Local Homotopy Theory

Basic References

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http://math.uwo.ca/ jardine/papers/LocalHom/index.shtml

[2] J.F. Jardine. *Local Homotopy Theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2015

[3] *Lecture Notes on Homotopy Theory*
http://math.uwo.ca/ jardine/papers/HomTh/index.shtml

[4] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition

9 Rigidity

Suppose that k is an algebraically closed field, and let ℓ be a prime such that $\ell \neq char(k)$.

We will be working with the big étale site $(\text{Sch}\,|_k)_{et}$ over the field k throughout this section.

Note the (standard) abuse: I should have written $(\text{Sch}\,|_{\text{Sp}(k)})_{et}$.

Fact: Every k-scheme X represents a sheaf on $(\text{Sch}\,|_k)_{et}$, by the theorem of *faithfully flat descent*. See any étale cohomology textbook, such as [9].

Examples: 1) I use the notation Gl_n to represent either the algebraic group

$$Gl_n = \text{Sp}(k[X_{ij}]_{det})$$

over k, or the sheaf of groups

$$Gl_n = \hom(\ ,Gl_n)$$

that it represents on the site $(\text{Sch}\,|_k)_{et}$.

2) Gl_1 is the multiplicative group $\mathbb{G}_m$. One sees the notation $\mu = \mathbb{G}_m$, and one always sees μ_ℓ for its ℓ-torsion part. μ_ℓ is the sheaf of ℓ^{th} roots of unity.

Since the prime $\ell \neq char(k)$, there is an isomorphism

$$\mu_\ell \cong \Gamma^*\mathbb{Z}/\ell = \mathbb{Z}/\ell.$$

$\Gamma^*\mathbb{Z}/\ell$ is the constant sheaf on the group $\mathbb{Z}/\ell$, and the displayed equality is a standard abuse.

Constant sheaves, global sections

The constant sheaf functor $A \mapsto \Gamma^*(A)$ is left adjoint to the global sections functor $X \mapsto \Gamma_* X$, where

$$\Gamma_* X = X(k),$$

and there's a geometric morphism

$$\Gamma : \mathrm{Shv}((\mathrm{Sch}\,|_k)_{et}) \to \mathbf{Set}.$$

This is a special case of a geometric morphism

$$\Gamma : \mathrm{Shv}(\mathscr{C}) \to \mathbf{Set}$$

defined by

$$\Gamma_*(X) = \varprojlim_{U \in \mathscr{C}} X(U),$$

which is the global sections functor for an arbitrary site $\mathscr{C}$.

The general version of Γ_* specializes to global sections for sheaves on $(\mathrm{Sch}\,|_k)_{et}$, because this site has a terminal object, namely $\mathrm{Sp}(k)$.

Remark 9.1. It's a special feature of the étale topology (and some others) that

$$\Gamma^* A(U) = \hom(\pi_0 U, A)$$

where $\pi_0(U)$ is the set of connected components of the k-scheme U, since $\mathrm{Sp}(k)$ is connected.

In effect, the k-scheme $\bigsqcup_A \mathrm{Sp}(k)$ represents $\Gamma^* A$, and there is an isomorphism

$$\hom_k(U, \bigsqcup_A \mathrm{Sp}(k)) \cong \hom(\pi_0 U, A).$$

Affine schemes and sheaves

The sheaf of groups Gl_n is defined on affine k-schemes $Sp(R)$ (ie. k-algebras R) by

$$Gl_n(\mathrm{Sp}(R)) = Gl_n(R),$$

where $Gl_n(R)$ is the group of invertible $n \times n$ matrices with entries in R.

There is a standard way to recover the sheaf Gl_n on $(\mathrm{Sch}|_k)_{et}$ from the matrix group description for affine schemes, by an equivalence

$$\mathrm{Shv}((\mathrm{Sch}|_k)_{et}) \simeq \mathrm{Shv}((\mathrm{Aff}|_k)_{et})$$

where $(\mathrm{Aff}|_k)_{et}$ is the étale site of affine k-schemes.

The homomorphisms $Gl_n(R) \to Gl_{n+1}(R)$ with

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$$

define a homomorphism $Gl_n \to Gl_{n+1}$ of sheaves of groups. The colimit presheaf

$$Gl = \varinjlim_n Gl_n \tag{9.1}$$

is the traditional infinite general linear group $Gl(R)$ in affine sections.

Warning: One typically also writes Gl for the associated sheaf, so that there is a relation of the form (9.1) in the category of sheaves of groups.

