Contents

13 Chain complexes

Suppose that $\mathscr{C}$ is a fixed small site.

We suppress $\mathscr{C}$ in notation: write $s\mathrm{Pre}$ for the category $s\mathrm{Pre}(\mathscr{C})$ of simplicial presheaves on $\mathscr{C}$, etc.

Suppose that R is a presheaf of commutative rings with unit on $\mathscr{C}$.

Write Pre_R for the category of R-modules, or abelian presheaves which have an R-module structure.

Then $s\mathrm{Pre}_R$ is the category of simplicial R-modules, $\mathrm{Ch}_+(\mathrm{Pre}_R)$ is the category of positively graded (ordinary) chain complexes in Pre_R, and $\mathrm{Ch}(\mathrm{Pre}_R)$ is the category of unbounded complexes in Pre_R.

Most of the time in applications, R is a constant presheaf of rings, like $\mathbb{Z}$ or $\mathbb{Z}/n$.

$\mathrm{Pre}_{\mathbb{Z}}$ is the category of presheaves of abelian groups, $s\mathrm{Pre}_{\mathbb{Z}}$ is presheaves of simplicial abelian groups, and $\mathrm{Ch}(\mathbb{Z})$ and $\mathrm{Ch}_+(\mathbb{Z})$ are categories of presheaves of chain complexes.

The category $\mathrm{Pre}_{\mathbb{Z}/n}$ is the category of n-torsion abelian presheaves, and so on.

All of these categories have corresponding sheaf categories, based on the category Shv_R of sheaves of R-modules.

$s\,\mathrm{Shv}_R$ is the category of simplicial sheaves in R-modules, $\mathrm{Ch}_+(\mathrm{Shv}_R)$ is the category of positively graded chain complexes in Shv_R, and $\mathrm{Ch}(\mathrm{Shv}_R)$ is the category of unbounded complexes.

There is a **free R-module functor**

$$R : s\,\mathrm{Pre} \to s\,\mathrm{Pre}_R,$$

written $X \mapsto R(X)$ for simplicial presheaves X, where $R(X)_n$ is the free R-module on the presheaf X_n. This functor is left adjoint to the forgetful functor

$$u : s\,\mathrm{Pre}_R \to s\,\mathrm{Pre}\,.$$

The sheaf associated to $R(X)$ is denoted by $\tilde{R}(X)$.

I also write $R(X)$ for the associated (presheaf of) Moore chains on X.

In general, each simplicial R-module A has an associated **Moore complex** in $\mathrm{Ch}_+(\mathrm{Pre}_R)$, also denoted by A, with n-chains A_n and boundary maps

$$\partial = \sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1}.$$

The **homology sheaf** $\tilde{H}_n(X,R)$ is the sheaf associated to the presheaf $H_n(R(X))$.

If A is an R-module, then $\tilde{H}_n(X,A)$ is the sheaf associated to the presheaf $H_n(R(X) \otimes A)$.

The normalized chains functor induces a functor

$$N : s\mathrm{Pre}_R \to \mathrm{Ch}_+(\mathrm{Pre}_R),$$

and there is an equivalence of categories (the Dold-Kan correspondence)

$$N : s\mathrm{Pre}_R \simeq \mathrm{Ch}_+(\mathrm{Pre}_R) : \Gamma.$$

Recall [2] that NA is the chain complex with

$$NA_n = \cap_{i=0}^{n-1} \ker(d_i)$$

and boundary

$$\partial = (-1)^n d_n : NA_n \to NA_{n-1}.$$

The natural inclusion $NA \subset A$ of NA in the Moore chains is split by collapsing by degeneracies, and this map induces an isomorphism

$$H_*(NA) \cong H_*(A),$$

and hence an isomorphism

$$\tilde{H}_*(NA) \cong \tilde{H}_*(A)$$

of homology sheaves.

A map $f : A \to B$ of $s(\mathrm{Pre}_R)$ is said to be a **local weak equivalence** if the underlying map of simplicial presheaves is a local weak equivalence.

Lemma 13.1. *Suppose that $f : X \to Y$ is a local weak equivalence of simplicial presheaves.*

Then the induced map $f_* : R(X) \to R(Y)$ of simplicial abelian presheaves is a local weak equivalence.

Proof. It's enough to show that if $f : X \to Y$ is a local equivalence of locally fibrant simplicial sheaves, then $f_* : \tilde{R}(X) \to \tilde{R}(Y)$ is a local equivalence of simplicial abelian sheaves, where R is a sheaf of rings (exercise).

