Lecture 011 (October 17, 2014)

26 K-theory of finite fields

We will sketch a proof of the following well-known result of Quillen [5]:

Theorem 26.1. *Suppose that $\mathbb{F}_q$ is the field with $q = p^n$ elements. Then there are isomorphisms*

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/(q^j - 1) & \textit{if } i = 2j-1, j > 0, \textit{ and} \\ 0 & \textit{if } i = 2j, j > 0. \end{cases}$$

The proof that is given here appears in [4].

The first step is to completely compute the groups $K_n(\overline{\mathbb{F}}_q)$ for the algebraic closure $\overline{\mathbb{F}}_q$. For this, we use the following form of the Gabber rigidity theorem [3]:

Theorem 26.2. *Suppose that $\mathcal{O}$ is a henselian local ring with residue field k, and that $1/n \in \mathcal{O}$. Then the residue map $\pi : \mathcal{O} \to k$ induces isomorphisms*

$$\pi_* : K_*(\mathcal{O}, \mathbb{Z}/n) \xrightarrow{\cong} K_*(k, \mathbb{Z}/n).$$

Examples of henselian local rings $\mathcal{O}$ include the Witt ring $W(\overline{\mathbb{F}}_q)$ of the field $\overline{\mathbb{F}}_q$. The residue map for $W(\overline{\mathbb{F}}_q)$ has the form $W(\overline{\mathbb{F}}_q) \to \overline{\mathbb{F}}_q$, and $W(\overline{\mathbb{F}}_q)$ is a complete DVR with a quotient field K of characteristic 0, since $\overline{\mathbb{F}}_q$ is perfect.

It is a consequence of Gabber rigidity that the ring homomorphisms

$$\overline{\mathbb{F}}_q \stackrel{\pi}{\leftarrow} W(\overline{\mathbb{F}}_q) \to \overline{K} \leftarrow \overline{\mathbb{Q}} \to \mathbb{C}$$

induce isomorphisms

$$\begin{aligned} K_*(\overline{\mathbb{F}}_q, \mathbb{Z}/n) &\cong K_*(W(\overline{\mathbb{F}}_q), \mathbb{Z}/n) \cong K_*(\overline{K}, \mathbb{Z}/n) \\ &\cong K_*(\overline{\mathbb{Q}}, \mathbb{Z}/n) \cong K_*(\mathbb{C}, \mathbb{Z}/n) \end{aligned}$$

for $(n,p) = 1$.

Some comments:

1) The map

$$H^*_{et}(BGl_{W(\overline{\mathbb{F}}_q)}, \mathbb{Z}/n) \to H^*_{et}(BGl_{\overline{K}}, \mathbb{Z}|/n)$$

is an isomorphism by smooth proper base change for algebraic groups [2], and the map

$$H^*_{et}(BGl_{W(\overline{\mathbb{F}}_q)}, \mathbb{Z}/n) \to H^*(BGl(W(\overline{\mathbb{F}}_q)), \mathbb{Z}/n)$$

is an isomorphism by the Gabber theorem (the homology sheaves $\tilde{H}_n(BGl_{W(\overline{\mathbb{F}}_q)}, \mathbb{Z}/n)$ are constant) and the fact that strict local hensel rings $\mathcal{O}$ have no étale cohomology (global sections on $\mathbf{Shv}(Sch|_{\mathcal{O}})_{et}$

is exact). From the diagram

$$\begin{array}{ccc} H^*_{et}(BGl_{\overline{K}}, \mathbb{Z}/n) & \xrightarrow{\cong} & H^*(BGl(\overline{K}), \mathbb{Z}/n) \\ & \searrow^{\cong} & \downarrow \\ & & H^*(BGl(W(\overline{\mathbb{F}}_q)), \mathbb{Z}/n)) \end{array}$$

we see that the map

$$H^*(BGl(\overline{K}), \mathbb{Z}/n) \to H^*(BGl(W(\overline{\mathbb{F}}_q), \mathbb{Z}/n))$$

is an isomorphism.