Classifying spaces

A presheaf of groups G has a **classifying simplicial presheaf** BG, with

$$BG(U) = B(G(U)), \ \ U \in \mathrm{Sch}|_k,$$

given by the standard simplicial set construction.

The object BG is a simplicial sheaf if G is a sheaf, because

$$BG_n = G \times \cdots \times G$$

(n factors) as a presheaf.

The classifying space construction commutes with filtered colimits, so we are entitled to a classifying simplicial sheaf (or presheaf) BGl with

$$BGl = \varinjlim_n BGl_n.$$

Homology sheaves, cohomology groups

Simplicial sheaves (or presheaves) X have cohomology groups and homology sheaves.

1) The **homology sheaves** $\tilde{H}_n(X,A)$ are easier to define: form the presheaf of chain complexes

$$\mathbb{Z}(X)\otimes A,$$

with

$$(\mathbb{Z}(X)\otimes A)(U) = \mathbb{Z}(X(U))\otimes A(U),$$

where $\mathbb{Z}(X(U))$ is the standard (functorial) Moore chain complex for the simplicial set $X(U)$. Then the sheaf $\tilde{H}_n(X,A)$ is the sheaf which is associated to the presheaf $H_n(\mathbb{Z}(X)\otimes A)$.

Example: The sheaf $\tilde{H}_n(X,\mathbb{Z}/\ell)$ is the sheaf associated to the presheaf $H_n(\mathbb{Z}/\ell(X))$.

2) Cohomology has a more interesting definition: the n^{th} (étale) **cohomology group** $H^n(X,A)$ of the simplicial presheaf X with coefficients in the abelian presheaf A is defined by

$$H^n(X,A) = [X, K(A,n)],$$

where the thing on the right is morphisms in the local homotopy category of simplicial presheaves on the étale site.

$K(A,n)$ is the presheaf $\Gamma(A[-n])$, where Γ is the Dold-Kan functor from chain complexes to simplicial abelian groups, and $A[-n]$ is the presheaf of chain complexes which consists of a copy of A concentrated in degree n.

Local homotopy theory

There is a model structure on simplicial presheaves (respectively, and Quillen equivalently, simplicial sheaves) on the site $(\mathrm{Sch}|_k)_{et}$, for which the weak equivalences are those maps $X \to Y$ which induce weak equivalences of simplicial sets in all stalks — I call these **local weak equivalences**, and for which the cofibrations are the monomorphisms.

This is a special case of a construction for arbitrary Grothendieck sites.

Example: The canonical map $\eta : X \to \tilde{X}$ from a simplicial presheaf to its associated simplicial sheaf is a local weak equivalence.

Remark 9.2. 1) If X is represented by a (simplicial) scheme having the same name, and A is a sheaf of abelian groups, then $H^n(X,A)$ coincides up to isomorphism with the étale cohomology group $H^n_{et}(X,A)$ of X, as it is normally defined.

In particular, if X is a k-scheme, and $A \to I^*$ is an injective resolution of A in sheaves of abelian groups, then there is an isomorphism

$$H^n(X,A) \cong H^n(I^*(X)) \cong \mathrm{Ext}^n(\tilde{\mathbb{Z}}(X),A).$$

We have, in effect, generalized the standard definition of étale cohomology groups of schemes to cohomology for arbitrary simplicial presheaves.

2) There is a spectral sequence [5] relating homology sheaves and cohomology groups, with

$$E_2^{p,q} = \mathrm{Ext}^p(\tilde{H}_q(X),A) \Rightarrow H^{p+q}(X,A).$$

There is also an ℓ-torsion version, with

$$E_2^{p,q} = \mathrm{Ext}^p(\tilde{H}_q(X,\mathbb{Z}/\ell),A) \Rightarrow H^{p+q}(X,A) \quad (9.2)$$

if A is an ℓ-torsion sheaf.

These spectral sequences both come from bicomplexes of the form

$$\hom(X_p, I^q),$$

where $A \to I^*$ is an injective resolution of A.

Thus, if $f : X \to Y$ is a map of simplicial presheaves which induces homology sheaf isomorphisms

$$f_* : \tilde{H}_n(X, \mathbb{Z}/\ell) \xrightarrow{\cong} \tilde{H}_n(Y, \mathbb{Z}/\ell),\ n \geq 0,$$

then f induces isomorphisms

$$f^* : H^n(Y, \mathbb{Z}/\ell) \xrightarrow{\cong} H^n(X, \mathbb{Z}/\ell)$$

in étale cohomology groups for all $n \geq 0$.

Fact: Local weak equivalences induce isomorphisms of homology sheaves, hence isomorphisms of cohomology groups.

