Contents

15 Cocycles

Let $\mathbf{M}$ be a closed model category such that

1) $\mathbf{M}$ is right proper
2) the class of weak equivalences is closed under products: if $f : X \to Y$ is a weak equivalence, so is any map $f \times 1 : X \times Z \to Y \times Z$

Examples include any of the model structures on $s\mathrm{Pre}$, $s\mathrm{Shv}$, $s\mathrm{Pre}_R$ or $s\mathrm{Shv}_R$ that we've seen, where the weak equivalences are local weak equivalences. Use Boolean localization arguments.

Suppose that X,Y are objects of $\mathbf{M}$, and write $H(X,Y)$ for the category whose objects are all pairs of maps (f,g)

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

where f is a weak equivalence.

A morphism

$$\alpha : (f,g) \to (f',g')$$

of $H(X,Y)$ is a commutative diagram

$$
\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, "\alpha"] & \\
X & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y
\end{tikzcd}
$$

$H(X,Y)$ is the **category of cocycles**, or **cocycle category** from X to Y.

Example: Every set X has an associated (homotopically) trivial groupoid $C(X)$ whose objects are the elements of X and whose morphisms are pairs of elements of X.

Suppose that a presheaf map $U \to *$ is a local epimorphism. Then the canonical simplicial presheaf map $BC(U) \to *$ is a local weak equivalence (it's a local trivial fibration).

$BC(U)$ is called the **Čech resolution** associated to the covering $U \to *$.

Given a covering $U \to *$ and a sheaf of groups G, a **normalized cocycle** on U with values in G is a groupoid morphism $C(U) \to G$, or (equivalently) a simplicial presheaf map $BC(U) \to BG$.

Such a map defines a cocycle

$$* \xleftarrow{\simeq} BC(U) \to BG$$

as defined above.

Normalized cocycles were the original examples.

Write $\pi_0 H(X,Y)$ for the class of path components of $H(X,Y)$.

There is a function

$$\phi : \pi_0 H(X,Y) \to [X,Y]$$

defined by $(f,g) \mapsto g \cdot f^{-1}$.

Lemma 15.1. *Suppose that $\alpha : X \to X'$ and $\beta : Y \to Y'$ are weak equivalences. Then the function*

$$(\alpha,\beta)_* : \pi_0 H(X,Y) \to \pi_0 H(X',Y')$$

is a bijection.

Proof. An object (f,g) of $H(X',Y')$ is a map $(f,g) : Z \to X' \times Y'$ such that f is a weak equivalence.

There is a factorization (map of cocycles)

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "(f,g)"'] & W \arrow[d, "(p_{X'},p_{Y'})"] \\ & X' \times Y' \end{tikzcd}$$

such that j is a trivial cofibration and $(p_{X'}, p_{Y'})$ is a fibration. The map $p_{X'}$ is a weak equivalence. Form the pullback

$$\begin{tikzcd} W_* \arrow[r, "(\alpha\times\beta)_*"] \arrow[d, "(p_X^*,p_Y^*)"'] & W \arrow[d, "(p_{X'},p_{Y'})"] \\ X \times Y \arrow[r, "\alpha\times\beta"'] & X' \times Y' \end{tikzcd}$$

Then the map (p_X^*, p_Y^*) is a fibration and $(\alpha \times \beta)_*$ is a local weak equivalence (since $\alpha \times \beta$ is a weak equivalence, with right properness). The map p_X^* is also a weak equivalence.

The assignment $(f,g) \mapsto (p_X^*, p_Y^*)$ defines a function

$$\pi_0 H(X', Y') \to \pi_0 H(X, Y)$$

which is inverse to $(\alpha, \beta)_*$. $\square$

Lemma 15.2. *Suppose that Y is fibrant and X is cofibrant.*

Then the canonical map

$$\phi : \pi_0 H(X, Y) \to [X, Y]$$

is a bijection.

Proof. The function $\pi(X,Y) \to [X,Y]$ relating naive homotopy classes to morphisms in the homotopy category is a bijection since X is cofibrant and Y is fibrant.