2) The map

$$BGl(\mathbb{C}) \to BGl(\mathbb{C})^{top} \simeq BU$$

induces a monomorphism

$$H^*(BU, \mathbb{Z}/\ell) \to H^*(BGl(\mathbb{C}), \mathbb{Z}/\ell).$$

In effect, the comparison

$$BT(\mathbb{C}) \to BT(\mathbb{C})^{top}$$

induces an isomorphism

$$H^*(BT(\mathbb{C})^{top}, \mathbb{Z}/\ell) \cong H^*(BT(\mathbb{C}), \mathbb{Z}/\ell)$$

for any complex torus T (exercise), and the map

$$H^*(BGl_n(\mathbb{C})^{top}, \mathbb{Z}/\ell) \to H^*(BT(\mathbb{C})^{top}, \mathbb{Z}/\ell)$$

which is induced by the inclusion $T \subset Gl_n$ of a maximal torus induces a monomorphism.

It follows that the map

$$H^p(BU, \mathbb{Z}/\ell) \to H^p(BGl(\mathbb{C}), \mathbb{Z}/\ell)$$

is a monomorphism of finite dimensional $\mathbb{Z}/\ell$-vector spaces of the same dimension (by Gabber rigidity), and is therefore an isomorphism, for all $p \geq 0$.

3) In complex K-theory, there is an isomorphism $\pi_2 BU \cong \pi_1(U) \cong \mathbb{Z}$, with generator β, and complex Bott periodicity says that cup product (induced by tensor product) with the generator β induces a map

$$\beta_* : ku[2] \simeq S^2 \wedge ku \to ku$$

which is an isomorphism in stable homotopy groups π_i for $i \geq 2$. Here, ku is connective (topological) complex K-theory, which is formed by group-completing the monoid

$$\bigsqcup_{n \geq 0} BU_n \simeq \bigsqcup_{n \geq 0} BGl_n(\mathbb{C})^{top},$$

or rather by taking a fibrant model of the spectrum associated to the Γ-space which arises from direct sum of matrices. In particular, there is a weak equivalence

$$ku^0 \simeq \mathbb{Z} \times BU.$$

The map of monoids

$$\bigsqcup_{n\geq 0} BGl_n(\mathbb{C}) \to \bigsqcup_{n\geq 0} BGl_n(\mathbb{C})^{top}$$

induces a map of (symmetric) spectra

$$\epsilon : K(\mathbb{C}) \to ku.$$

Both spectra are ring spectra with ring structure induced by tensor product, so that ϵ is a map of ring spectra.

The map

$$BGl(\mathbb{C}) \to BU$$

induces an isomorphism

$$H_*(BGl(\mathbb{C}), \mathbb{Z}/\ell) \xrightarrow{\cong} H_*(BU, \mathbb{Z}/\ell).$$

In effect, the map is dual to an isomorphism in mod ℓ cohomology.

It follows that the induced map

$$\epsilon : K(\mathbb{C})/\ell \to ku/\ell$$

is a stable equivalence. To see this, one shows that the homotopy fibre F of the map ϵ has uniquely ℓ-divisible homotopy groups, by an argument similar to that for Theorem 20.4 in Lecture 008.

It follows in particular that the map

$$\pi_2 K(\mathbb{C})/\ell \to \pi_2 ku/\ell$$

is an isomorphism, and therefore takes Bott element to Bott element.

The Bott element $\beta \in \pi_2 ku/\ell$ is the image of the Bott element in $\pi_2 ku$ under the map

$$\pi_2 ku \to \pi_2 ku/\ell.$$

There is a comparison of cofibre sequences

$$\begin{array}{ccccc} S^2 \wedge ku & \longrightarrow & S^2 \wedge ku & \longrightarrow & S^2 \wedge ku/\ell \\ \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_*} \\ ku & \longrightarrow & ku & \longrightarrow & ku/\ell \end{array}$$

in which all vertical maps are defined by (left) multiplication by the Bott elelement. It follows that the map

$$\beta_* : S^2 \wedge ku/\ell \to ku/\ell$$

is an isomorphism in π_i for $i \geq 2$.