We can assume that the map $f : X \to Y$ is a morphism of locally fibrant simplicial sheaves on a complete Boolean algebra $\mathscr{B}$, since the inverse image functor p^* for a Boolean localization $p : \mathrm{Shv}(\mathscr{B}) \to \mathrm{Shv}(\mathscr{C})$ commutes with the free R-module functor (p_* preserves module structures).

In this case, $f : X \to Y$ is a sectionwise weak equivalence, so $f_* : R(X) \to R(Y)$ is a sectionwise weak equivalence, and so $f_* : \tilde{R}(X) \to \tilde{R}(Y)$ is a local weak equivalence. □

[In the above, $f_* : \tilde{R}(X) \to \tilde{R}(Y)$ is a sectionwise equivalence of simplicial sheaves on $\mathscr{B}$, since the simplicial sheaves $\tilde{R}(X)$ and $\tilde{R}(Y)$ are locally fibrant.]

Remark 13.2. At one time, Lemma 13.1 was called the **Illusie conjecture**. There are various proofs of this result in the literature: the earliest, by van Osdol [5] in 1977, is one of the first applications of Boolean localization.

Suppose that A is a simplicial abelian group. Then A is a Kan complex, and there is a natural isomorphism

$$\pi_n(A,0) \cong H_n(NA)$$

for $n \geq 0$. There is a canonical isomorphism

$$\pi_n(A,0) \xrightarrow{\cong} \pi_n(A,a)$$

which is defined for any $a \in A_0$ by $[\alpha] \mapsto [\alpha + a]$ where we have written a for the composite

$$\Delta^n \to \Delta^0 \xrightarrow{a} A$$

The collection of these isomorphisms, taken together, define a natural isomorphism

$$\begin{tikzcd} \pi_n(A,0) \times A_0 \arrow[rr, "\cong"] \arrow[dr, "pr"'] & & \pi_n A \arrow[dl] \\ & A_0 & \end{tikzcd}$$

Lemma 13.3. *A map $A \to B$ of simplicial R-modules is a local weak equivalence if and only if the map $NA \to NB$ induces an isomorphism in all homology sheaves.*

Proof. If $NA \to NB$ induces an isomorphism in all homology sheaves, then the map $\tilde{\pi}_0(A) \to \tilde{\pi}_0(B)$ and all maps $\tilde{\pi}_n(A,0) \to \tilde{\pi}_n(B,0)$ are isomorphisms of sheaves.

The diagram of sheaves

$$\begin{tikzcd}
\tilde{\pi}_n(A,0) \times A_0 \arrow[r] \arrow[d] & \tilde{\pi}_n(B,0) \times B_0 \arrow[d] \\
A_0 \arrow[r] & B_0
\end{tikzcd}$$

is a pullback. □

Lemma 13.4. *Suppose given a pushout diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "i_*"] \\
B \arrow[r] & D
\end{tikzcd}$$

where the map i is a monomorphism and a homology sheaf isomorphism. Then the induced map i_* is a homology sheaf isomorphism.

Proof. The cokernel of the monomorphism i_* is B/A, which is **acyclic**: $\tilde{H}_*(B/A) = 0$.

The Moore chains functor is exact, and the short exact sequence

$$0 \to C \xrightarrow{i_*} D \to B/A \to 0$$

of presheaves induces a long exact sequence

$$\begin{aligned} \ldots \to \tilde{H}_n(C) \xrightarrow{i_*} \tilde{H}_n(D) \to \tilde{H}_n(B/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \ldots \\ \xrightarrow{\partial} \tilde{H}_0(C) \xrightarrow{i_*} \tilde{H}_0(D) \to \tilde{H}_0(B/A) \to 0 \end{aligned}$$

It follows that all maps

$$\tilde{H}_n(C) \xrightarrow{i_*} \tilde{H}_n(D)$$

are isomorphisms. □

Say that a map $f : A \to B$ of simplicial R-modules is an **injective fibration** if the simplicial presheaf map $u(A) \to u(B)$ is an injective fibration.

A **cofibration** of simplicial R-modules is a map which has the left lifting property with respect to all trivial injective fibrations.

In view of Lemma 13.3, $f : A \to B$ is a local weak equivalence if and only if the induced maps $NA \to NB$ and $A \to B$ of normalized and Moore chains, respectively, are homology sheaf isomorphisms.

Following Grothendieck and Illusie, homology sheaf isomorphisms are often called **quasi-isomorphisms**.

If $i : A \to B$ is a cofibration of simplicial presheaves, then the induced map $i_* : R(A) \to R(B)$ is a cofibration of simplicial R-modules.

The map i_* is a monomorphism, because the free R-module functor preserves monomorphisms.