If $f, g : X \to Y$ are homotopic, there is a diagram

$$\begin{tikzcd}
& X \arrow[dl, "1"'] \arrow[d, "d_0"'] \arrow[dr, "f"] & \\
X & X \otimes I \arrow[l, "s"'] \arrow[r, "h"] & Y \\
& X \arrow[ul, "1"] \arrow[u, "d_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

where h is the homotopy. Thus, sending $f : X \to Y$ to the class of $(1_X, f)$ defines a function

$$\psi : \pi(X,Y) \to \pi_0 H(X,Y)$$

and there is a diagram

$$\begin{tikzcd} \pi(X,Y) \arrow[r, "\psi"] \arrow[dr, "\cong"'] & \pi_0 H(X,Y) \arrow[d, "\phi"] \\ & {[X,Y]} \end{tikzcd}$$

It suffices to show that ψ is surjective, or that any object $X \xleftarrow{f} Z \xrightarrow{g} Y$ is in the path component of some a pair $X \xleftarrow{1} X \xrightarrow{k} Y$ for some map k.

The weak equivalence f has a factorization

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "f"'] & V \arrow[d, "p"] \\ & X \end{tikzcd}$$

where j is a trivial cofibration and p is a trivial fibration. The object Y is fibrant, so the dotted arrow θ exists in the diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "j"] & \\ X & & Y \\ & V \arrow[ul, "p"] \arrow[ur, dotted, "\theta"'] & \end{tikzcd}$$

Since X is cofibrant, the trivial fibration p has a section σ, and so there is a commutative diagram

$$\begin{tikzcd}
& X \arrow[dl, "1"'] \arrow[dr, "\theta\sigma"] \arrow[dd, "\sigma"] & \\
X & & Y \\
& V \arrow[ul, "p"] \arrow[ur, dotted, "\theta"'] &
\end{tikzcd}$$

Then the composite $\theta\sigma$ is the required map k. □

Theorem 15.3. *Suppose that the model category* **M** *has the properties 1) and 2) listed above.*

Then the canonical map

$$\phi : \pi_0 H(X,Y) \to [X,Y]$$

is a bijection for all objects X and Y of **M**.

Proof. There are weak equivalences $\pi : X' \to X$ and $j : Y \to Y'$ such that X' and Y' are cofibrant and fibrant, respectively, and there is a commutative diagram

$$\begin{tikzcd}
\pi_0 H(X,Y) \arrow[r, "\phi"] \arrow[d, "(1,j)_*"', "\cong"] & {[X,Y]} \arrow[d, "\cong"', "j_*"] \\
\pi_0 H(X,Y') \arrow[r, "\phi"] & {[X,Y']} \arrow[d, "\cong"', "\pi^*"] \\
\pi_0 H(X',Y') \arrow[u, "(\pi,1)_*", "\cong"'] \arrow[r, "\cong", "\phi"'] & {[X',Y']}
\end{tikzcd}$$

The functions $(1,j)_*$ and $(\pi,1)_*$ are bijections by Lemma 15.1, and the bottom map ϕ is a bijection by Lemma 15.2. □

Remark 15.4. Cocycle categories have appeared before, in the context of Dwyer-Kan hammock localizations [3], [2].

One of the main results in the area, which holds for arbitrary model categories $\mathbf{M}$, says roughly that the nerve $BH(X,Y)$ is a model for the function space of maps from X to Y if Y is fibrant. This result implies Theorem 15.3 if the target object Y is fibrant.

On the other hand (see below), the most powerful applications of Theorem 15.3 in local homotopy theory involve target objects Y which are not fibrant in general.

16 Sheaf cohomology

Suppose that A is a sheaf of abelian groups, and let $A \to J$ be an injective resolution of A, thought of as a $\mathbb{Z}$-graded chain complex, concentrated in negative degrees.

Write $A[-n]$ for the chain complex consisting of A concentrated in degree n, and consider the chain map $A[-n] \to J[-n]$.

$K(A,n) = \Gamma A[-n]$ defines the **Eilenberg-Mac Lane** simplicial sheaf associated to A. Let

$$K(J,n) = \Gamma \mathrm{Tr}_0(J[-n])$$

where $\mathrm{Tr}_0(J[-n])$ is the good truncation of $J[-n]$ in non-negative degrees.

