filtered ring such that $gr(A)$ is Noetherian. Then A is Noetherian.*

Proof. Every ideal I of A has a filtration

$$F_pI = I \cap F_pA.$$

Then $gr(I)$ is an ideal of $gr(A)$ and is therefore finitely generated, so that I is finitely generated by Lemma 21.1. □

Every graded ring $B = (B_n)$ has an associated (ungraded) ring $\oplus B_n$, with multiplication defined by

$$(a_n)(b_n) = (\sum_{i+j=n} a_i b_j).$$

Any graded module $M = (M_n)$ over the graded ring B determines a module $\oplus M_n$ over the ring $\oplus B_n$, in the obvious way. In particular, every graded ideal I of B determines an ideal $\oplus I_n$ of the ring $\oplus B_n$.

Lemma 21.3. *Suppose that M is a graded B-module, and suppose that $\oplus M_n$ is a finitely generated $\oplus B_n$-module. Then M is a finitely generated B-module*

Proof. The module $\oplus M_n$ is generated over $\oplus B_n$ by a finite collection of homogeneous elements

$$x_1, \dots, x_m$$

with $x_i \in M_{n_i}$. There is a graded B-module homomorphism

$$\phi_i : B(-n_i) \to M$$

with $\phi_i(1) = x_i$. The sum of these maps

$$\phi = \sum_{i=1}^{m} \phi_i : B(-n_i) \to M$$

is a surjective homomorphism of graded modules because the induced map $\oplus\phi$ is surjective. $\square$

Corollary 21.4. *Suppose that B is a graded ring. Then the ring $\oplus B_n$ is Noetherian if and only if B is Noetherian.*

Proof. The ring $\oplus B_n$ has a filtration with

$$F_p(\oplus B_n) = \oplus_{0 \leq n \leq p} B_n,$$

and the associated graded ring for this filtration is isomorphic to B. Corollary 21.2 then says that $\oplus B_n$ is Noetherian if B is Noetherian.

Suppose that the ring $\oplus B_n$ is Noetherian. If I is a graded ideal in B, then the ideal $\oplus I_n$ of $\oplus B_n$ is finitely generated, so that I is a finitely generated B-module, by Lemma 21.3. $\square$

Lemma 21.5. *Suppose that A is a filtered ring such that $gr(A)$ is Noetherian. Then the graded ring (F_pA) is Noetherian.*

Proof. The ring $\oplus F_pA$ is isomorphic to the subring $A' \subset A[z]$ consisting of those polynomials

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

with $a_i \in F_i(A)$. We show that A' is Noetherian, and then invoke Corollary 21.4.

Filter A' by requiring F_pA' to consist of those polynomials $f(z)$ with all coefficients in F_pA. The associated graded ring of this filtration has

$$F_pA'/F_{p-1}A' \cong \bigoplus_{n \geq p} gr(A)_p z^n.$$

There is an isomorphism of graded rings

$$\bigoplus_{n \geq 0} gr(A)_p z^n \to \bigoplus_{n \geq p} gr(A)_p z^n$$

which is defined at level p by multiplication by z^p. There is an isomorphism of rings

$$\oplus(\bigoplus_{n \geq 0} gr(A)_p z^n) \cong (\oplus gr(A)_n)[z]$$

and the ring $(\oplus gr(A)_n)[z]$ is Noetherian by Corollary 21.4 (and the Hilbert basis theorem). It follows, again by Corollary 21.4, that the graded ring $(F_pA'/F_{p-1}A')$ is Noetherian, so that A' is Noetherian by Corollary 21.2. □

22 K-theory of graded rings

Suppose that $B = (B_n)$ is a graded ring, and let $k = B_0$.

For a graded B-module $N = (N_n)$, define

$$T_i(N) = \mathrm{Tor}_i^B(k, N),$$

as a graded B-module, where k is a graded B-module via the augmentation. Then in particular there are natural isomorphisms

$$T_0(N)_r \cong N_r/(B_1N_{r-1} + \cdots + B_rN_0).$$

In other words, T_0N is N mod decomposables in all degrees.