If E is a ring spectrum and the Moore spectrum S/n has a ring spectrum structure, then the composite

$$E \wedge S/n \wedge E \wedge S/n \xrightarrow{1 \wedge \tau \wedge 1} E \wedge E \wedge S/n \wedge S/n \xrightarrow{m \wedge m} E \wedge S/n$$

defines a ring spectrum structure on

$$E/n = E \wedge S/n.$$

The Moore spectrum S/n has such a ring spectrum structure if $n = \ell^\nu$ where $\ell > 3$, $\nu \geq 2$ if $\ell = 3$, and $\nu \geq 4$ if $\ell = 2$ [7, p.544]. Assume that n is such a prime power henceforth.

There is a commutative diagram

$$\begin{array}{ccccc}
S^2 \wedge ku/n & \xrightarrow{\beta\wedge 1} & ku \wedge ku/n & & \\
& \searrow^{\beta\wedge 1} & \downarrow{\scriptstyle p\wedge 1} & \searrow^{m\wedge 1} & \\
{\scriptstyle 1\wedge\epsilon}\uparrow{\scriptstyle \simeq} & & ku/n \wedge ku/n & \xrightarrow[m]{} & ku/n \\
& & {\scriptstyle \epsilon\wedge\epsilon}\uparrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\uparrow{\scriptstyle \epsilon} \\
S^2 \wedge K(\mathbb{C})/n & \xrightarrow[\beta\wedge 1]{} & K(\mathbb{C})/n \wedge K(\mathbb{C})/n & \xrightarrow[m]{} & K(\mathbb{C})/n
\end{array}$$

The top composite $(m \wedge 1)(\beta \wedge 1)$ is the composite β_* above, and is an isomorphism in π_i for $i \geq 2$. It follows that the composite

$$S^2 \wedge K(\mathbb{C})/n \xrightarrow{\beta\wedge 1} K(\mathbb{C})/n \wedge K(\mathbb{C})/n \xrightarrow{m} K(\mathbb{C})/n$$

is an isomorphism in π_i for $i \geq 2$.

We have proved:

Lemma 26.3. *Suppose that $n = \ell^\nu$, subject to the constraints on the prime ℓ and the power ν listed above. Then all powers of the Bott element $\beta^k \in K_{2k}(\mathbb{C}, \mathbb{Z}/n)$ are non-trivial, and there is an isomorphism*

$$K_*(\mathbb{C}, \mathbb{Z}/n) \cong \mathbb{Z}/n[\beta].$$

Corollary 26.4. *Suppose that* $n = \ell^\nu$*, subject to the constraints on the prime* ℓ *and the power* ν *listed above. Suppose that* k *is an algebraically closed field with* $(n, char(k)) = 1$*. Then there is an isomorphism*

$$K_*(k, \mathbb{Z}/n) \cong \mathbb{Z}/n[\beta].$$

For the record, the following is proved with a transfer argument:

Lemma 26.5. *Suppose that* L *is a separably closed field and that* $(n, char(L)) = 1$*. Then the inclusion* $L \subset \overline{L}$ *induces an isomorphism*

$$K_*(L, \mathbb{Z}/n) \cong K_*(\overline{L}, \mathbb{Z}/n).$$

Now let's talk about finite fields.

Lemma 26.6. $K_i(\mathbb{F}_q)$ *is a finite abelian group of order prime to* p *if* $i \geq 0$*.*

Proof. Homological stability [6] says that there is an isomorphism

$$H_i(BGl(\mathbb{F}_q), \mathbb{Z}) \cong H_i(BGl_n(\mathbb{F}_q), \mathbb{Z})$$

for n sufficiently large. It follows that the reduced homology $\tilde{H}_*(BGl(\mathbb{F}_q), \mathbb{Z})$ consists of finite groups.