Suppose that C is an ordinary chain complex and that I is an unbounded chain complex which is 0 in non-negative degrees. Form the bicomplex

$$\hom(C,I)_{p,q} = \hom(C_{-p}, I_q)$$

with the obvious induced differentials:

$$\partial' = \partial_C^* : \hom(C_{-p}, I_q) \to \hom(C_{-p-1}, I_q)$$
$$\partial'' = (-1)^p \partial_{I*} : \hom(C_{-p}, I_q) \to \hom(C_{-p}, I_{q-1}).$$

Then $\hom(C,I)$ is a third quadrant bicomplex with associated total complex

$$\begin{aligned} \mathrm{Tot}_{-n} \hom(C,I) &= \bigoplus_{p+q=-n} \hom(C_{-p}, I_q) \\ &= \bigoplus_{0 \leq p \leq n} \hom(C_p, I_{-n+p}), \end{aligned}$$

for $n \geq 0$, concentrated in negative degrees.

Exercise: Show that there are natural isomorphisms

$$\begin{aligned} H_{-n}(\mathrm{Tot} \hom(C,I)) &\cong \pi(C(0), I[-n]) \\ &\cong \pi(C, \mathrm{Tr}_0 I[-n]), \end{aligned}$$

where $\pi(C(0), I[-n])$ denotes chain homotopy classes of maps from the unbounded complex $C(0)$ canonically associated to C to the shifted complex $I[-n]$, and $\pi(C, \mathrm{Tr}_0 I[-n])$ is chain homotopy classes of maps in the bounded complex category.

Example: If $A \to J$ is an injective resolution of an abelian sheaf A, then the bicomplex $\hom(C, J)$ determines a spectral sequence with

$$E_2^{p,q} = \mathrm{Ext}^q(H_p(C), A) \Rightarrow \pi(C, \mathrm{Tr}_0 J[-p-q]). \tag{16.1}$$

Lemma 16.1. *Every local weak equivalence $f : X \to Y$ induces an isomorphism*

$$\pi_{ch}(N\tilde{\mathbb{Z}}Y, \mathrm{Tr}_0 J[-n]) \xrightarrow{\cong} \pi_{ch}(N\tilde{\mathbb{Z}}X, \mathrm{Tr}_0 J[-n])$$

in chain homotopy classes for all $n \geq 0$.

Proof. The map f induces a homology sheaf isomorphism $N\tilde{\mathbb{Z}}X \to N\tilde{\mathbb{Z}}Y$, and then a comparison of spectral sequences

$$E_2^{p,q} = \mathrm{Ext}^q(\tilde{H}_p(X), A) \Rightarrow \pi_{ch}(N\tilde{\mathbb{Z}}X, \mathrm{Tr}_0 J[-p-q]) \tag{16.2}$$

gives the desired result. □

If two chain maps $f, g : N\tilde{\mathbb{Z}}X \to \mathrm{Tr}_0 J[-n]$ are chain homotopic, then there is a right homotopy

$$\begin{tikzcd} & Z \arrow[d, "p"] \\ X \arrow[ur] \arrow[r, "(f_*,g_*)"'] & K(J,n)\times K(J,n) \end{tikzcd}$$

for some path object Z over $K(J,n)$ in the projective model structure for $\mathscr{C}^{op}$-diagrams of simplicial sets.

Choose a sectionwise trivial fibration $\pi : W \to X$ such that W is projective cofibrant.

Then $f_*\pi$ is left homotopic to $g_*\pi$ for some choice of cylinder object $W \otimes I$ for W, again in the projective structure.

This means that there is a diagram

$$\begin{tikzcd} & & W \arrow[dl, "1"'] \arrow[d, "i_0"] \arrow[r, "\pi"] & X \arrow[dr, "f_*"] & \\ X & W \arrow[l, "\pi"'] & W\otimes I \arrow[l, "s"'] \arrow[rr, "h"] & & K(J,n) \\ & & W \arrow[ul, "1"] \arrow[u, "i_1"'] \arrow[r, "\pi"'] & X \arrow[ur, "g_*"'] & \end{tikzcd}$$

where the maps s, i_0, i_1 are all part of the cylinder object structure for $W \otimes I$, and are sectionwise weak equivalences.

Thus,

$$(1, f_*) \sim (\pi, f_*\pi) \sim (\pi s, h) \sim (\pi, g_*\pi) \sim (1, g_*)$$

in $\pi_0 H(X, K(J,n))$.

It follows that there is a well defined abelian group homomorphism

$$\phi : \pi_{ch}(N\tilde{\mathbb{Z}}X, \mathrm{Tr}_0 J[-n]) \to \pi_0 H(X, K(J,n)).$$

This map is natural in X.