Let $F_p(N)$ be the graded submodule of N which is generated by N_n with $n \leq p$. Then there is an isomorphism

$$T_0(N)_p \cong (F_p(N)/F_{p-1}(N))_p \qquad (1)$$

for each $p \geq 0$. There are also isomorphisms

$$T_0(F_pN) = \begin{cases} 0 & \text{if } n > p, \text{ and} \\ T_0(N)_n & \text{if } n \leq p. \end{cases} \tag{2}$$

The natural isomorphism (1) induces a natural surjective homomorphism

$$\phi_p : B(-p) \otimes_k T_0(N)_p \twoheadrightarrow F_p(N)/F_{p-1}(N)$$

of graded B-modules.

Lemma 22.1. *Suppose that N is a graded B-module such that $T_1(N) = 0$ and*

$$\mathrm{Tor}_i^k(B, T_0(N)) = 0$$

for all $i > 0$. Then the map ϕ_p is an isomorphism.

Proof. Suppose that $P_* \to M$ is a projective resolution of a graded k-module M. Then

$$\mathrm{Tor}_i^k(B, M) = 0$$

for all $i > 0$ means that $B \otimes_k P_* \to B \otimes_k M$ is a B-projective resolution of $B \otimes_k M$. But then

$$T_i(B \otimes_k M) = H_i(k \otimes_B B \otimes_k P_*) = 0$$

for $i > 0$. It follows that

$$T_i(B \otimes_k T_0(N)) = 0$$

for $i > 0$.

Form the exact sequence of graded B-modules

$$0 \to K \to B(-p) \otimes_k T_0(N)_p \xrightarrow{\phi_p} F_p(N)/F_{p-1}(N) \to 0.$$

We show that $K = 0$ by showing that $T_0(K) = 0$.

There is an isomorphism of graded k-modules

$$T_0(N) \cong \bigoplus_{p \geq 0} T_0(N)_p(-p),$$

so that there are isomorphisms

$$\begin{aligned} T_i(B \otimes_k T_0(N)) &\cong \bigoplus_{p \geq 0} T_i(B \otimes_k T_0(N)_p(-p)) \\ &\cong \bigoplus_{p \geq 0} T_i(B(-p) \otimes_k T_0(N)_p). \end{aligned}$$

It follows that

$$T_i(B(-p) \otimes_k T_0(N)_p) = 0$$

for all $i > 0$ and for all $p \geq 0$.

The induced map

$$T_0(B(-p) \otimes_k T_0(N)_p) \xrightarrow{k \otimes \phi_p} T_0(F_p(N)/F_{p-1}(N))$$

is isomorphic to a shift of the isomorphism (1), and is therefore an isomorphism. It follows that the boundary map

$$T_1(F_p(N)/F_{p-1}(N)) \xrightarrow{\partial} T_0(K)$$

is an isomorphism, so it suffices to show that

$$T_1(F_p(N)/F_{p-1}(N)) = 0$$

In the exact sequence

$$T_1(F_p(N)) \to T_1(F_p(N)/F_{p-1}(N)) \xrightarrow{\partial} T_0(F_{p-1}N) \xrightarrow{i_*} T_0(F_p(N)),$$

the map i_* is always monic on account of the isomorphisms (2), and so it suffices to show that $T_1(F_p(N)) = 0$ for all p.

Fix a number $s \geq 0$. We show by descending induction on p that

$$T_1(F_p(N))_n = 0 \tag{3}$$

for $n \leq s$.

The condition (3) holds for large p. In effect,

$$(N/F_p(N))_n = 0$$

for $n \leq p$, so that

$$T_i(N/F_p(N))_n = \mathrm{Tor}_i^B(k, N/F_p(N)) = 0$$

for $n \leq p$, by suitable choice of projective resolution for $N/F_p(N)$. It follows that the map

$$T_1(F_p(N))_n \to T_1(N)_n$$

is an isomorphism for $n \leq p$. We assume that $T_1(N) = 0$, so it follows that

$$T_1(F_p(N))_n = 0$$

if $n \leq p$. Thus, (3) holds for $p \geq s$.

