21 The Verdier hypercovering theorem

Suppose that $\mathscr{C}$ is a small Grothendieck site. As before, $s\mathrm{Pre}(\mathscr{C})$ is the category of simplicial presheaves on the site $\mathscr{C}$.

The discussion that follows will be confined to simplicial presheaves. It has an exact analog for simplicial sheaves.

Let A be a fixed choice of simplicial presheaf.

The slice category $A/s\mathrm{Pre}(\mathscr{C})$ has all morphisms $x : A \to X$ as objects, and all diagrams

$$\begin{array}{ccc} & A & \\ {}^{x}\swarrow & & \searrow^{y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

as morphisms.

The intuition is that $x : A \to X$ is a "base point" of X (geometric points for the étale topology are good examples to keep in mind) even though A could be non-trivial homotopically.

The category $A/s\mathrm{Pre}(\mathscr{C})$ inherits a local model structure from $s\mathrm{Pre}(\mathscr{C})$, in that a morphism

$$\begin{array}{ccc} & A & \\ {}^{x}\swarrow & & \searrow^{y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a local weak equivalence (respectively cofibration, fibration) if and only if the underlying map $f : X \to Y$ is a local weak equivalence (respectively cofibration, injective fibration) of simplicial presheaves.

Remark 21.1. 1) Not all objects of the slice category are cofibrant: the identity morphsm $1 : A \to A$ is initial, and so an object $x : A \to X$ is cofibrant if an only if the map x is a cofibration of simplicial presheaves.

2) The unique map $A \to *$ taking values in the terminal simplicial presheaf $*$ is the terminal object of $A/s\mathrm{Pre}(\mathscr{C})$, and it follows that an object $x : A \to X$ is fibrant if and only if X is an injective fibrant simplicial presheaf.

Say that a map $f : x \to y$ in the slice category $A/s\mathrm{Pre}(\mathscr{C})$ is a **hypercover** if the underlying simplicial presheaf map $f : X \to Y$ is a hypercover (or a local trivial fibration).

More generally, $f : x \to y$ is a **local fibration** if the map $f : X \to Y$ is a local fibration of simplicial presheaves.

In particular, $x : A \to X$ is locally fibrant if X is locally fibrant.

The theory of cocycle categories of [3] applies without change to the model category $A/s\mathrm{Pre}(\mathscr{C})$.

A cocycle (g, f) from x to y is a diagram

$$x \xleftarrow[\simeq]{g} z \xrightarrow{f} y$$

in the slice category, or a diagram of simplicial presheaf maps

$$\begin{array}{ccccc} & & A & & \\ & {}^{x}\swarrow & \downarrow{\scriptstyle z} & \searrow^{y} & \\ X & \xleftarrow[g]{\simeq} & Z & \xrightarrow[f]{} & Y \end{array}$$

for which the map g is a weak equivalence.

The cocycles are the objects of a category $H(x, y)$ which has morphisms $\theta : (g, f) \to (g, f')$ given by the commutative diagrams

$$\begin{array}{ccccc} & {}^{g}\swarrow & z & \searrow^{f} & \\ x & & \downarrow{\scriptstyle \theta} & & y \\ & {}_{g'}\nwarrow & z' & \nearrow_{f'} & \end{array}$$

The category $H(x,y)$ is the category of cocycles from x to y.

The model structure on $A/s\,\mathrm{Pre}(\mathscr{C})$ is right proper, and weak equivalences in this structure are closed under finite products, because these properties both hold for the category of simplicial presheaves.

Thus, Theorem 15.3 (Lecture 07) implies the following:

Lemma 21.2. *The function*

$$\phi : \pi_0 H(x,y) \to [x,y],$$

which is defined by $(g,f) \mapsto f \cdot g^{-1}$ *for a cocycle* (g,f) *in the slice category* $A/s\,\mathrm{Pre}(\mathscr{C})$*, is a bijection.*

Suppose that $f,g : x \to y$ are morphisms of the slice category $A/s\,\mathrm{Pre}(\mathscr{C})$.

A (naive) pointed homotopy from f to g is a commutative diagram

$$\begin{array}{ccc} A \times \Delta^1 & \xrightarrow{pr} & A \\ {\scriptstyle x\times\Delta^1}\downarrow & & \downarrow{\scriptstyle y} \\ X \times \Delta^1 & \xrightarrow[h]{} & Y \end{array}$$

such that h is a simplicial homotopy from f to g in the usual sense.