It is known [5] that $\tilde{H}_*(BGl(\mathbb{F}_q), \mathbb{Z}/p) = 0$. Thus $\tilde{H}_*(BGl(\mathbb{F}_q), \mathbb{Z})$ consists of uniquely p-divisible finite abelian groups. □

Lemma 26.7. *There are isomorphisms*

$$K_i(\overline{\mathbb{F}}_q) \cong \begin{cases} \mathbb{Q}_{(p)}/\mathbb{Z} & \textit{if } i = 2j-1, j \geq 1 \textit{ and} \\ 0 & \textit{if } i = 2j, j \geq 1. \end{cases}$$

Proof. $K_i(\overline{\mathbb{F}}_q) = \varinjlim K_i(\mathbb{F}_{q'})$ is a torsion abelian group with no p-torsion by the last Lemma.

$K_{2j+1}(\overline{\mathbb{F}}_q, \mathbb{Z}/n) = 0$ for all $j \geq 1$ if $(n,p) = 1$. It follows that $K_{2j}(\overline{\mathbb{F}}_q)$ is a torsion group with $\mathrm{Tor}(\mathbb{Z}/n, K_{2j}(\overline{\mathbb{F}}_q)) = 0$ for $(n,p) = 1$, so that $K_{2j}(\overline{\mathbb{F}}_q) = 0$ for all $j \geq 1$.

It follows that there are isomorphisms

$$\mathbb{Z}/n \cong K_{2j}(\overline{\mathbb{F}}_q, \mathbb{Z}/n) \cong \mathrm{Tor}(\mathbb{Z}/n, K_{2j-1}(\overline{\mathbb{F}}_q))$$

for $j \geq 1$. These isomorphisms are functorial in the poset of numbers n with $(n,p) = 1$ and with $n \leq m$ if $n|m$.

An element of $\mathbb{Q}_{(p)}$ is a fraction $\frac{m}{n}$ such that p does not divide n, and this element is in $\mathbb{Z}$ if $n = 1$. The maps

$$\mathbb{Z}/n \to \mathbb{Q}_{(p)}/\mathbb{Z}$$

defined by $1 \mapsto \frac{1}{n}$ define the isomorphism

$$\varinjlim_n \mathbb{Z}/n \xrightarrow{\cong} \mathbb{Q}_{(p)}/\mathbb{Z}.$$

□

Note that

$$K_1(\overline{\mathbb{F}}_q) \cong \overline{\mathbb{F}}_q^* \cong \mathbb{Q}_{(p)}/\mathbb{Z}.$$

Corollary 26.8. *The group $K_{2j-1}(\overline{\mathbb{F}}_q)$ is n-divisible for all n with $(n,p) = 1$, if $j \geq 1$.*

Now recall that the Frobenius automorphism

$$\phi : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q$$

is defined by $\phi(\alpha) = \alpha^q$. Recall that ϕ is the identity on $\mathbb{F}_q$, since $\mathbb{F}_q$ is the splitting field of the polynomial $x^q - x$.

The Frobenius induces a morphism of spectra $\phi : K(\overline{\mathbb{F}}_q) \to K(\overline{\mathbb{F}}_q)$, and there is a commutative diagram of spectra

$$\begin{array}{ccc} K(\mathbb{F}_q) & \xrightarrow{i} & K(\overline{\mathbb{F}}_q) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \Delta} \\ K(Fq) & \xrightarrow[(\phi,1)]{} & K(\overline{\mathbb{F}}_q) \times K(\overline{\mathbb{F}}_q) \end{array}$$

where i is induced by the inclusion $\mathbb{F}_q \subset \overline{\mathbb{F}}_q$.

Here's the main theorem:

Theorem 26.9. *The square*

$$\begin{array}{ccc} K(\mathbb{F}_q) & \xrightarrow{i} & K(\overline{\mathbb{F}}_q) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \Delta} \\ K(\overline{\mathbb{F}}_q) & \xrightarrow[(\phi,1)]{} & K(\overline{\mathbb{F}}_q) \times K(\overline{\mathbb{F}}_q) \end{array}$$

is homotopy cartesian.