Lemma 16.2. *The map*

$$\phi : \pi_{ch}(N\tilde{\mathbb{Z}}X, \mathrm{Tr}_0 J[-n]) \to \pi_0 H(X, K(J,n)).$$

is an isomorphism.

Proof. Suppose that

$$X \xleftarrow{f} Z \xrightarrow{g} K(J,n)$$

is an object of $H(X, K(J,n))$.

Then there is a unique chain homotopy class $[v] : N\tilde{\mathbb{Z}}X \to J[-n]$ such that $[v_* f] = [g]$ since f is a local weak equivalence.

This chain homotopy class $[v]$ is also independent of choice of representative for the component of (f,g).

We therefore have a well defined function

$$\psi : \pi_0 H(X, K(J,n)) \to \pi_{ch}(N\tilde{\mathbb{Z}}X, \mathrm{Tr}_0 J[-n]).$$

The composites $\psi \cdot \phi$ and $\phi \cdot \psi$ are identity morphisms. $\square$

We have proved

Theorem 16.3. *Suppose that A is a sheaf of abelian groups on $\mathscr{C}$, and let $A \to J$ be an injective resolution of A in the category of abelian sheaves. Let X be a simplicial presheaf on $\mathscr{C}$.*

Then there is an isomorphism

$$\pi_{ch}(N\tilde{\mathbb{Z}}X, \mathrm{Tr}_0 J[-n]) \cong [X, K(A,n)].$$

This isomorphism is natural in X.

Suppose that A is an abelian (pre)sheaf on $\mathscr{C}$ and that X is a simplicial presheaf.

Set

$$H^n(X,A) := [X, K(A,n)],$$

and say that this group is the n^{th} **cohomology group** of X with coeffients in A.

The following is an immediate consequence of Theorem 16.3 (but it's easier than that — exercise):

Corollary 16.4. *Suppose that $f : X \to Y$ induces a homology sheaf isomorphism*

$$\tilde{H}_*(X) \cong \tilde{H}_*(Y).$$

Then the induced map in cohomology

$$H^*(Y,A) \to H^*(X,A)$$

is an isomorphism for all coefficient presheaves A.

Proof. The induced map $\mathbb{Z}(X) \to \mathbb{Z}(Y)$ is a local weak equivalence. $\square$

There is also a torsion coeffients version:

Corollary 16.5. *If $f : X \to Y$ induces a homology sheaf isomorphism*

$$\tilde{H}_*(X, \mathbb{Z}/n) \cong \tilde{H}_*(Y, \mathbb{Z}/n)$$

then f induces an isomorphism

$$H^*(Y, A) \to H^*(X, A)$$

for all n-torsion presheaves A.

Remark 16.6. 1) The associated sheaf map

$$K(A, n) \to K(\tilde{A}, n)$$

is a local weak equivalence, so that

$$H^n(X, A) \cong H^n(X, \tilde{A}).$$

2) One can (and does) define sheaf cohomology $H^n(\mathscr{C}, A)$ for an abelian sheaf A on a site $\mathscr{C}$ by

$$H^n(\mathscr{C}, A) = H_{-n}(\Gamma_* J)$$

where $A \to J$ is an injective resolution of A concentrated in negative degrees and Γ_* is global sections (ie. inverse limit).

But $\Gamma_* Y = \hom(*, Y)$ for any Y, and so

$$H^n(\mathscr{C}, A) \cong \pi_{ch}(\tilde{\mathbb{Z}}*, \mathrm{Tr}_0 J[-n]) \cong [*, K(A, n)]$$

by Theorem 16.3.

3) Putting together (16.2) and Theorem 16.3 gives a **universal coefficients spectral sequence**

$$E_2^{p,q} = \mathrm{Ext}^q(\tilde{H}_p(X), \tilde{A}) \Rightarrow H^{p+q}(X, A) \qquad (16.3)$$

for abelian presheaves A and simplicial presheaves X.

The n-torsion analog is a corresponding spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^q(\tilde{H}_p(X, \mathbb{Z}/n), \tilde{A}) \Rightarrow H^{p+q}(X, A) \qquad (16.4)$$

for n-torsion presheaves A.