To complete the induction, observe that the condition (3) implies that $T_0(K)_n = 0$ for $n \leq s$, so that $K_n = 0$ for $n \leq s$. But then $T_2(K)_n = 0$ for $n \leq s$, so that

$$0 = T_2(B(-p) \otimes_k T_0(N)_p) \cong T_2(F_p(N)/F_{p-1}(N))_n$$

for $n \leq s$. It follows that

$$T_1(F_{p-1}(N))_n = 0$$

for $n \leq s$. $\square$

Remark 22.2. Suppose given a graded ring homomorphism

$$f : A \to B,$$

where A and B are Noetherian. Suppose further that B has finite Tor dimension as an A-module — this means that there is some integer n such that $\mathrm{Tor}_i^A(B, N) = 0$ for all $i > n$, and for all A-modules N. Let $\mathbf{N}_d \subset \mathbf{M}_g(A)$ be the full subcategory of those modules N for which $\mathrm{Tor}_i^A(B, N) = 0$ for $i > d$. Then the inclusions

$\mathbf{N}_d \subset \mathbf{N}_{d+1}$ satisfy the hypotheses of the Resolution Theorem (Theorem 14.1, Lecture 006), so that the inclusion

$$\mathbf{N}_0 \subset \mathbf{M}_g(A)$$

induces a stable equivalence

$$K(\mathbf{N}_0) \simeq K(\mathbf{M}_g(A)).$$

The functor $M \mapsto B \otimes_A M$ is exact for all $M \in \mathbf{N}_0$, and hence induces a map

$$K(\mathbf{N}_0) \to K(\mathbf{M}_g(B)).$$

It follows that there is a map $f_* : K(\mathbf{M}_g(A)) \to K(\mathbf{M}_g(B))$ in the stable category, which is defined by the diagram

$$\begin{array}{ccc} K(\mathbf{N}_0) & \longrightarrow & K(\mathbf{M}_g(B)) \\ \simeq\downarrow & \nearrow_{f_*} & \\ K(\mathbf{M}_g(A)) & & \end{array}$$

Suppose that a graded ring B is Noetherian, and let $\mathbf{M}_g(B)$ be the exact category of finitely generated graded B-modules. Suppose that B is flat as a k-module, where $k = B_0$. Then the functor $M \mapsto B \otimes_k N$ is exact on $\mathbf{M}(k)$, and induces a map

$$(B\otimes)_* : K'(k) \to K(\mathbf{M}_g(B))$$

of associated K-theory spectra.

The shift operator $N \mapsto N(-1)$ defines an action of the ring $\mathbb{Z}[t]$ on the group $K_n(\mathbf{M}_g(B))$.

Theorem 22.3. *Suppose that B is a graded Noetherian ring such that B is flat as a k-module, where $k = B_0$. Suppose that k has finite* Tor *dimension as a B-module. Then the map*

$$(B\otimes)_* : K'_n(k) \to K_n(\mathbf{M}_g(B))$$

induces an isomorphism of $\mathbb{Z}[t]$-modules

$$\mathbb{Z}[t] \otimes_{\mathbb{Z}} K'_n(k) \cong K_n(\mathbf{M}_g(B)).$$

Proof. Let $\mathbf{N}_n \subset \mathbf{M}_g(B)$ be the full exact subcategory of objects N such that $T_i(N) = 0$ for $i > n$. Then, as in Remark 22.2, the inclusion

$$\mathbf{N}_0 \subset \mathbf{M}_g(B)$$

induces a stable equivalence

$$K(\mathbf{N}_0) \simeq K(\mathbf{M}_g(B)).$$

Write $\mathbf{N}_0^{(k)}$ for the full subcategory of $\mathbf{N}_0$ which consists of those graded modules N such that

$$F_k(N) = N$$

(meaning that N has a set of generators in degrees $\leq k$). This category is closed under taking kernels and quotients, and is therefore exact. In effect, if

$$0 \to K \to M \to N \to 0$$

is an exact sequence of $\mathbf{N}_0$ then the sequence

$$0 \to T_0(K) \to T_0(M) \to T_0(N) \to 0$$

is exact, and $T_0(K)_r = T_0(N)_r = 0$ for $r > k$ if $T_0(M)_r = 0$ for $r > k$.