This result is often paraphrased by saying that the spectrum $K(\mathbb{F}_q)$ is the homotopy fixed points of the Frobenius.

Corollary 26.10. *There are isomorphisms*

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/(q^j - 1) & \textit{if } i = 2j-1, j > 0, \textit{ and} \\ 0 & \textit{if } i = 2j, j > 0. \end{cases}$$

Proof. The squares in the diagram

$$\begin{array}{ccccc} K(\mathbb{F}_q) & \xrightarrow{i} & K(\overline{\mathbb{F}}_q) & \longrightarrow & * \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \Delta} & & \downarrow \\ K(\overline{\mathbb{F}}_q) & \xrightarrow[(\phi,1)]{} & K(\overline{\mathbb{F}}_q) \times K(\overline{\mathbb{F}}_q) & \xrightarrow[(1,-1)]{} & K(\overline{\mathbb{F}}_q) \end{array}$$

are homotopy cartesian, so that there is a fibre sequence

$$K(\mathbb{F}_q) \xrightarrow{i} K(\overline{\mathbb{F}}_q) \xrightarrow{\phi - 1} K(\overline{\mathbb{F}}_q). \qquad (1)$$

It follows from the Suslin calculations that the map

$$\phi_* : K_{2j}(\overline{\mathbb{F}}_q) \to K_{2j}(\overline{\mathbb{F}}_q)$$

is multiplication by q^j. In effect,

$$\phi_* : K_2(\overline{\mathbb{F}}_q, \mathbb{Z}/n) \to K_2(\overline{\mathbb{F}}_q, \mathbb{Z}/n)$$

is multiplication by q (ie. $\beta \mapsto q\beta$) since

$$K_2(\overline{\mathbb{F}}_q, \mathbb{Z}/n) \cong \mathrm{Tor}(\mathbb{Z}/n, \overline{\mathbb{F}}_q^*).$$

It follows that the map

$$\phi_* : K_{2j}(\overline{\mathbb{F}}_q, \mathbb{Z}/n) \to K_{2j}(\overline{\mathbb{F}}_q, \mathbb{Z}/n)$$

is multiplication by q^j ($\beta^j \mapsto (q\beta)^j$). Thus, the map

$$\phi_* : \mathrm{Tor}(\mathbb{Z}/n, K_{2j-1}(\overline{\mathbb{F}}_q)) \to \mathrm{Tor}(\mathbb{Z}/n, K_{2j-1}(\overline{\mathbb{F}}_q))$$

is also multiplication by q^j. But $K_{2j-1}(\overline{\mathbb{F}}_q)$ consists of torsion prime to p, so that $\phi_* : K_{2j-1}(\overline{\mathbb{F}}_q) \to K_{2j-1}(\overline{\mathbb{F}}_q)$ is multiplication by q^j.

It follows that the map

$$\phi_* - 1 : K_{2j-1}(\overline{\mathbb{F}}_q) \to K_{2j-1}(\overline{\mathbb{F}}_q)$$

is multiplication by $q^j - 1$. This number is prime to p, so that the map $\phi_* - 1$ is surjective. It follows from the long exact sequence for the fibre sequence (1) that

$$K_{2j-1}(\mathbb{F}_q) = \mathrm{Tor}(\mathbb{Z}/(q^j - 1), K_{2j-1}(\overline{\mathbb{F}}_q)) \cong \mathbb{Z}/(q^j - 1).$$

and $K_{2j}(\mathbb{F}_q) = 0$. $\square$

To prove Theorem 26.9, form the homotopy pullback diagram

$$\begin{array}{ccc} F_\phi(\overline{\mathbb{F}}_q) & \longrightarrow & K(\overline{\mathbb{F}}_q) \\ \downarrow & & \downarrow \Delta \\ K(\overline{\mathbb{F}}_q) & \xrightarrow[(\phi,1)]{} & K(\overline{\mathbb{F}}_q) \times K(\overline{\mathbb{F}}_q) \end{array}$$

The game is to show that the induced map

$$K(\mathbb{F}_q) \to F_\phi(\overline{\mathbb{F}}_q)$$

is a stable equivalence.