Exercise: Show that if $p : F \to F'$ is a local epimorphism of presheaves on $(\mathrm{Sch}|_k)_{et}$, then the induced map $F(k) \to F'(k)$ in global sections is surjective, since k is an algebraically closed field.

It follows that the associated sheaf map $\eta : F \to \tilde{F}$ induces a bijection $F(k) \xrightarrow{\cong} \tilde{F}(k)$ in global sections.

It also follows that the global sections functor on $\mathrm{Shv}((\mathrm{Sch}|_k)_{et})$ is exact on abelian sheaves.

Warning: Global sections is usually not exact.

There are isomorphisms

$$H^n_{et}(k,A) \cong \begin{cases} A(k) & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

More generally, the map $A \to I^*$ of chain complexes defined by an injective resolution with I^* in negative degrees induces a natural isomorphism

$$H^n(X, A(k)) \cong H^n(\Gamma^* X, A)$$

for any simplicial set X and abelian sheaf A.

Rigidity

The canonical map

$$\varepsilon : \Gamma^* \Gamma_* BGl \to BGl$$

has the form

$$\varepsilon : \Gamma^* BGl(k) \to BGl$$

up to isomorphism, and that the induced map

$$\varepsilon^* : H^n(BGl, \mathbb{Z}/\ell) \to H^n(\Gamma^* BGl(k), \mathbb{Z}/\ell)$$

can be written as

$$\varepsilon^* : H^n_{et}(BGl, \mathbb{Z}/\ell) \to H^n(BGl(k), \mathbb{Z}/\ell), \quad (9.3)$$

where the object on the right is a standard cohomology group of the simplicial set $BGl(k)$ with coefficients in the abelian group $\mathbb{Z}/\ell$.

The map (9.3) is a **comparison map** of étale with discrete cohomology for the group Gl.

Theorem 9.3. *Suppose that k is an algebraically closed field, and that ℓ is prime such that $\ell \neq char(k)$. Then the comparison map*

$$\varepsilon^* : H^n_{et}(BGl, \mathbb{Z}/\ell) \to H^n(BGl(k), \mathbb{Z}/\ell)$$

is an isomorphism.

Remark 9.4. This theorem gives a calculation

$$H^*(BGl(k), \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_1, c_2, \dots],$$

since standard results in étale cohomology theory imply that $H^*_{et}(BGl, \mathbb{Z}/\ell)$ is a polynomial ring in Chern classes c_i, with $deg(c_i) = 2i$.

Proof of Theorem 9.3. The idea is to show that the map ε induces isomorphisms

$$\tilde{H}_n(\Gamma^* BGl(k), \mathbb{Z}/\ell) \xrightarrow{\cong} \tilde{H}_n(BGl, \mathbb{Z}/\ell)$$

in all homology sheaves, and then invoke a comparison of spectral sequences (9.2).

The category $\mathrm{Shv}((\mathrm{Sch}|_k)_{et})$ has a good theory of stalks, and it's enough to compare stalks at all closed points $x \in U$ of all k-schemes U (which are locally of finite type over k).

The map ε_* at the stalk for such a point x is the map

$$H_n(BGl(k),\mathbb{Z}/\ell) \to H_n(BGl(\mathcal{O}_x^{sh}),\mathbb{Z}/\ell),$$

where $\mathcal{O}_x^{sh}$ is the strict Henselization of the local ring $\mathcal{O}_x$ of U at x, and the indicated map is induced by the k-algebra structure map $k \to \mathcal{O}_x^{sh}$.

The Gabber Rigidity Theorem [2], [3] asserts that the residue field homomorphism $\pi : \mathcal{O}_x^{sh} \to k$ induces an isomorphism

$$\pi_* : H_n(BGl(\mathcal{O}_x^{sh}),\mathbb{Z}/\ell) \xrightarrow{\cong} H_n(BGl(k),\mathbb{Z}/\ell).$$

The desired result follows. □

Remarks

1) The Gabber Rigidity Theorem is a consequence of a mod ℓ K-theory rigidity statement, namely that the residue map induces isomorphisms

$$\pi_* : K_*(\mathcal{O}_x^{sh},\mathbb{Z}/\ell) \xrightarrow{\cong} K_*(k,\mathbb{Z}/\ell)$$

As such, it is a stable statement that depends on the existence of the K-theory transfer, as well as the homotopy property ($K_*(A) \cong K_*(A[t])$ for regular rings A).

2) An axiomatic approach to rigidity has evolved in the intervening years, which first appeared in [11], and achieved its modern form for torsion presheaves with transfers satisfying the homotopy property in [12].