Cup products

Suppose that

$$X \xleftarrow{\simeq} X' \to K(A, n), \quad Y \xleftarrow{\simeq} Y' \to K(B, m)$$

are cocycles. Then the adjoint simplicial abelian presheaf maps

$$\mathbb{Z}X' \to K(A, n), \quad \mathbb{Z}Y' \to K(B, n)$$

have a (simplicial abelian group) tensor product

$$\mathbb{Z}(X' \times Y') \cong \mathbb{Z}X' \otimes \mathbb{Z}Y' \to K(A, n) \otimes K(B, n)$$

and there is a natural weak equivalence

$$K(A,n) \otimes K(B,m) \simeq K(A \otimes B, n+m).$$

in simplicial abelian groups, hence in simplicial abelian presheaves (Exercise: suppose first that $A = B = \mathbb{Z}$).

The adjoint

$$X \times Y \xleftarrow{\simeq} X' \times Y' \to K(A \otimes B, n+m)$$

represents the external cup product of the classes represented by the two cocycles.

We have defined an **external cup product**

$$H^n(X,A) \times H^m(Y,B) \to H^{n+m}(X \times Y, A \otimes B).$$

If A happens to be a presheaf of rings this construction specializes to the cup product pairing

$$\begin{aligned} H^n(X,A) \times H^m(X,A) &\to H^{n+m}(X \times X, A) \\ &\xrightarrow{\Delta^*} H^{n+m}(X,A). \end{aligned}$$

where $\Delta : X \to X \times X$ is the diagonal map.

Cohomology operations

A *cohomology operation* is a map

$$K(A,n) \to K(B,m)$$

in the homotopy category.

The *Steenrod operation* Sq^i is a morphism

$$K(\mathbb{Z}/2,n) \to K(\mathbb{Z}/2,n+i)$$

in the ordinary homotopy category. The constant presheaf functor preserves weak equivalences, and so Sq^i induces a morphism

$$K(\Gamma^*\mathbb{Z}/2,n) \to K(\Gamma^*\mathbb{Z}/2,n+i)$$

in the homotopy category of simplicial presheaves on an arbitrary small site $\mathscr{C}$. It therefore induces a homomorphism

$$\mathrm{Sq}^i : H^n(X,\mathbb{Z}/2) \to H^{n+i}(X,\mathbb{Z}/2)$$

which is natural in simplicial presheaves X. The collection of Steenrod operations $\{\mathrm{Sq}^i\}$ for simplicial presheaves has the same basic list of properties as the Steenrod operations for ordinary spaces.

Steenrod operations for mod 2 étale cohomology were introduced by Breen [1]; the definition given here for mod 2 cohomology of arbitrary simplicial presheaves is a vast generalization.

The first calculational application was in questions concerning Hasse-Witt classes for non-degenerate symmetric bilinear forms in the mod 2 Galois cohomology of fields — see [5] and [6].

The definition of Steenrod operations that is given here has its uses, but it is relatively naive. Voevodsky introduced and made very effective use of a much more sophisticated construction for motivic homotopy theory in his proof of the Milnor conjecture [10], [11].

17 Descent spectral sequences

Proposition 17.1. *Suppose that A is a presheaf of abelian groups, and that*

$$j : K(A,n) \to GK(A,n)$$

is an injective fibrant model of $K(A,n)$.

Then there are isomorphisms

$$\pi_j GK(A,n)(U) \cong \begin{cases} H^{n-j}(\mathscr{C}/U, \tilde{A}|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

for all $U \in \mathscr{C}$.

Exercise 17.2. Suppose given a diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & X' \arrow[dl, "p'"] \\ & Y & \end{tikzcd}$$

where p and p' are local fibrations and f is a local weak equivalence. Suppose that $Z \to Y$ is a map of simplicial presheaves.

Show that the induced map

$$Z \times_Y X \xrightarrow{f_*} Z \times_Y X'$$

is a local weak equivalence — use Boolean localization.

Suppose that $U \in \mathscr{C}$ and write $X|_U$ for the restriction of X along the functor

$$\mathscr{C}/U \to \mathscr{C}.$$

Lemma 17.3. *The restriction functor $X \mapsto X|_U$ preserves injective fibrations and local weak equivalences, and therefore preserves injective fibrant models.*

Proof. The restriction functor $X \mapsto X|_U$ has a left adjoint j_U^* where

$$j_U^*(Y)(V) = \bigsqcup_{V \to U} Y(V).$$

Then j_U^* clearly preserves cofibrations and sectionwise weak equivalences.

The functor j_U^* also preserves local trivial fibrations (exercise) and therefore preserves local weak equivalences.

