Lecture 010 (October 27, 2014)

24 Rigidity

We will prove Suslin's first rigidity theorem [7]:

Theorem 24.1. *Suppose that $i : k \subset L$ is an inclusion of algebraically closed fields, and that n is a number such that $(n, char(k)) = 1$. Then the induced map*

$$i_* : K_*(k, \mathbb{Z}/n) \to K_*(L, \mathbb{Z}/n)$$

is an isomorphism.

This is now a baby rigidity theorem by comparison with more recent results, but the line of argument which Suslin introduced to prove Theorem 24.1 essentially survives in all subsequent proofs, albeit in settings of progressively increasing interest. The proof of Theorem 24.1 uses all of the basic results and calculational tools of algebraic K-theory which have been introduced in this course.

How would you prove such a thing?

1) First of all, it suffices, by a Zorn's Lemma argument, to assume that L has transcendance degree one over k.

The field L is a filtered colimit

$$L \cong \varinjlim_{A \subset L} A$$

of its finitely generated k-subalgebras A. We can assume that each such A is integrally closed in its field of fractions, and therefore defines an irreducible smooth affine curve $\mathrm{Sp}(A)$ over k.

Every point x of $\mathrm{Sp}(A)$ defines a k-rational point $x : \mathrm{Sp}(k) \to \mathrm{Sp}(A)$, which corresponds to a k-algebra homomorphism $x : A \to k$ which splits the inclusion $k \subset A$.

It follows that there is a stable equivalence

$$K(L)/n \simeq \varinjlim_{A \subset L} K(A)/n$$

and splitting

$$K(k)/n \to K(A)/n \xrightarrow{x_*} K(k)/n$$

corresponding to each rational point x of each affine curve $\mathrm{Sp}(A)$. It also follows that the induced homomorphisms

$$i_* : K_*(k, \mathbb{Z}/n) \to K_*(L, \mathbb{Z}/n)$$

in stable homotopy groups are monomorphisms of groups.

2) Here is the central idea of the proof: we show that any two rational points $x, y : \mathrm{Sp}(k) \to \mathrm{Sp}(A)$ of a smooth affine curve A induce the same maps

$$x_* = y_* : K_*(A, \mathbb{Z}/n) \to K_*(k, \mathbb{Z}/n).$$

Why is this enough? Suppose that $x : A \to k$ is a point of A. Then x and the inclusion $j : A \subset L$ together induce a diagram of ring homomorphisms

$$\begin{array}{ccccc} k & \xleftarrow{x} & A & \xrightarrow{j} & L \\ {\scriptstyle i}\downarrow & & \downarrow & & \downarrow{\scriptstyle =} \\ L & \xleftarrow[x_*]{} & A \otimes_k L & \xrightarrow[j_*]{} & L \end{array}$$

Then $\mathrm{Sp}(A \otimes_K L)$ is smooth over L, and

$$x_* = j_* : K_*(A \otimes_k L, \mathbb{Z}/n) \to K_*(L, \mathbb{Z}/n)$$

if we can prove the claim. If so, and $\alpha \in K_*(A, \mathbb{Z}/n)$, then

$$j_*(\alpha) = i_*(x_*(\alpha)),$$

in $K_*(L, \mathbb{Z}/n)$, and so the map

$$i_* : K_*(k, \mathbb{Z}/n) \to K_*(L, \mathbb{Z}/n)$$

is surjective.

3) The next step is to make the problem birational.

Suppose that $x \in Y(k)$ is a rational point for a smooth curve Y/k. Then the local ring $\mathcal{O}_x = \mathcal{O}_{x,Y}$

is a discrete valuation ring with quotient field $k(Y)$ (function field of Y) and residue map $p : \mathcal{O}_x \to k$. Let π_x be a choice of uniformizing parameter (aka. generator for the maximal ideal) for $\mathcal{O}_x$. Then it is a consequence of the results of Section 18 and 19 that the K-theory (symmetric) spectrum $K(\mathcal{O}_x)$ acts on the localization sequence

$$K'(k)/n \xrightarrow{p_*} K'(\mathcal{O}_x)/n \xrightarrow{j^*} K'(k(Y))/n$$

in such a way that there is a commutative diagram of pairings

$$\begin{array}{ccc}
K_p(\mathcal{O}_x) \otimes K'_q(k, \mathbb{Z}/n) & \xrightarrow{\cup\cdot(p^*\otimes 1)} & K'_{p+q}(k, \mathbb{Z}/n) \\
\downarrow{\scriptstyle 1\otimes p_*} & & \downarrow{\scriptstyle p_*} \\
K_p(\mathcal{O}_x) \otimes K'_q(\mathcal{O}_x, \mathbb{Z}/n) & \xrightarrow{\cup} & K'_{p+q}(\mathcal{O}_x, \mathbb{Z}/n) \\
\downarrow{\scriptstyle 1\otimes j^*} & & \downarrow{\scriptstyle j^*} \\
K_p(\mathcal{O}_x) \otimes K'_q(k(Y), \mathbb{Z}/n) & \xrightarrow[\cup\cdot(j^*\otimes 1)]{} & K'_{p+q}(k(Y), \mathbb{Z}/n)
\end{array}$$