Analogous definitions are available for maps of simplicial sheaves of R-modules:

Say that a map $f : A \to B$ in $s\,\mathrm{Shv}_R$ is a **local weak equivalence** (respectively **injective fibration**) if the underlying simplicial sheaf map $u(A) \to u(B)$ is a local weak equivalence (respectively injective fibration). **Cofibrations** are defined by a left lifting property with respect to trivial fibrations.

If $i : A \to B$ is a cofibration of simplicial presheaves, then the induced map $i_* : \tilde{R}(A) \to \tilde{R}(B)$ is a cofibration and a monomorphism of $s\,\mathrm{Shv}_R$.

Proposition 13.5. *1) With these definitions, the category $s\,\mathrm{Pre}_R$ of simplicial R-modules satisfies the axioms for a proper closed simplicial model category.*

2) With these definitions, the category $s\,\mathrm{Shv}_R$ of simplicial sheaves of R-modules satisfies the axioms for a proper closed simplicial model category.

3) The inclusion and associated sheaf functors define a Quillen equivalence

$$L^2 : s\,\mathrm{Pre}_R \leftrightarrows s\,\mathrm{Shv}_R : i$$

between the (injective) model structures of parts 1) and 2).

Proof. The injective model structure on $s\,\mathrm{Pre}$ is cofibrantly generated.

It follows from Corollary 13.4 that every map $f : A \to B$ of $s\,\mathrm{Pre}_R$ has factorizations

$$\begin{tikzcd} & C \arrow[dr, "p"] & \\ A \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "j"'] & & B \\ & D \arrow[ur, "q"'] & \end{tikzcd}$$

such that p is an injective fibration, i is a trivial cofibration which has the left lifting property with respect to all fibrations, q is a trivial injective fibration, j is a cofibration, and both i and j are monomorphisms. This proves the factorization axiom **CM5**.

It follows that every trivial cofibration is a retract of a map of the form i and therefore has the left lifting property with respect to all fibrations, giving **CM4**.

The remaining closed model axioms for the category $s\mathrm{Pre}_R$ of simplicial R-modules are easy to verify.

The simplicial structure is given by the function complexes $\mathbf{hom}(A,B)$, where $\mathbf{hom}(A,B)_n$ is the abelian group of homomorphisms

$$A \otimes R(\Delta^n) \to B.$$

Left properness is proved with a comparison of long exact sequences in homology sheaves. Right properness follows from the corresponding property for simplicial presheaves.

The proof of statement 2), for simplicial sheaves of R-modules is completely analogous, and the verification of 3) follows the usual pattern. $\square$

I often write

$$A \otimes K = A \otimes R(K)$$

(degreewise tensor product) for a simplicial R-module A and a simplicial presheaf K.

The Dold-Kan correspondence

$$N : s\mathrm{Pre}_R \simeq \mathrm{Ch}_+(\mathrm{Pre}_R) : \Gamma.$$

induces an injective model structure on the category $\mathrm{Ch}_+(\mathrm{Pre}_R)$ of presheaves of chain complexes from the model structure on the category $s\mathrm{Pre}_R$ of simplicial modules given by Proposition 13.5.

In particular, a morphism $f : C \to D$ of $\mathrm{Ch}_+(\mathrm{Pre}_R)$ is a **local weak equivalence** if it is a homology sheaf isomorphism.

The map f is an **injective fibration** (resp. **cofibration**) if the induced map $f_* : \Gamma C \to \Gamma D$ is an injective fibration (respectively cofibration) of $s\mathrm{Pre}_R$.

Similar definitions can be made for chain complexes in sheaves of R-modules.

Corollary 13.6. *1) With these definitions, the category* $\mathrm{Ch}_+(\mathrm{Pre}_R)$ *of chain complexes in R-modules satisfies the axioms for a proper closed simplicial model category.*

2) The category $\mathrm{Ch}_+(\mathrm{Shv}_R)$ *of chain complexes in sheaves of R-modules satisfies the axioms for a proper closed simplicial model category.*

3) The inclusion and associated sheaf functors define a Quillen equivalence

$$L^2 : \mathrm{Ch}_+(\mathrm{Pre}_R) \leftrightarrows \mathrm{Ch}_+(\mathrm{Shv}_R) : i$$

between the model structures of 1) and 2).

Remark: An injective fibration $p : C \to D$ corresponds to an injective fibration $p_* : \Gamma C \to \Gamma D$ of simplicial modules.

The map p_* is a Kan fibration in each section, so that the maps $p : C_n \to D_n$ are surjective in all sections for $n \geq 1$.