3) Theorem 9.3 implies that an inclusion of algebraically closed fields $k \to L$ of characteristic $\neq \ell$ induces an isomorphism

$$i^* : H^*(BGl(L), \mathbb{Z}/\ell) \cong H^*(BGl(k), \mathbb{Z}/\ell), \quad (9.4)$$

since there is an isomorphism of the corresponding étale cohomology rings by a smooth base change argument.

The map i^* is an isomorphism if and only if the map

$$i_* : K_*(k, \mathbb{Z}/\ell) \to K_*(L, \mathbb{Z}/\ell)$$

is an isomorphism, by H-space tricks, so that Theorem 9.3 implies Suslin's first rigidity theorem [10].

The proof of Suslin's second rigidity theorem, for local fields [13], uses Gabber rigidity explicitly.

3) The outcome of that result, that there are isomorphisms

$$K_n(\mathbb{C},\mathbb{Z}/\ell) \cong \pi_n KU/\ell$$

for $n \geq 0$, is also a consequence of Theorem 9.3.

4) The comparison map

$$\varepsilon^* : H^n_{et}(BGl,\mathbb{Z}/\ell) \to H^n(BGl(k),\mathbb{Z}/\ell)$$

is a special case of a natural comparison map

$$\varepsilon^* : H^n(X,\mathbb{Z}/\ell) \to H^n(X(k),\mathbb{Z}/\ell)$$

which one can can construct for an arbitrary simplicial presheaf X on the big site $(\mathrm{Sch}|_k)_{et}$.

There are versions of Theorem 9.3 for all of the classical infinite families of algebraic groups. In particular, there are comparison isomorphisms

$$\varepsilon^* : H^*_{et}(BSl,\mathbb{Z}/\ell) \xrightarrow{\cong} H^*(BSl(k),\mathbb{Z}/\ell),$$
$$\varepsilon^* : H^*_{et}(BSp,\mathbb{Z}/\ell) \xrightarrow{\cong} H^*(BSp(k),\mathbb{Z}/\ell),$$
$$\varepsilon^* : H^*_{et}(BO,\mathbb{Z}/\ell) \xrightarrow{\cong} H^*(BO(k),\mathbb{Z}/\ell),$$

for the infinite special linear, symplectic and orthogonal groups, respectively.

The special linear case follows from Theorem 9.3, by a fibre sequence argument.

The symplectic and orthogonal group statements follow from a rigidity statement for Karoubi L-theory which is deduced from Gabber rigidity with a Karoubi peridicity argument [7].

4) There is also a comparison map

$$\varepsilon^* : H^n_{et}(BG, \mathbb{Z}/\ell) \to H^n(BG(k), \mathbb{Z}/\ell) \qquad (9.5)$$

for an arbitrary algebraic group G over k.

The **Friedlander-Milnor conjecture** (aka. the **isomorphism conjecture**) asserts that this comparison map is an isomorphism if G is reductive.

This conjecture specializes to a conjecture of Milnor when the underlying field is the complex numbers, in which case the étale cohomology groups $H^n(BG, \mathbb{Z}/\ell)$ correspond with the ordinary singular cohomology groups of the (simplicial analytic) classifying space $BG(\mathbb{C})$.

Remarks:

a) The isomorphism conjecture holds when $k = \overline{\mathbb{F}}_p$ is the algebraic closure of the finite field $\mathbb{F}_p$ with $p \neq \ell$. This is a result of Friedlander and Mislin [1] which depends strongly on the Lang isomorphism for algebraic groups defined over $\mathbb{F}_p$.

b) The isomorphism conjecture is not known to hold, in general, for any other algebraically closed field. It is not even known to hold for any of the general linear groups Gl_n outside of a stable range in homology. See Kevin Knudson's book [8].

c) This conjecture is perhaps the most important unsolved classical problem of algebraic K-theory.

It was known since the 1970s that a calculation of the form

$$H^*(BGl_n(k), \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_1, \ldots, c_n]$$

would imply the Lichtenbaum conjecture that

$$K_*(k, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\beta]$$

where $\beta \in K_2(k, \mathbb{Z}/\ell)$ is the Bott element.

The Lichtenbaum conjecture was proved by Suslin (the rigidity theorems).

The Lichtenbaum conjecture is part of the Lichtenbaum-Quillen complex of conjectures that relate the torsion part of algebraic K-theory to étale cohomology.

The Lichtenbaum-Quillen conjectures are consequences of the Bloch-Kato conjecture, which has been proved by Rost and Voevodsky [14].

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