Furthermore, for $\alpha \in K_p(\mathcal{O}_x)$ and $\beta \in K'_q(k(Y), \mathbb{Z}/n)$, we have the relation

$$\partial_x(j^*(\alpha) \cup \beta) = p^*(\alpha) \cup \partial_x(\beta)$$

in $K'_{p+q-1}(k, \mathbb{Z}/n)$.

Now consider the composition

$$
\begin{array}{ccc}
K_1(\mathcal{O}_x) \otimes K'_q(k(Y), \mathbb{Z}/n) & \xrightarrow{j^* \otimes 1} & K_1(k(Y)) \otimes K'_q(k(Y), \mathbb{Z}/n) \\
 & & \downarrow \cup \\
 & & K'_{q+1}(k(Y), \mathbb{Z}/n) \\
 & & \downarrow \partial_x \\
 & & K'_q(k, \mathbb{Z}/n)
\end{array}
$$

This composite is 0 since

$$\partial_x(j^*(\alpha) \cup \beta) = p^*(\alpha) \cup \partial_x(\beta)$$

and $p^*(\alpha) \in K_1(k)$, which is n-divisible. The sequence

$$K_1(\mathcal{O}_x) \xrightarrow{j^*} K_1(k(Y)) \xrightarrow{v_x} \mathbb{Z} \to 0$$

is exact, so there is a uniquely determined map

$$s_x : K'_q(k(Y), \mathbb{Z}/n) \to K'_q(k, \mathbb{Z}/n),$$

called the *specialization* at x, such that the diagram

$$
\begin{array}{ccc}
K_1(k(Y)) \otimes K'_q(k(Y), \mathbb{Z}/n) & \xrightarrow{v_x \otimes 1} & \mathbb{Z} \otimes K'_q(k(Y), \mathbb{Z}/n) \\
\cup \downarrow & & \downarrow s_x \\
K'_{q+1}(k(Y), \mathbb{Z}/n) & \xrightarrow[\partial_x]{} & K'_q(k, \mathbb{Z}/n)
\end{array}
$$

commutes. In equational terms,

$$s_x(\alpha) = \partial_x(\pi_x \otimes \alpha)$$

for each $\alpha \in K'_q(k(Y), \mathbb{Z}/n)$.

The composite

$$\begin{array}{ccc} K_1(k(Y)) \otimes K'_q(\mathcal{O}_x, \mathbb{Z}/n) & \xrightarrow{1 \otimes j^*} & K_1(k(Y)) \otimes K'_q(k(Y), \mathbb{Z}/n) \\ & & \downarrow \cup \\ & & K'_{q+1}(k(Y), \mathbb{Z}/n) \end{array}$$

is isomorphic to the composite

$$\begin{array}{ccc} K'_1(k(Y)) \otimes K_q(\mathcal{O}_x, \mathbb{Z}/n) & \xrightarrow{1 \otimes j^*} & K'_1(k(Y)) \otimes K_q(k(Y), \mathbb{Z}/n) \\ & & \downarrow \cup \\ & & K'_{q+1}(k(Y), \mathbb{Z}/n) \end{array}$$

which arises from the pairing of fibre sequences

$$\begin{array}{ccc} K'(k) \wedge K(\mathcal{O}_x)/n & \xrightarrow{\cup \cdot (1 \wedge p^*)} & K'(k)/n \\ {\scriptstyle p_* \wedge 1} \downarrow & & \downarrow {\scriptstyle p_*} \\ K'(\mathcal{O}_x) \wedge K(\mathcal{O}_x)/n & \xrightarrow{\cup} & K'(\mathcal{O}_x)/n \\ {\scriptstyle j_* \wedge 1} \downarrow & & \downarrow {\scriptstyle j^*} \\ K'(k(Y)) \wedge K(\mathcal{O}_x)/n & \xrightarrow[\cup \cdot (1 \wedge j^*)]{} & K'(k(Y))/n \end{array}$$

and it follows (by twisting the action, and thus introducing signs) that

$$s_x(j^*(\gamma)) = \partial_x(\pi_x \cup j^*(\gamma)) = (-1)^q p^*(\gamma)$$

for $\gamma \in K'_q(\mathcal{O}_x, \mathbb{Z}/n)$.