There are exact functors

$$\mathbf{M}(k)^{\times(n+1)} \xrightarrow{b} \mathbf{N}_0^{(n)} \xrightarrow{c} \mathbf{M}(k)^{\times(n+1)}$$

where b is defined by

$$(F_0, \ldots, F_n) \mapsto \bigoplus_{j=0}^{n} B(-j) \otimes_k F_j$$

and c is defined by

$$N \mapsto (T_0(N)_0, \ldots T_0(N)_n).$$

The ring B is a flat (graded) k-module, so that

$$\mathrm{Tor}_i^B(k, B(-j) \otimes_k F_j) = 0$$

for $i > 0$. To see this, one tensors a projective resolution $P_* \to F_j$, first by the free module $B(-j)$ to obtain a shifted projective resolution

$$B(-j) \otimes_k P_* \to B(-j) \otimes_k F_j$$

Then tensoring with k gives the shifted resolution $P_*(-j) \to F_j(-j)$. In particular, the definition of the functor b makes sense.

One also sees that

$$T_0(B(-j) \otimes_k F_j) \cong k(-j) \otimes_k F_j \cong F_j(-j),$$

so that $c \cdot b \cong 1$. On the other hand, as maps of K-theory spectra

$$\begin{aligned} b_* c_* &= \sum_{j=0}^{n} (B(-j) \otimes_k T_0(\ )_j)_* \\ &= \sum_{j=0}^{n} F_j/F_{j-1}(\ )_* \\ &= 1 \end{aligned}$$

by Lemma 22.1 and additivity (Corollary 7.1 of Lecture 003).

Taking a filtered colimit of these equivalences over all subcategories $\mathbf{N}_0^{(n)}$ of $\mathbf{N}_0$ finishes the proof. □

23 The homotopy property

The following result is variously called the "Fundamental Theorem of Algebraic K-theory", or the "homotopy property". It is presented here in a simplified form.

Theorem 23.1. *Suppose that A is a Noetherian ring. Then tensoring with $A[t]$ induces a stable equivalence*

$$K'(A) \simeq K'(A[t]).$$

Note that $A[t]$ is a flat A-module, so the statement of the Theorem makes sense.

Proof. Filter $A[t]$ by degree, let

$$B = (F_p(A[t])/F_{p-1}(A[t]))_{p\geq 0}$$

be the associated graded ring, and let A' be the graded ring

$$A' = (F_p(A[t]))_{p\geq 0}.$$

Then B and A' are Noetherian, by Corollary 21.4 and Lemma 21.5, respectively.

Let $z = 1 \in F_1(A[t])$, identified with an element of homogeneous degree 1 in A'. There is an exact functor

$$L : \mathbf{M}_g(A') \to \mathbf{M}(A[t])$$

of abelian categories, which takes a graded module $M = (M_p)$ to the colimit of the system

$$M_0 \xrightarrow{\times z} M_1 \xrightarrow{\times z} M_2 \xrightarrow{\times z} \dots$$

There are various details to check, but this functor is localizing with kernel consisting of those graded

A'-modules which are annihilated by some power of z. There is a graded ring isomorphism

$$A'/zA' \cong B,$$

so by dévissage (Theorem 15.1) and the localization theorem (Theorem 16.1), there is a fibre homotopy sequence

$$K(\mathbf{M}_g(B)) \xrightarrow{\pi_*} K(\mathbf{M}_g(A')) \xrightarrow{L} K'(A[t])$$

of K-theory spectra, where π_* is transfer along the surjective graded ring homomorphism $\pi : A' \to B$.