Restriction preserves sectionwise equivalences and local trivial fibrations, and therefore preserves local weak equivalences. $\square$

Proof of Proposition 17.1. There are isomorphisms

$$\begin{aligned}\pi_0 GK(A,n)(U) &\cong [*, GK(A,n)(U)] \\ &\cong [*, GK(A|_U,n)]_{\mathscr{C}/U} \\ &\cong H^n(\mathscr{C}/U, \tilde{A}|_U).\end{aligned}$$

$GK(A,n)|_U$ is an injective fibrant model of $K(A|_U,n)$ by Lemma 17.3, giving the second and third isomorphisms.

The associated sheaf map

$$\eta : K(A,0) \to K(\tilde{A},0)$$

is an injective fibrant model for the constant simplicial presheaf $K(A,0)$ (see Section 11.2) , and

$$\pi_j K(\tilde{A},0)(U) = 0$$

for $j > 0$.

There is a sectionwise fibre sequence

$$\begin{aligned}K(A,n-1) \to & WK(A,n-1) \\ & \to \overline{W}K(A,n-1) = K(A,n)\end{aligned}$$

where $WK(A,n-1)$ is sectionwise contractible.

Take an injective fibrant model

$$\begin{tikzcd}
WK(A,n-1) \arrow[r, "j"] \arrow[d] & GWK(A,n-1) \arrow[d, "p"] \\
K(A,n) \arrow[r, "j"'] & GK(A,n)
\end{tikzcd}$$

where the maps labelled j are local weak equivalences, $GK(A,n)$ is injective fibrant and p is an injective fibration.

Let $F = p^{-1}(0)$. Then F is injective fibrant and the induced map

$$K(A,n-1) \to F$$

is a local weak equivalence, by Exercise 17.2. Write $GK(A,n-1)$ for F.

We have sectionwise fibre sequences

$$\begin{aligned} GK(A,n-1)(U) \to & GWK(A,n-1)(U) \\ & \to GK(A,n)(U) \end{aligned}$$

for all $U \in \mathscr{C}$. The map

$$GWK(A,n-1) \to *$$

is a trivial injective fibration, and is therefore a sectionwise trivial fibration.

It follows that

$$\pi_j GK(A,n)(U) \cong \pi_{j-1} GK(A,n-1)(U)$$

for $j \geq 1$, so that

$$\pi_j GK(A,n)(U) \cong H^{n-j}(\mathscr{C}/U, \tilde{A}|_U)$$

for $1 \leq j \leq n$ by induction on n. $\square$

Example: Suppose $\mathscr{C}$ is the big site $(Sch|_S)_{et}$ for a scheme S with the étale topology and that U is an S-scheme in this site.

Then $\mathscr{C}/U$ is isomorphic to the site $(Sch|_U)_{et}$.

If A is a sheaf on the big étale site for S, and if $K(A,n) \to GK(A,n)$ is an injective fibrant model for $K(A,n)$, then the presheaves of homotopy groups for $GK(A,n)$ have the form

$$\pi_j GK(A,n)(U) \cong \begin{cases} H^{n-j}_{et}(U, \tilde{A}|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

for all $U \in \mathscr{C}$.

Similar statements obtain for all other geometric topologies on categories of S-schemes.

Suppose that X is a presheaf of locally connected pointed Kan complexes, and form the Postnikov tower

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] \\
& P_2X \arrow[r, "j"] \arrow[d] & GP_2X \arrow[d, "p"] \\
& P_1X \arrow[r, "j"] \arrow[d] & GP_1X \arrow[d, "p"] \\
X \arrow[r] \arrow[ur] \arrow[uur] & P_0X \arrow[r, "j"'] & GP_0X
\end{tikzcd}$$

where all maps labelled j are injective fibrant models and the maps p are injective fibrations.

The fibre of $GP_nX \to GP_{n-1}X$ is sectionwise equivalent to $GK(\tilde{\pi}_nX, n)$, where

$$\tilde{\pi}_nX = \tilde{\pi}_n(X, *)$$

is the n^{th} homotopy group sheaf, based at the global base point.