Here are some consequences:

a) The diagram

$$\begin{array}{ccc} K'_q(Y,\mathbb{Z}/n) & & \\ {\scriptstyle j^*}\downarrow & \searrow^{(-1)^q x^*} & \\ K'_q(k(Y),\mathbb{Z}/n) & \xrightarrow[s_x]{} & K'_q(k,\mathbb{Z}/n) \end{array}$$

commutes for all k-rational points $x \in Y$.

b) The map x_* is a split surjection, so that s_x is surjective, as is the boundary map

$$\partial_x : K'_{q+1}(k(Y),\mathbb{Z}/n) \to K'_q(k,\mathbb{Z}/n).$$

In particular, the transfer map

$$K'_q(k,\mathbb{Z}/n) \xrightarrow{p_*} K'_q(\mathcal{O}_x,\mathbb{Z}/n)$$

is the 0 map, and the localization homomorphism

$$K'_q(\mathcal{O}_x,\mathbb{Z}/n) \xrightarrow{j^*} K'_q(k(Y),\mathbb{Z}/n)$$

is injective.

It therefore suffices to prove the following:

Lemma 24.2. *Suppose that Y is a smooth curve over an algebraically closed field k, and suppose that $(n, char(k)) = 1$. Then the specialization maps*

$$s_x : K'_q(k(Y),\mathbb{Z}/n) \to K'_q(k,\mathbb{Z}/n)$$

associated to the closed points $x \in Y$ all coincide.

Some remarks are in order:

a) It suffices to prove Lemma 24.2 for smooth complete curves Y, because any other smooth curve over k has a smooth compactification.

b) The smooth complete curves Y over k are in one to one correspondence with the field extensions L/k of transcendance degree 1. Given Y the corresponding field is the function field $k(Y)$, and given L one can make a smooth complete curve $Y(L)$ whose closed points are all the discrete valuation rings $\mathcal{O}$ with function field L, and whose closed subsets are the finite subsets. Finally

$$\Gamma(U, \mathcal{O}_{Y(L)}) = \cap_{R \in U} R$$

defines the sheaf of rings $\mathcal{O}_{Y(L)}$ for $Y(L)$.

Example: The projective line $\mathbb{P}^1$ over k is the curve associated to the field $k(t)$. The DVRs in $k(t)$ are of the form $k[t]_{(t-a)}$ or $k[\frac{1}{t-a}]_{(\frac{1}{t-a})}$. Define the points 0 and ∞ by $0 = k[t]_{(t)}$ and $\infty = k[\frac{1}{t}]_{(\frac{1}{t})}$, respectively.

We need another calculational tool:

Suppose that $\phi : Y \to X$ is a finite (surjective), flat morphism of complete smooth curves over k. Localization commutes with transfer, so that there is a comparison of long exact sequences

$$\begin{array}{ccccccc}
\bigoplus_{y\in Y} K'_q(k,\mathbb{Z}/n) & \longrightarrow & K'_q(Y,\mathbb{Z}/n) & \xrightarrow{j^*} & K'_q(k(Y),\mathbb{Z}/n) & \xrightarrow{\partial} & \dots \\
\downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} & & \\
\bigoplus_{x\in X} K'_q(k,\mathbb{Z}/n) & \longrightarrow & K'_q(X,\mathbb{Z}/n) & \xrightarrow[j^*]{} & K'_q(k(X),\mathbb{Z}/n) & \xrightarrow[\partial]{} & \dots
\end{array}$$

The fibre

$$\mathrm{Sp}(k) \times_X Y$$

over $x : \mathrm{Sp}(k) \to X$ is finite over $\mathrm{Sp}(k)$, and therefore has reduced subscheme $\sqcup_{\phi^{-1}(x)} \mathrm{Sp}(k)$. It follows that the transfer map

$$\phi_* : K'_q(\mathrm{Sp}(k) \times_X Y, \mathbb{Z}/n) \to K'_q(k, \mathbb{Z}/n)$$

is isomorphic to the fold map

$$\nabla : \bigoplus_{\phi^{-1}(x)} K'_q(k, \mathbb{Z}/n) \to K'_q(k, \mathbb{Z}/n).$$

This means that there is a commutative diagram

$$\begin{array}{ccc}
K'_q(k,\mathbb{Z}/n) & \xrightarrow{in_y} & \bigoplus_{y\in Y} K'_q(k,\mathbb{Z}/n) \\
& {\scriptstyle in_{\phi(y)}} \searrow & \downarrow{\scriptstyle \phi_1} \\
& & \bigoplus_{x\in X} K'_q(k,\mathbb{Z}/n)
\end{array}$$

which defines ϕ_1.