It suffices to do this at the level of 1-connected covers. In effect, if $\tilde{K}(R) \to K(R)$ denotes the 1-connected cover (ie. fibre of the map $K(R) \to P_1K(R)$), form the homotopy pullback

$$\begin{array}{ccc} \tilde{F}_\phi(\overline{\mathbb{F}}_q) & \longrightarrow & \tilde{K}(\overline{\mathbb{F}}_q) \\ \downarrow & & \downarrow \Delta \\ \tilde{K}(\overline{\mathbb{F}}_q) & \xrightarrow[(\phi,1)]{} & \tilde{K}(\overline{\mathbb{F}}_q) \times \tilde{K}K(\overline{\mathbb{F}}_q) \end{array} \tag{2}$$

Then $\tilde{F}_\phi(\overline{\mathbb{F}}_q) \to F_\phi(\overline{\mathbb{F}}_q)$ is the 1-connected cover, while we already know that the map $\pi_i K(\mathbb{F}_q) \to \pi_i F_\phi$ is an isomorphism for $i = 0, 1$.

Note that the map $\overline{\mathbb{F}}_q^* \to \overline{\mathbb{F}}_q^*$ defined by $\alpha \mapsto \alpha/\phi(\alpha)$ is an isomorphism (Lang isomorphism) so that $\tilde{F}_\phi$ is simply-connected.

We therefore want to show that the map

$$\tilde{K}(\mathbb{F}_q) \to \tilde{F}_\phi(\overline{\mathbb{F}}_q)$$

is a stable equivalence. Both spectra are 1-connected, so it suffices to show that the map

$$\tilde{K}^0(\mathbb{F}_q) \to \tilde{F}^0_\phi(\overline{\mathbb{F}}_q)$$

of pointed simplicial sets is a weak equivalence. Both spaces are simply-connected and have homotopy groups which are finite and of order prime to p, so it is enough to show that the maps

$$H_*(\tilde{K}^0(\mathbb{F}_q), \mathbb{Z}/\ell) \to H_*(\tilde{F}^0_\phi(\overline{\mathbb{F}}_q), \mathbb{Z}/\ell)$$

are isomorphisms for all primes ℓ with $(\ell, p) = 1$.

Suppose that $E(R) \subset Gl(R)$ is the subgroup of elementary transformations, and recall that $E(R) = [Gl(R), Gl(R)]$, naturally in rings R. There is map

$$BE \to \tilde{K}^0$$

of simplicial presheaves on the big étale site $(Sch|_{\overline{\mathbb{F}}_q})_{et}$ which is an $H_*(\ ,\mathbb{Z})$-isomorphism on affine patches ($\tilde{K}^0(R) = BE(R)^+$). It therefore suffices to show that the composition map

$$BE(\mathbb{F}_q) \to \tilde{K}^0(\mathbb{F}_q) \to \tilde{F}^0_\phi(\overline{\mathbb{F}}_q)$$

is an $H_*(\ ,\mathbb{Z}/\ell)$-isomorphism for $(\ell, p) = 1$.

The Frobenius homomorphism induces a natural map $\phi : E \to E$ for presheaves of spectra on $(Sch|_{\overline{\mathbb{F}}_q})_{et}$, and we can form the sectionwise homotopy cartesian diagram

$$\begin{array}{ccc} \tilde{F}_\phi & \longrightarrow & \tilde{K} \\ \downarrow & & \downarrow \Delta \\ \tilde{K} & \xrightarrow[(\phi,1)]{} & \tilde{K} \times \tilde{K} \end{array}$$

The diagram (2) is global sections of this diagram of presheaves of spectra. There is a corresponding pointwise homotopy cartesian diagram

$$\begin{array}{ccc} \tilde{F}^0_\phi & \longrightarrow & \tilde{K}^0 \\ \downarrow & & \downarrow \Delta \\ \tilde{K}^0 & \xrightarrow[(\phi,1)]{} & \tilde{K}^0 \times \tilde{K}^0 \end{array}$$

of pointed simplicial presheaves.