Now take $U \in \mathscr{C}$ and consider the tower of fibrations

$$GP_0X(U) \leftarrow GP_1X(U) \leftarrow GP_2X(U) \leftarrow \dots$$

The fibre $GK(\tilde{\pi}_nX, n)(U)$ of the map

$$GP_nX(U) \to GP_{n-1}X(U)$$

has homotopy groups

$$\pi_j GK(\tilde{\pi}_n X, n)(U) \cong \begin{cases} H^{n-j}(\mathscr{C}/U, \tilde{\pi}_n X|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

and so the tower of fibrations spectral sequence (with the Thomason re-indexing trick [9, 5.54]) determines a spectral sequence with

$$E_2^{s,t}(U) = H^s(\mathscr{C}/U, \tilde{\pi}_s X|_U)$$

This is the (unstable) **descent spectral sequence** — it is actually a presheaf of spectral sequences.

This spectral sequence is often called either a cohomological or topological descent spectral sequence.

There are two issues:

1) the spectral sequence might or might not converge to

$$\pi_{t-s} \varprojlim GP_n X(U)$$

2) it can be a bit of work to show that the map $X \to \varprojlim_n GP_n X$ is a local weak equivalence.

Both issues can be resolved (ie. the spectral sequence converges and the map of 2) is a local weak equivalence) if X is locally connected in the sense that $\tilde{\pi}_0 X \cong *$ and there is a uniform bound on cohomological dimension for all sheaves $\tilde{\pi} X|_U$. See [4].

Why would you care?

By construction, the object $\varinjlim GP_n$ is injective fibrant (exercise), so the real question is whether or not the map $X \to \varprojlim_n GP_n X$ is an injective fibrant model for X. Is this map a local weak equivalence?

If so, and if the spectral sequence converges, it is calculating the homotopy groups $\pi_n(Z(U))$ of an injective fibrant model $X \to Z$ in sections.

If X satisfies descent, then $X|_U$ satisfies descent.

In this case, this means that $X(U) \to Z(U)$ is a weak equivalence for any injective fibrant model $X \to Z$, and we are computing the homotopy groups $\pi_n(X(U))$ in sections, from sheaf cohomological data.

The possibility of such calculations is the motivation behind all descent questions, and is the source of the name for the spectral sequence.

The probability that all these things will work out at once is akin to that of finding an earthlike exoplanet orbiting a particular star.

There are also “finite” descent spectral sequences, which are Bousfield-Kan spectral sequences arising from function complexes $\mathbf{hom}(V,Z)$, where $V \to *$ is a local weak equivalence and Z is injective fibrant.

V could be the Čech resolution $C(U) \to *$ which is associated to a local epimorphism $U \to *$ of sheaves (or presheaves).

Example: Suppose that L/k is a finite Galois extension of a field k with Galois group G.

Then, by Galois theory, there is an isomorphism

$$G \times \mathrm{Sp}(L) \xrightarrow{\cong} \mathrm{Sp}(L) \times \mathrm{Sp}(L)$$

of k-schemes which induces an isomorphism

$$EG \times_G \mathrm{Sp}(L) \cong C(\mathrm{Sp}(L))$$

on simplical sheaves (even simplicial schemes) on any of the **étale** sites for the field k.

It follows that the canonical map

$$EG \times_G \mathrm{Sp}(L) \to *$$

is a local weak equivalence for the étale topology.

If Z is injective fibrant, the map

$$Z(k) \cong \mathbf{hom}(*, Z) \to \mathbf{hom}(EG \times_G \mathrm{Sp}(L), Z)$$

is a weak equivalence of simplicial sets.

The Bousfield-Kan spectral sequence for the function complex on the right has the form

$$E_2^{s,t} = H^s(G, \pi_t Z(L)) \Rightarrow \pi_{t-s} Z(k).$$

This is the **finite Galois descent spectral sequence** for the homotopy groups of the global section $Z(k)$ of Z.

The full Galois (or étale) cohomological descent spectral sequence for Z has the form

$$E_2^{s,t} = H^s(\Omega, \tilde{\pi}_t Z) \Rightarrow \pi_{t-s} Z(k),$$

where Ω is the absolute Galois group of k.

One often says that a simplicial presheaf X on an étale site for k satisfies **finite descent** if the map

$$X(k) \cong \mathbf{hom}(*, X) \to \mathbf{hom}(EG \times_G \mathrm{Sp}(L), X)$$

is a weak equivalence for every finite Galois extension L/k.

The question of whether a given simplicial presheaf X (like an algebraic K-theory presheaf) satisfies finite descent is also called the **homotopy fixed points problem**.

Warning: You may be tempted (many were) to say that finite descent for X implies that X satisfies descent for the étale topology on k, but you would be wrong.

Such claims hold only in very special cases — see [9], [7], [8].

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