Suppose that Y is a smooth complete curve over k, as in the statement of Lemma 24.2. The degree homomorphism

$$deg : \mathrm{Div}(Y) = \bigoplus_{y \in Y} \mathbb{Z} \to \mathbb{Z}$$

on the group of divisors of Y is just the standard fold map ∇ on the direct sum. Write $\mathrm{Div}^0(Y)$ for the kernel of the degree homomorphism. Observe that if x, y are rational points (aka. divisors) of Y then the divisor $x - y$ is an element of $\mathrm{Div}^0(Y)$.

If $\phi : Y \to X$ is a finite flat map of smooth complete curves over k then the inverse image functor ϕ^* is exact on coherent sheaves, and there is an induced comparison of localization sequences

$$\begin{array}{ccccccc} k(X)^* & \xrightarrow{(v_x)} & \bigoplus_{x \in X} \mathbb{Z} & \longrightarrow & K_0(X) & \longrightarrow & K_0(k(X)) \\ \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi_2} & & \downarrow & & \downarrow \\ k(Y)^* & \xrightarrow[(v_y)]{} & \bigoplus_{y \in Y} \mathbb{Z} & \longrightarrow & K_0(Y) & \longrightarrow & K_0(k(Y)) \end{array} \tag{1}$$

One isolates $\phi_2(x)$ in the component corresponding to y such that $\phi(y) = x$ by localizing the sequences at x and y, and then one sees that

$$\phi_2(x) = v_y \phi^*(\pi_x) = n_y$$

where $\phi^*(\pi_x) = \pi_y^{n_y}$, and so n_y is the ramification index of x at y. It follows that

$$\phi_2(x) = \sum_{\phi(y)=x} n_y y$$

for points $x \in X$ in $\mathrm{Div}(Y)$.

There is a commutative diagram

$$\begin{array}{ccccc} k(X)^* & \xrightarrow{(v_x)} & \bigoplus_{x\in X} \mathbb{Z} & \xrightarrow{deg} & \mathbb{Z} \\ \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle d} \\ k(Y)^* & \xrightarrow[(v_y)]{} & \bigoplus_{y\in Y} \mathbb{Z} & \xrightarrow[deg]{} & \mathbb{Z} \end{array}$$

where d is multiplication by the degree

$$d = [k(Y); k(X)]$$

of the extension — it is the common dimension of all fibres of the map ϕ, and therefore coincides with the dimension of the fibre $\mathrm{Sp}(k) \times_X Y$ over x, which is $\sum_{\phi(y)=x} n_y$, as well as the dimension of the generic fibre, which is d.

It is a standard fact that the valuation map

$$v : k(Y)^* \to \mathrm{Div}(Y)$$

takes values in $\mathrm{Div}^0(Y)$: one sees this for $\alpha \in k(Y)^*$ by choosing a homomorphism $k(t) \to k(Y)$ with $t \mapsto \alpha$, and then using the instance of diagram (1) for the corresponding morphism $\phi : Y \to$

$\mathbb{P}^1$. The punch line is that

$$deg(t) = 0 - \infty \in \mathrm{Div}(\mathbb{P}^1).$$

The *Jacobian* $J(Y)$ is the cokernel of the map

$$v : k(Y)^* \to \mathrm{Div}^0(Y)$$

The group $J(Y)$ is the group of points of an abelian variety over k, and is uniquely n-divisible.

For the proof of Lemma 24.2, the specialization maps

$$s_y : K'_q(k(Y), \mathbb{Z}/n) \to K'_q(k, \mathbb{Z}/n)$$

together determine a homomorphism

$$s : \mathrm{Div}(Y) \otimes K'_q(k(Y), \mathbb{Z}/n) \to K'_q(k, \mathbb{Z}/n),$$

and restrict to a homomorphism

$$s : \mathrm{Div}^0(Y) \otimes K'_q(k(Y), \mathbb{Z}/n) \to K'_q(k, \mathbb{Z}/n),$$

in the obvious way. Lemma 24.2 will be proved if we can show that the composite

$$\begin{array}{ccc} k(Y)^* \otimes K'_q(k(Y), \mathbb{Z}/n) & \xrightarrow{v \otimes 1} & \mathrm{Div}^0(Y) \otimes K'_q(k(Y), \mathbb{Z}/n) \\ & \searrow & \downarrow s \\ & & K'_q(k, \mathbb{Z}/n) \end{array}$$

is 0, for then s factors through the group

$$J(Y) \otimes K'_q(k(Y), \mathbb{Z}/n) \cong 0,$$

and $x - y \in \mathrm{Div}^0(Y)$.