The projection map $pr : A \times \Delta^1 \to A$ onto A defines the constant homotopy on A.

Equivalently, such a pointed homotopy is a map

$$h : (X \times \Delta^1) \cup_{A \times \Delta^1} A \to Y.$$

In the pushout diagram

$$\begin{array}{ccc} A \times \Delta^1 & \xrightarrow{pr} & A \\ {\scriptstyle x \times \Delta^1} \downarrow & & \downarrow \\ X \times \Delta^1 & \xrightarrow[pr_*]{} & (X \times \Delta^1) \cup_{A \times \Delta^1} A \end{array}$$

the map pr_* is a weak equivalence if the map $x : A \to X$ is a cofibration, or if x is a cofibrant object of $A/s\mathrm{Pre}(\mathscr{C})$.

In that case, the pushout object is a cylinder for x in the slice category.

Every object $x : A \to X$ has a cofibrant model, meaning a diagram

$$\begin{array}{ccc} A & \xrightarrow{v} & Z \\ & {\scriptstyle x} \searrow & \downarrow {\scriptstyle p} \\ & & X \end{array}$$

such that v is a cofibration and p is a weak equivalence.

If the maps $f,g : x \to y$ are pointed homotopic and $p : v \to x$ is a cofibrant model of x, then the composites fp and gp are pointed homotopic, and therefore represent the same map in the homotopy category since v is cofibrant.

But then p is an isomorphism in that category, so that $f = g$ in the homotopy category.

The objects of the category $Triv/x$ are the pointed homotopy classes of maps $[p] : z \to x$ which are represented by hypercovers $p : z \to x$.

The morphisms of this category are commutative triangles of pointed homotopy classes of maps, in the obvious sense.

There is a contravariant set-valued functor which takes an object $[p] : z \to x$ of $Triv/x$ to the set $\pi(z,y)$ of pointed homotopy classes.

There is a function

$$\phi_h : \varinjlim_{[p]:z\to x} \pi(z,y) \to [x,y]$$

which is defined by sending the diagram

$$x \xleftarrow{[p]} z \xrightarrow{[f]} y$$

to the morphism $f \cdot p^{-1}$ in the homotopy category.

The colimit

$$\varinjlim_{[p]:z\to x} \pi(z,y)$$

is the set of path components of a category $H_h(x,y)$ whose objects are the pictures of pointed homotopy classes

$$x \xleftarrow{[p]} z \xrightarrow{[f]} y,$$

such that $p : z \to x$ is a hypercover, and whose morphisms are the commutative diagrams

$$\begin{array}{ccccc} & & z & & \\ & \swarrow^{[p]} & \downarrow [\theta] & \searrow^{[f]} & \\ x & & & & y \\ & \nwarrow_{[p']} & z' & \nearrow_{[f']} & \end{array} \tag{21.1}$$

in pointed homotopy classes of maps. The map ϕ_h therefore has the form

$$\phi_h : \pi_0 H_h(x,y) \to [x,y]$$

The following result is a generalized Verdier hypercovering theorem:

Theorem 21.3. *The function*

$$\phi_h : \pi_0 H_h(x,y) \to [x,y]$$

is a bijection if y is locally fibrant.

Remark 21.4. Theorem 21.3 specializes to a generalization of the standard form of the Verdier hypercovering theorem [1, p.425], [2] if $A = \emptyset$, for the unique map $x : \emptyset \to X$.

The object X is not required to be locally fibrant.

There are multiple variations of the category $H_h(x,y)$:

1) Write $H_h'(x,y)$ for the category whose objects are pictures

$$x \xleftarrow{p} z \xrightarrow{[f]} y$$

where p is a hypercover and $[f]$ is a pointed homotopy class of maps.

The morphisms of $H_h'(x,y)$ are diagrams

$$\begin{array}{ccccc} & & z & & \\ & \swarrow^{p} & & \searrow^{[f]} & \\ x & & \downarrow [\theta] & & y \\ & \nwarrow_{p'} & & \nearrow_{[f']} & \\ & & z' & & \end{array} \tag{21.2}$$

such that $[\theta]$ is a fibrewise pointed homotopy class of maps over x, and $[f'][\theta] = [f]$ as pointed homotopy classes.

There is a functor

$$\omega : H_h'(x,y) \to H_h(x,y),$$

which is defined by the assignment $(p,[f]) \mapsto ([p],[f])$, and which sends the morphism (21.2) to the morphism (21.1).