Lemma 26.11. *The simplicial presheaf $\tilde{F}^0_\phi$ is rigid in the sense that the map*

$$\Gamma^* \tilde{F}^0_\phi(\overline{\mathbb{F}}_q) \to \tilde{F}^0_\phi$$

induces an isomorphism in homology sheaves $\tilde{H}_*(\ ,\mathbb{Z}/\ell)$ for all $(\ell, p) = 1$.

Proof. The K-theory presheaf is rigid in mod ℓ stable homotopy groups (Gabber rigidity), and $P_1 K$

is rigid (calculation), so that $\tilde{K}$ is rigid and then $\tilde{F}_\phi$ is rigid, as presheaves of spectra. Extract the homology statement in the usual way. $\square$

In particular, there is an isomorphism

$$H^*_{et}(\tilde{F}^0_\phi, \mathbb{Z}/\ell) \cong H^*(\tilde{F}^0_\phi(\overline{\mathbb{F}}_q), \mathbb{Z}/\ell),$$

and it suffices to show that the maps

$$H^*_{et}(\tilde{F}^0_\phi, \mathbb{Z}/\ell) \to H^*_{et}(\Gamma^* BE(\overline{\mathbb{F}}_q), \mathbb{Z}/\ell) \cong H^*(BE(\overline{\mathbb{F}}_q), \mathbb{Z}/\ell)$$

are isomorphisms.

The natural inclusion $E_n(R) \subset Sl_n(R)$ induces local weak equivalences

$$BE_n \to BSl_n, \quad BE \to BSl$$

of simplicial presheaves on $(Sch|_{\overline{\mathbb{F}}_q})_{et}$ since the groups in question coincide on local rings.

Lemma 26.12. *There is a homotopy cartesian diagrams of simplicial presheaves*

$$\begin{array}{ccc} \Gamma^* BSl_n(\mathbb{F}_q) & \longrightarrow & BSl_n \\ \downarrow & & \downarrow \Delta \\ BSl_n & \xrightarrow[(\phi,1)]{} & BSl_n \times BSl_n \end{array}$$

Proof. Any inclusion $G \subset H$ of groups determines a homotopy fibre sequence

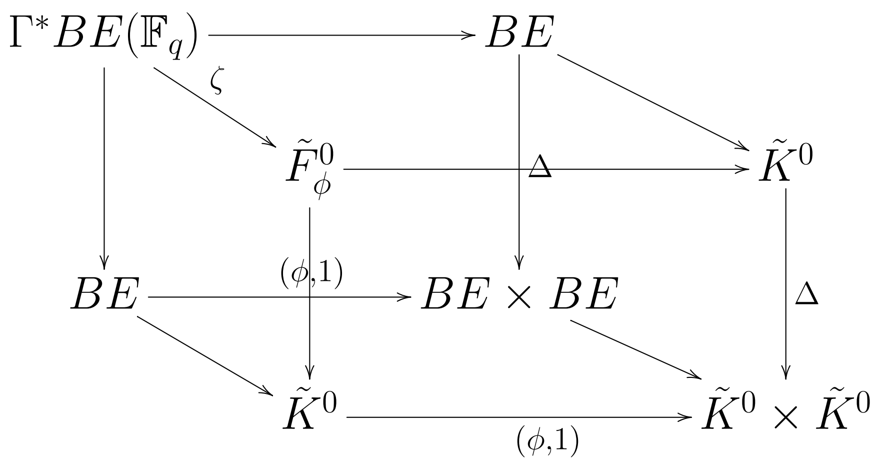
$$EG \times_G H \to BG \to BH$$

and all homotopy groups $\pi_i(EG \times_G H)$ are trivial for $i \geq 1$. Checking that the diagram above is homotopy cartesian amounts to showing that the induced map on homotopy fibres is a local equivalence, but this amounts to showing that the induced map