Pick $\alpha \in k(Y)^*$ and let $\phi : Y \to \mathbb{P}^1$ be the finite flat morphism of smooth complete curves such that $t \mapsto \alpha$ under the induced map $k(t) \to k(Y)$. Let x be an element of $\mathbb{P}^1(k)$. Let γ be an element of $K'_q(k(Y), \mathbb{Z}/n)$.

We have a commutative diagram

$$\begin{array}{ccc} K'_{q+1}(k(Y), \mathbb{Z}/n) & \xrightarrow{(\partial_y)} & \bigoplus_{y \in \phi^{-1}(x)} K'_q(k, \mathbb{Z}/n) \\ \downarrow \phi_* & & \downarrow \nabla \\ K'_{q+1}(k(t), \mathbb{Z}/n) & \xrightarrow[\partial_x]{} & K'_q(k, \mathbb{Z}/n) \end{array}$$

and the projection formula

$$\begin{array}{ccccc} & & K_1(k(Y)) \otimes K_q(k(Y), \mathbb{Z}/n) & \xrightarrow{\cup} & K'_{q+1}(k(Y), \mathbb{Z}/n) \\ & \nearrow^{\phi^* \otimes 1} & & & \\ K_1(k(t)) \otimes K'_q(k(Y), \mathbb{Z}/n) & & & & \downarrow \phi_* \\ & \searrow_{1 \otimes \phi_*} & & & \\ & & K_1(k(t)) \otimes K_q(k(t), \mathbb{Z}/n) & \xrightarrow[\cup]{} & K'_{q+1}(k(t), \mathbb{Z}/n) \end{array}$$

It follows that

$$\begin{aligned} s_x(\phi_*(\gamma)) &= \partial_x(\pi_x \cup \phi_*(\gamma)) \\ &= \sum_{y\in\phi^{-1}(x)} \partial_y(\phi^*(\pi_x)\cup\gamma) \\ &= \sum_{y\in\phi^{-1}(x)} n_y\partial_y(\pi_y\cup\gamma) \\ &= \sum_{y\in\phi^{-1}(x)} n_y s_y(\gamma). \end{aligned}$$

It follows that the diagram

$$\begin{array}{ccc} \mathrm{Div}^0(\mathbb{P}^1)\otimes K_q'(k(Y),\mathbb{Z}/n) & \xrightarrow{1\otimes\phi_*} & \mathrm{Div}^0(\mathbb{P}^1)\otimes K_q'(k(t),\mathbb{Z}/n) \\ \downarrow{\scriptstyle \phi_2\otimes 1} & & \downarrow{\scriptstyle s_{\mathbb{P}^1}} \\ \mathrm{Div}^0(Y)\otimes K_q'(k(Y),\mathbb{Z}/n) & \xrightarrow[s_Y]{} & K_q'(k,\mathbb{Z}/n) \end{array}$$

commutes. Thus,

$$\begin{aligned} s_Y(v\otimes 1)(\alpha\otimes\gamma) &= s_Y(v_Y\otimes 1)(\phi^*\otimes 1)(t\otimes\gamma) \\ &= s_{\mathbb{P}^1}(1\otimes\phi_*)(v_{\mathbb{P}^1}\otimes 1)(t\otimes\gamma) \\ &= s_0(\phi_*(\gamma)) - s_\infty(\phi_*(\gamma)). \end{aligned}$$

It follows that $s_Y\cdot(v_Y\otimes 1)=0$ on

$$\mathrm{Div}^0(Y)\otimes K_q'(k(Y),\mathbb{Z}/n)$$

if $s_0 = s_\infty$ on $K_q'(k(t),\mathbb{Z}/n)$, and this is what we shall prove.

From the localization sequence

$$\cdots \to K'_q(k[t], \mathbb{Z}/n) \to K'_q(k(t), \mathbb{Z}/n) \to \bigoplus_{a \in k} K'_{q-1}(k, \mathbb{Z}/n) \to \dots$$

and the isomorphism

$$K'_q(k, \mathbb{Z}/n) \cong K'_q(k[t], \mathbb{Z}/n)$$

(homotopy property) one sees that

$$K'_q(k(t), \mathbb{Z}/n) \cong K'_q(k, \mathbb{Z}/n) \oplus (\bigoplus_{a \in k} K'_{q-1}(k, \mathbb{Z}/n)),$$

where the splitting on the summand corresponding to $a \in K$ is defined for $\gamma \in K'_{q-1}(k, \mathbb{Z}/n)$ by the assignment $\gamma \mapsto (t-a) \cup \overline{\gamma}$, where $\gamma \mapsto \overline{\gamma}$ under the homomorphism

$$K_{q-1}(k, \mathbb{Z}/n) \to K_{q-1}(k(t), \mathbb{Z}/n).$$

Then

$$\begin{aligned} s_x((t-a) \cup \overline{\gamma}) &= \partial_x(\pi_x \cup (t-a) \cup \overline{\gamma}) \\ &= (-1)^{q-1} \partial_x(\pi_x \cup (t-a)) \cup \gamma \\ &= 0 \end{aligned}$$

for all x and a, since $\partial_x(\pi_x \cup (t-a)) \in K_1(k)$, and $K_1(k)$ is n-divisible.