By Theorem 22.3 (for the graded rings A' and B), there is a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[t] \otimes K'_i(A) & \xrightarrow{\cong} & K_i(\mathbf{M}_g(B)) \\ \downarrow & & \downarrow \pi_* \\ \mathbb{Z}[t] \otimes K'_i(A) & \xrightarrow[\cong]{} & K_i(\mathbf{M}_g(A') \end{array}$$

The isomorphism

$$\mathbb{Z}[t] \otimes K'_i(A) \xrightarrow{\cong} K_i(\mathbf{M}_g(B))$$

restricts to the map

$$B(-n) \otimes_A (\ )_* : K'_i(A) \to K_i(\mathbf{M}_g(B))$$

on the summand corresponding to t^n. There is an exact sequence of exact functors

$$0 \to A'(-n-1) \otimes_A (\ ) \to A'(-n) \otimes_A (\ ) \to B(-n) \otimes_A (\ ) \to 0$$

Thus,

$$\pi_*(B(-n)\otimes_A(\ ))_* = A'(-n)\otimes_A(\ )_* - A'(-n-1)\otimes_A(\ )_*$$

by additivity, so that the dotted arrow is multiplication by $1 - t$.

It follows that the homomorphism π_* is injective, and the short split exact sequence

$$0 \to \mathbb{Z}[t]\otimes K_i'(A) \xrightarrow{1-t} \mathbb{Z}[t]\otimes K_i'(A) \xrightarrow{\epsilon} K_i'(A) \to 0$$

(the map ϵ is defined by $t \mapsto 1$) defines an isomorphism

$$K_i'(A) \cong K_i'(A[t]). \tag{4}$$

The evaluation map ϵ is split by the map

$$A' \otimes (\ )_* : K_i'(A) \to K_i(\mathbf{M}_g(A')),$$

and so the isomorphism (4) coincides with the the map $K_i'(A) \to K_i'(A[t])$ which is defined by tensoring with the A-module $A[t]$. □

Corollary 23.2. *Suppose that A is a regular Noetherian ring. Then the ring homomorphism $A \to A[t]$ induces a stable equivalence*

$$K(A) \xrightarrow{\simeq} K(A[t]).$$

Proof. The polynomial ring $A[t]$ is a regular Noetherian ring if A is. Use the resolution theorem (Corollary 14.4) with Theorem 23.1. □

Corollary 23.3. *Suppose that A is a Noetherian ring. Then there is a short split exact sequence*

$$0 \to K'_i(A) \to K'_i(A[t,t^{-1}]) \to K'_{i-1}(A) \to 0$$

for each $i \geq 1$.

Proof. The localization sequence for the localizing functor

$$\mathbf{M}(A[t]) \to \mathbf{M}(A[t,t^{-1}])$$

has the form

$$\dots \to K'_i(A) \to K'_i(A[t]) \to K'_i(A[t,t^{-1}]) \to \dots$$

The A-algebra homomorphism

$$\epsilon : A[t,t^{-1}] \to A$$

defined by $t \mapsto 1$ makes A an $A[t,t^{-1}]$-module of Tor dimension 1, and so there is a map

$$\epsilon_* : K'(A[t,t^{-1}]) \to K'(A)$$

in the stable category. The map

$$\mathbf{M}(A) \to \mathbf{M}(A[t,t^{-1}])$$

defined by tensoring with the flat A-module $A[t,t^{-1}]$ takes values in the subcategory $\mathbf{N}_0$ of $\mathbf{M}(A[t,t^{-1})$ for which the module $\epsilon : A[t,t^{-1}] \to A$ is flat, and the composite

$$\mathbf{M}(A) \to \mathbf{N}_0 \to \mathbf{M}(A)$$

is isomorphic to the identity on $\mathbf{M}(A)$. It follows that the diagram

$$\begin{array}{ccc} K_i'(A[t]) & \longrightarrow & K_i'(A[t,t^{-1}]) \\ \cong\uparrow & & \downarrow\epsilon_* \\ K_i'(A) & \xrightarrow[1]{} & K_i'(A) \end{array}$$

commutes, giving the required split exact sequence. □

Corollary 23.4. *Suppose that A is a regular Noetherian ring. Then there is a short split exact sequence*

$$0 \to K_i(A) \to K_i(A[t,t^{-1}]) \to K_{i-1}(A) \to 0$$

for each $i \geq 1$.

References

[1] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.