2) Write $H_h''(x,y)$ for the category whose objects are the pictures

$$x \xleftarrow{p} z \xrightarrow{[f]} y$$

where p is a hypercover and $[f]$ is a pointed simplicial homotopy class of maps.

The morphisms of $H_h''(X,Z)$ are commutative diagrams

$$\begin{array}{ccc} & & z \\ & \overset{p}{\swarrow} & \\ x & & \downarrow \theta \\ & \underset{p'}{\nwarrow} & \\ & & z' \end{array}$$

such that $[f' \cdot \theta] = [f]$.

There is a canonical functor

$$H_h''(x,y) \xrightarrow{\omega'} H_h'(x,y)$$

which is the identity on objects, and takes morphisms θ to their associated fibrewise pointed homotopy classes.

3) Let $H_{hyp}(x,y)$ be the full subcategory of $H(x,y)$ whose objects are the cocycles

$$x \xleftarrow{p} z \xrightarrow{f} y$$

with p a hypercover.

There is a functor

$$\omega'' : H_{hyp}(x,y) \to H_h''(x,y)$$

which takes a cocycle (p,f) to the object $(p,[f])$.

Lemma 21.5. *Suppose that y is locally fibrant.*

Then the inclusion functor $i : H_{hyp}(x,y) \subset H(x,y)$ is a homotopy equivalence.

Proof. Objects of the cocycle category $H(x,y)$ can be identified with maps $(g,f) : z \to x \times y$ such that the morphism g is a weak equivalence, and morphisms of $H(x,y)$ are commutative triangles in the obvious way.

Maps of the form (g,f) have functorial factorizations

$$\begin{array}{ccc} z & \xrightarrow{j} & v \\ & {\scriptstyle (g,f)} \searrow & \downarrow {\scriptstyle (p,g')} \\ & & x \times y \end{array} \tag{21.3}$$

such that j is a pointwise trivial cofibration and (p,g') is a pointwise Kan fibration.

It follows that (p,g') is a local fibration and the map p, or rather the composite

$$z \xrightarrow{(p,g')} x \times y \xrightarrow{pr} x,$$

is a local weak equivalence.

The projection map pr is a local fibration since

y is locally fibrant, so the map p is also a local fibration, and hence a hypercover.

It follows that the assignment $(u,g) \mapsto (p,g')$ defines a functor

$$\psi' : H(x,y) \to H_h(x,y).$$

The weak equivalences j of the diagram (21.3) define homotopies $p' \cdot i \simeq 1$ and $i \cdot \psi' \simeq 1$. $\square$

Proof of Theorem 21.3. The composite

$$H(x,y) \xrightarrow{\psi'} H_{hyp}(x,y) \xrightarrow{\omega''} H''_h(x,y) \xrightarrow{\omega'} H'_h(x,y) \xrightarrow{\omega} H_h(x,y)$$

is the functor ψ, and the composite

$$\begin{aligned}\pi_0 H(x,y) \xrightarrow{\psi'_*} \pi_0 H_{hyp}(x,y) \xrightarrow{\omega''_*} \pi_0 H''_h(x,y) \xrightarrow{\omega'_*} \pi_0 H'_h(x,y) \\ \xrightarrow{\omega_*} \pi_0 H_h(x,y) \xrightarrow{\phi_h} [x,y]\end{aligned} \tag{21.4}$$

is the bijection ϕ of Lemma 21.2.

The function ψ'_* is a bijection by Lemma 21.5, and the functions ω''_*, ω'_* and ω_* are surjective, as is the function ϕ_h.

The functions which make up the string (21.4) are therefore all bijections. $\square$

The following corollary of the proof of Theorem 21.3 deserves independent mention:

Corollary 21.6. *Suppose that the object* $y : A \to Y$ *of* $A/s\mathrm{Pre}(\mathscr{C})$ *is locally fibrant. Then the induced functions*

$$\pi_0 H_{hyp}(x,y) \xrightarrow{\omega''_*} \pi_0 H''_h(x,y) \xrightarrow{\omega'_*} \pi_0 H'_h(x,y) \xrightarrow{\omega_*} \pi_0 H_h(x,y)$$

are bijections, and all of these sets are isomorphic to the set $[x,y]$ *of morphisms* $x \to y$ *in the homotopy category* $\mathrm{Ho}(s/\mathrm{Pre}(\mathscr{C}))$.

The bijections of the path component objects in the statement of Corollary 21.6 with the set $[x,y]$ all represent specific variants of the Verdier hypercovering theorem.

References

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