Finally, $s_\infty = s_0$ on the image of the map

$$K_q(k, \mathbb{Z}/n) \cong K_q(k[t], \mathbb{Z}/n) \to K_q(k(t), \mathbb{Z}/n),$$

and Theorem 24.1 is proved.

25 Gabber rigidity

In all that follows, suppose that k is an algebraically closed field and that ℓ is a prime number such that $(\ell, char(k)) = 1$.

Here is an obvious consequence of Theorem 24.1 (with Theorem 20.4):

Corollary 25.1. *Suppose that $i : k \subset L$ is an inclusion of algebraically closed Then the induced maps*

$$i_* : H_*(BGl(k), \mathbb{Z}/\ell) \to H_*(BGl(L), \mathbb{Z}/\ell),$$
$$i^* : H^*(BGl(L), \mathbb{Z}/\ell) \to H^*(BGl(k), \mathbb{Z}/\ell)$$

are isomorphisms.

The homology statement is equivalent to the mod ℓ version of Theorem 24.1. Furthermore, the collection of mod ℓ versions of Theorem 24.1, for all primes ℓ such that $(\ell, char(k)) = 1$, implies the full statement of Theorem 24.1.

Suslin's rigidity theorem was followed rather closely by the Gabber rigidity theorem. The following is the first published statement of this result, which appeared in a paper of Gillet and Thomason [2]:

Theorem 25.2. *Suppose that X is a smooth variety over k, and let $x \in X(k)$ be a k-rational* *point. Suppose that $\mathcal{O}_x^h$ is the (strict) henselization of the local ring $\mathcal{O}_x = \mathcal{O}_{x,X}$. Then the residue homomorphism $\mathcal{O}_x^h \to k$ induces an isomorphism*

$$K_*(\mathcal{O}_x^h, \mathbb{Z}/\ell) \xrightarrow{\cong} K_*(k, \mathbb{Z}/\ell).$$

A more modern version and proof of this result appears in [9]. The proof given in [8] is a more axiomatic and more general. Gabber's full rigidity result (which is about henselian pairs), was published in [1].

Here's the obvious corollary:

Corollary 25.3. *Suppose given the conditions of Theorem 25.2. Then the residue homomorphism $\mathcal{O}_x^h \to k$ induces an isomorphism*

$$H_*(BGl(\mathcal{O}_x^h), \mathbb{Z}/\ell) \xrightarrow{\cong} H_*(BGl(k), \mathbb{Z}/\ell).$$

Of course, Corollary 25.3 is equivalent to Theorem 25.2, via Theorem 20.4. Corollary 25.3 is also equivalent to the assertion that the k-algebra structure map $k \to \mathcal{O}_x^h$ induces an isomorphism

$$H_*(BGl(k), \mathbb{Z}/\ell) \xrightarrow{\cong} H_*(BGl(\mathcal{O}_x^h), \mathbb{Z}/\ell). \quad (2)$$

Now to say something about what you can do with this result.

The algebraic group Gl_n represents a sheaf of groups on the étale site $(Sm|_k)_{et}$ of smooth k-schemes, which sheaf will have the same name. Taking the filtered colimit of the presheaves

$$Gl_1 \subset Gl_2 \subset Gl_3 \subset \dots$$

along the standard inclusions defines a presheaf of groups

$$Gl = \varinjlim_n Gl_n$$

and so one is entitled to a simplicial presheaf BGl which is defined by applying the classifying space functor B to all of the groups of sections $Gl(U)$. The maps $U \to \mathrm{Sp}(k)$ (U smooth) define transition maps

$$BGl(k) \to BGl(U),$$

which together define a canonical morphism

$$\epsilon : \Gamma^* BGl(k) \to BGl$$

of simplicial presheaves, where $\Gamma^* BGl(k)$ is the constant simplicial presheaf on the simplicial set $BGl(k)$. Simplicial presheaves X have mod ℓ homology presheaves $H_n(X, \mathbb{Z}/\ell)$ defined by

$$U \to H_n(X(U), \mathbb{Z}/\ell)$$

and associated homology sheaves $\tilde{H}_n(X, \mathbb{Z}/\ell)$. The stalks of the induced morphism

$$\tilde{H}_n(\Gamma^* BGl(k), \mathbb{Z}/\ell) \to \tilde{H}_n(BGl, \mathbb{Z}/\ell)$$

are the group homomorphisms

$$H_n(BGl(k), \mathbb{Z}/\ell) \to H_n(BGl(\mathcal{O}_x^h), \mathbb{Z}/\ell),$$

which are isomorphisms by Gabber rigidity (Corollary 25.3).