14 The derived category

An ordinary chain complex C can be identified with an unbounded complex $C(0)$ by putting 0 in negative degrees.

The right adjoint of the functor $C \mapsto C(0)$ is the **good truncation** $D \mapsto \mathrm{Tr}_0 D$ at level 0, where

$$\mathrm{Tr}_0 D_n = \begin{cases} \ker(\partial : D_0 \to D_{-1}) & \text{if } n = 0, \text{ and} \\ D_n & \text{if } n > 0. \end{cases}$$

If D is an unbounded complex and $n \in \mathbb{Z}$, then the **shifted complex** $D[n]$ is defined by

$$D[n]_p = D_{p+n}.$$

If C is an ordinary chain complex and $n \in \mathbb{Z}$, define the shifted complex $C[n]$ by

$$C[n] = \mathrm{Tr}_0(C(0)[n]).$$

Example: Suppose that $n > 0$.

$C[-n]$ is the complex with $C[-n]_p = C_{p-n}$ for $p \geq n$ and $C[-n]_p = 0$ for $p < n$.

$C[n]$ is the complex with $C[n]_p = C_{p+n}$ for $p > 0$ and

$$C[n]_0 = \ker(\partial : C_n \to C_{n-1}).$$

There is an adjunction isomorphism

$$\hom(C[-n], D) \cong \hom(C, D[n])$$

for all $n \geq 0$.

The functor $C \mapsto C[-1]$ is a **suspension** functor for ordinary chain complexes, while $C \mapsto C[1]$ is a **loop** functor. The suspension functor is left adjoint to the loop functor.

A **spectrum** D **in chain complexes** consists of chain complexes D^n, $n \geq 0$, together with chain complex maps

$$\sigma : D^n[-1] \to D^{n+1}$$

called **bonding homomorphisms**.

A **morphism** $f : D \to E$ of spectra in chain complexes consists of chain complex maps $f : D^n \to E^n$ which respect structure in the sense that the diagrams

$$\begin{tikzcd} D^n[-1] \arrow[r, "\sigma"] \arrow[d, "{f[-1]}"'] & D^{n+1} \arrow[d, "f"] \\ E^n[-1] \arrow[r, "\sigma"'] & E^{n+1} \end{tikzcd}$$

commute.

I write $\mathbf{Spt}(\mathrm{Ch}_+(\ ))$ to denote the corresponding category of spectra, wherever it occurs.

For example, $\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Pre}_R))$ is the category of spectra in chain complexes of R-modules.

Example: Suppose E is an unbounded complex.

There is a canonical map

$$\sigma : (\mathrm{Tr}_0 E)[-1] \to \mathrm{Tr}_0(E[-1])$$

which is defined by the diagram

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ E_1 \arrow[r] \arrow[d] & E_1 \arrow[d] \\ \ker(\partial) \arrow[r] \arrow[d] & E_0 \arrow[d] \\ 0 \arrow[r] & \ker(\partial) \end{tikzcd}$$

Replacing E by $E[-n]$ gives maps

$$\sigma : (\mathrm{Tr}_0(E[-n]))[-1] \to \mathrm{Tr}_0(E[-n-1]).$$

These are the bonding maps for a spectrum $\mathrm{Tr}(E)$ with

$$\mathrm{Tr}(E)^n = \mathrm{Tr}_0(E[-n]).$$

Thus, every unbounded chain complex E defines a spectrum $\mathrm{Tr}(E)$ in chain complexes.

Example: If F is a spectrum in chain complexes, the maps

$$F^n(0)[-1] = F^n[-1](0) \to F^{n+1}(0)$$

have adjoints $F^n(0) \to F^{n+1}(0)[1]$ in the category of unbounded chain complexes.

Write $F(0)$ for the colimit of the maps

$$F^0(0) \to F^1(0)[1] \to F^2(0)[2] \to \dots$$

in the unbounded chain complex category.

Then $\mathrm{Tr}(F(0))^n$ is naturally isomorphic to the colimit of the diagram

$$F^n \to F^{n+1}[1] \to F^{n+2}[2] \to \dots$$

and the adjoint bonding maps

$$\mathrm{Tr}(F(0))^n \to \mathrm{Tr}(F(0))^{n+1}[1]$$

are the isomorphisms determined by the diagrams

$$\begin{tikzcd}
F^n \arrow[r] \arrow[d] & F^{n+1}[1] \arrow[r] \arrow[d] & F^{n+2}[2] \arrow[r] \arrow[d] & \dots \\
F^{n+1}[1] \arrow[r] & F^{n+2}[2] \arrow[r] & F^{n+3}[3] \arrow[r] & \dots
\end{tikzcd}$$