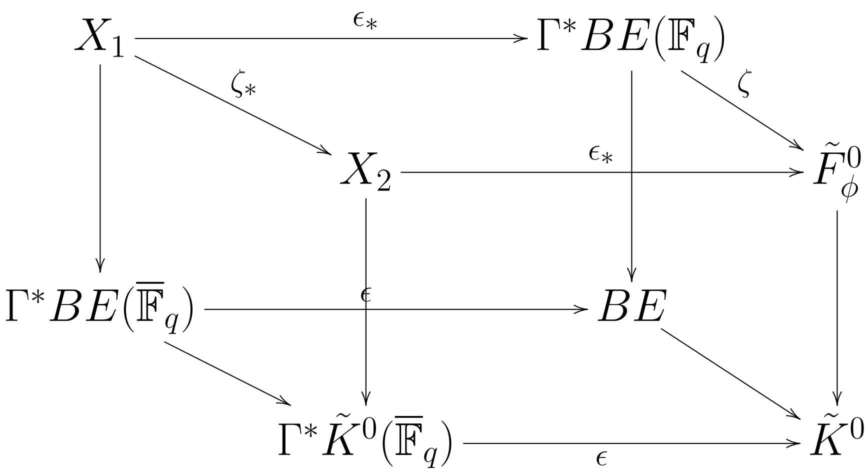
$$Sl_n/\Gamma^* Sl_n(\mathbb{F}_q) \to Sl_n$$

defined by $A \mapsto \phi(A)A^{-1}$ is an isomorphism. The fact that the displayed map is an isomorphism is well known — this map is called the *Lang isomorphism* [1, Prop. 2]. □

Form the comparisons of homotopy cartesian diagrams



and



We want to show that the map ζ induces an isomorphism in étale cohomology, and we do this by showing that the maps ϵ_* and ζ_* are isomorphisms in étale cohomology.

1) The map ϵ_* is a mod ℓ homology sheaf isomorphism, because the object $\tilde{K}^0$ is rigid. Use a comparison of Serre spectral sequences in stalks to see this.

2) There is a weak equivalence

$$Y_1 \simeq ESl(\overline{\mathbb{F}}_q) \times_{Sl(\overline{\mathbb{F}}_q)} \times Sl \times_{(Sl(\mathbb{F}_q)} ESl(\mathbb{F}_q).$$

The idea is to show that

$$\tilde{H}^*_{et}(ESl(\overline{\mathbb{F}}_q) \times_{Sl(\overline{\mathbb{F}}_q)} Sl, \mathbb{Z}/\ell) = 0$$

But there is an isomorphism

$$H^*_{et}(ESl(\overline{\mathbb{F}}_q) \times_{Sl(\overline{\mathbb{F}}_q)} Sl, \mathbb{Z}/\ell) \cong H^*(ESl(\mathbb{C}) \times_{Sl(\mathbb{C})} Sl^{top}, \mathbb{Z}/\ell)$$

by rigidity (and GAGA), and $ESl(\mathbb{C}) \times_{Sl(\mathbb{C})} Sl^{top}$ is the homotopy fibre of $BSl(\mathbb{C}) \to BSl^{top}$, which fibration is a mod ℓ cohomology isomorphism (by rigidity) with simply connected base. The fibre is therefore mod ℓ cohomologically acyclic.

3) To show that γ is an étale cohomology isomorphism, it suffices to show that the map $E \to \Omega\tilde{K}^0$ is a mod ℓ étale cohomology isomorphism. One uses a comparison of Serre-type spectral sequences, eg.

$$H^p(\tilde{K}^0(\overline{\mathbb{F}}_q), H^q_{et}(\Omega\tilde{K}^0, \mathbb{Z}/\ell)) \Rightarrow H^{p+q}_{et}(X_2, \mathbb{Z}/\ell)$$

to see this. But again the question about $E \to \Omega\tilde{K}^0$ base changes to topology, where there is an actual weak equivalence.

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