The map

$$\epsilon : \Gamma^* BGl(k) \to BGl$$

is therefore a mod ℓ homology sheaf isomorphism, by Gabber rigidity — in fact, the two statements are equivalent. The mod ℓ étale cohomology groups $H^n(X, \mathbb{Z}/\ell)$ for a simplicial presheaf X are defined by morphisms

$$H^n(X, \mathbb{Z}/\ell) = [X, K(\mathbb{Z}/\ell, n)]$$

in the homotopy category of simplicial presheaves on the site $(Sm|_k)_{et}$ associated to the (injective) model structure on simplicial presheaves, for which the cofibrations are monomorphisms and the weak equivalences are defined stalkwise. Then $H^n(X, \mathbb{Z}/\ell)$ coincides with the étale cohomology group $H^n_{et}(X, \mathbb{Z}/\ell)$ if X is represented by a scheme, and there is a similar coincidence for simplicial presheaves represented by simplicial schemes. There are $\varinjlim^1$ exact sequences for the étale cohomology of filtered colimits, just as for spaces, as well as naturally defined cup product structures. It follows that there are isomorphisms

$$H^*(BGl, \mathbb{Z}/\ell) \cong H^*(BU, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_1, c_2, \dots]$$

where the i^{th} *Chern class* c_i has degree $2i$. At the same time, the simplicial sheaf associated to the constant simplicial presheaf $\Gamma^* BGl(k)$ is represented by the simplicial scheme $\bigsqcup_{BGl(k)} \mathrm{Sp}(k)$. Algebraically closed fields are points in the eyes of étale cohomology, so there is an isomorphism

$$H^*(\Gamma^* BGl(k), \mathbb{Z}/\ell) \cong H^*(BGl(k), \mathbb{Z}/\ell).$$

Homology sheaf isomorphisms induce cohomology group isomorphisms, so we have proved

Theorem 25.4. *Suppose that k is an algebraically closed field, and let ℓ be a prime such that $(char(k), \ell) = 1$. Then the canonical map $\epsilon : \Gamma^* BGl(k) \to BGl$ induces isomorphisms*

$$H^*(BGl(k), \mathbb{Z}/\ell) \cong H^*_{et}(BGl, \mathbb{Z}/\ell).$$

In other words, the discrete and étale mod ℓ cohomology groups coincide for the general linear group

Gl.

Suslin's first rigidity theorem (Theorem 24.1) is a consequence:

Corollary 25.5. *Suppose that $i : k \subset L$ is an inclusion of algebraically closed fields, and let ℓ be a prime such that $(char(k), \ell) = 1$. Then the induced maps*

$$H_*(BGl(k), \mathbb{Z}/\ell) \to H_*(BGl(L), \mathbb{Z}/\ell)$$
$$K_*(k, \mathbb{Z}/\ell) \to K_*(L, \mathbb{Z}/\ell)$$

are isomorphisms.

Proof. There is a commutative diagram

$$\begin{array}{ccc} H^*_{et}(BGl_L, \mathbb{Z}/\ell) & \xrightarrow[\cong]{\epsilon^*} & H^*(BGl(L), \mathbb{Z}/\ell) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ H^*_{et}(BGl_k, \mathbb{Z}/\ell) & \xrightarrow[\epsilon_*]{\cong} & H^*(BGl(k), \mathbb{Z}/\ell) \end{array}$$

in which the base change morphism i^* in étale cohomology is an isomorphism by standard theory. It follows that the map

$$i^* : H^*(BGl(L), \mathbb{Z}/\ell) \to H^*(BGl(k), \mathbb{Z}/\ell)$$

is an isomorphism of finite dimensional $\mathbb{Z}/\ell$-vector spaces in each degree, and so the map

$$i_* : H_*(BGl(k), \mathbb{Z}/\ell) \to H_*(BGl(L), \mathbb{Z}/\ell)$$

is also an isomorphism of finite dimensional $\mathbb{Z}/\ell$-vector spaces. $\square$