There is a canonical map

$$\eta : F \to \mathrm{Tr}(F(0)),$$

defined by maps to colimits. Set

$$QF = \mathrm{Tr}(F(0)).$$

Lemma 14.1. *The suspension functor $C \mapsto C[-1]$ preserves cofibrations of ordinary chain complexes.*

Proof. It's enough to show that the functor $X \mapsto NR(X)[-1]$ takes cofibrations of simplicial presheaves X to cofibrations of $\mathrm{Ch}_+(\mathrm{Pre}_R)$.

$$R(X) = R_*(X_+),$$

where $R_*(X_+)$ is the reduced part of the complex $R(X_+)$ associated to $X_+ = X \sqcup \{*\}$, pointed by $*$.

The functor $Y \mapsto R_*Y$ is left adjoint to the forgetful functor from $s\,\mathrm{Pre}_R$ to pointed simplicial presheaves, and therefore preserves cofibrations.

Also,

$$\overline{W}(R_*Y) \cong R_*(\Sigma Y),$$

where ΣY is the Kan suspension of the pointed simplicial presheaf Y, and the Kan suspension preserves cofibrations of pointed simplicial sets (or presheaves).

Finally

$$N(\overline{W}(R_*Y)) \cong NR_*Y[-1].$$

$\square$

Say that a map $f: E \to F$ of spectra in chain complexes is a **strict weak equivalence** (respectively **strict fibration**) if all maps $f: E^n \to F^n$ are weak equivalences (respectively fibrations).

A **cofibration** is a map $i: A \to B$ of spectrum objects such that

1) the map $A^0 \to B^0$ is a cofibration of chain complexes, and
2) all induced maps

$$B^n[-1] \cup_{A^n[-1]} A^{n+1} \to B^{n+1}$$

are cofibrations.

It follows from Lemma 14.1 that if $i: A \to B$ is a cofibration of spectrum objects then all component maps $i: A^n \to B^n$ are cofibrations of chain complexes.

Lemma 14.2. *With the definitions of strict equivalence, strict fibration and cofibration given above, the category* $\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Pre}_R))$ *satisfies the axioms for a proper closed simplicial model category.*

The proof of Lemma 14.2 is a formality — it's a standard exercise from stable homotopy theory.

Say that a map $f : A \to B$ of spectrum objects in chain complexes is a **stable equivalence** if the induced map $f_* : QA \to QB$ is a strict equivalence.

In view of the examples above, this means precisely that the induced map $f_* : A(0) \to B(0)$ of unbounded complexes is a homology isomorphism.

A map $g : E \to F$ of unbounded complexes induces a stable equivalence $g_* : \mathrm{Tr}(E) \to \mathrm{Tr}(F)$ if and only if g is a homology sheaf isomorphism.

A map $p : C \to D$ of spectrum objects is a **stable fibration** if and only if it has the right lifting property with respect to all maps which are cofibrations and stable equivalences.

Proposition 14.3. *The classes of cofibrations, stable equivalences and stable fibrations give the category* $\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Pre}_R))$ *the structure of a proper closed simplicial model category.*

Proof. The proof follows the "Bousfield-Friedlander script" [1] — see also [2, X.4]. It is a formal consequence of the following assertions:

A1 The functor Q preserves strict weak equivalences.

A2 The maps η_{QC} and $Q(\eta_C)$ are strict equivalences for all spectrum objects C.

A3 The class of stable equivalences is closed under pullback along all stable fibrations, and is closed under pushout along all cofibrations.

Only the last of these statements is potentially interesting, but it is a consequence of long exact sequence arguments in homology in the unbounded chain complex category.

One uses Lemma 14.1 to show the cofibration statement.

The fibration statement is proved by showing that every stable fibration $p : C \to D$ is a strict fibration, and so the induced map $C(0) \to D(0)$ of unbounded complexes is a local epimorphism in all degrees. $\square$

The model structure of Proposition 14.3 is the **stable model structure** for spectrum objects in chain complexes of R-modules.

The associated homotopy category

$$\mathrm{Ho}(\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Pre}_R)))$$

is the derived category for the category of R-modules (presheaves and/or sheaves).

References

[1] A. K. Bousfield and E. M. Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II*, volume 658 of *Lecture Notes in Math.*, pages 80–130. Springer, Berlin, 1978.

[2] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[3] J.F. Jardine. *Local Homotopy Theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2015.

[4] D. H. Van Osdol. Simplicial homotopy in an exact category. *Amer. J. Math.*, 99(6):1193–1204, 1977.