On the topological side, there is a map of simplicial spaces

$$\epsilon : BGl(\mathbb{C}) \to BGl(\mathbb{C})_{top} \simeq BU.$$

The displayed weak equivalence is due to the fact that the unitary group U_n is the maximal compact subgroup of the topological group $Gl_n(\mathbb{C})$. Write H_n for the subgroup of diagonal matrices in Gl_n. Then it's relatively easy to see that the map

$$\epsilon : BH_n(\mathbb{C}) \to BH_n(\mathbb{C})_{top}$$

is a mod ℓ homology isomorphism, since both spaces have the mod ℓ homology of the space $B(\mu_{\ell^\infty})^{\times n}$ where μ_{ℓ^∞} is the group of ℓ-primary roots of unity in $\mathbb{C}$. It follows that the induced map

$$\epsilon^* : H^*(BU, \mathbb{Z}/\ell) \to H^*(BGl(\mathbb{C}), \mathbb{Z}/\ell)$$

is a monomorphism. But ϵ^* is, by Theorem 25.4, a monomorphism of $\mathbb{Z}/\ell$-vector spaces having the same dimension in each degree, and is therefore an isomorphism. We have essentially proved

Corollary 25.6. *The canonical map* $\epsilon : BGl(\mathbb{C}) \to$

$BGl(\mathbb{C})_{top}$ induces isomorphisms

$$H^*(BU, \mathbb{Z}/\ell) \xrightarrow{\cong} H^*(BGl(\mathbb{C}), \mathbb{Z}/\ell)$$
$$H_*(BGl(\mathbb{C}), \mathbb{Z}/\ell) \xrightarrow{\cong} H_*(BU, \mathbb{Z}/\ell)$$
$$K_*(\mathbb{C}, \mathbb{Z}/\ell) \xrightarrow{\cong} \pi_*(BU/\ell).$$

One has to be a little careful about the last statement but it follows from the homology isomorphism by an argument similar to that for Theorem 20.4 — see [3].

We also know that there is an isomorphism of Galois group modules

$$K_q(\overline{\mathbb{F}}_p, \mathbb{Z}/\ell) \cong \begin{cases} \mu_\ell^{\otimes r} & \text{if } q = 2r, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

for the algebraic closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$ (for $p \neq \ell$), by Quillen's calculation of the K-theory of finite fields [5] [6] (see also [4]). A modern proof of Quillen's result for finite fields is sketched in Lecture 011.

The last isomorphism in the statement of Corollary 25.6 was one of the major outcomes of Suslin's paper [10]. With this result, we have a complete calculation of the algebraic K-theory of an algebraically closed field k with torsion coefficients:

Theorem 25.7 (Suslin)**.** *Suppose that k is an algebraically closed field and that ℓ is a prime such that $(char(k), \ell) = 1$. Then there are isomorphisms*

$$K_q(k, \mathbb{Z}/\ell) \cong \begin{cases} \mu_\ell^{\otimes r} & \text{if } q = 2r, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Note that, when the Moore spectrum S/ℓ is a ring spectrum (see the remarks in Section 19), Suslin's calculation devolves to a ring isomorphism

$$K_*(k, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\beta],$$

so that the the mod ℓ K-theory of k is a polynomial ring which is generated over $\mathbb{Z}/\ell$ by the Bott element, most of the time.

References

[1] Ofer Gabber. K-theory of Henselian local rings and Henselian pairs. In *Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989)*, volume 126 of *Contemp. Math.*, pages 59–70. Amer. Math. Soc., Providence, RI, 1992.

[2] Henri A. Gillet and Robert W. Thomason. The K-theory of strict Hensel local rings and a theorem of Suslin. In *Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983)*, volume 34, pages 241–254, 1984.

[3] J. F. Jardine. Simplicial objects in a Grothendieck topos. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, pages 193–239. Amer. Math. Soc., Providence, RI, 1986.

[4] J. F. Jardine. The K-theory of finite fields, revisited. *K-Theory*, 7(6):579–595, 1993.

[5] Daniel Quillen. The Adams conjecture. *Topology*, 10:67–80, 1971.

[6] Daniel Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Ann. of Math. (2)*, 96:552–586, 1972.

[7] A. Suslin. On the K-theory of algebraically closed fields. *Invent. Math.*, 73(2):241–245, 1983.

[8] A. A. Suslin. Algebraic K-theory of fields. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 222–244, Providence, RI, 1987. Amer. Math. Soc.

[9] Andrei Suslin and Vladimir Voevodsky. Singular homology of abstract algebraic varieties. *Invent. Math.*, 123(1):61–94, 1996.

[10] Andrei A. Suslin. On the K-theory of local fields. In *Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983)*, volume 34, pages 301–318, 1984.