25 T-spectra

Suppose that T is a pointed simplicial presheaf on a small site $\mathcal{C}$.

A *T-spectrum* X is a collection of pointed simplicial presheaves X^n, $n \geq 0$, with pointed maps $\sigma : T \wedge X^n \to X^{n+1}$. A map $f : X \to Y$ of T-spectra consists of pointed simplicial presheaf maps $f : X^n \to Y^n$ which respect structure in the sense that the diagrams

$$\begin{array}{ccc} T \wedge X^n & \xrightarrow{\sigma} & X^{n+1} \\ {\scriptstyle T\wedge f}\downarrow & & \downarrow{\scriptstyle f} \\ T \wedge Y^n & \xrightarrow[\sigma]{} & Y^{n+1} \end{array}$$

commute. Write $\mathrm{Spt}_T(\mathcal{C})$ for the category of T-spectra.

Say that a map $f : X \to Y$ of T-spectra is a *strict weak equivalence* (respectively *strict fibration*) if all maps $f : X^n \to Y^n$ are local weak equivalences (respectively injective fibrations) of pointed simplicial presheaves on $\mathcal{C}$.

A *cofibration* of T-spectra is a map $i : A \to B$ such that

- $i : A^0 \to B^0$ is a cofibration of simplicial presheaves, and

- all maps
$$(T \wedge B^n) \cup_{(T \wedge A^n)} A^{n+1} \to B^{n+1}$$
are cofibrations of simplicial presheaves.

If K is a pointed simplicial presheaf and X is a T-spectrum, then $X \wedge K$ has the obvious meaning:
$$(X \wedge K)^n = X^n \wedge K.$$

The *function complex* $\mathbf{hom}(X, Y)$ for T-spectra X and Y is the pointed simplicial set with
$$\mathbf{hom}(X, Y)_n = \{ \ X \wedge \Delta^n_+ \to Y \ \}.$$

Lemma 25.1. *With these definitions, the category of* $\mathrm{Spt}_T(\mathcal{C})$ *of* T*-spectra on* $\mathcal{C}$ *satisfies the definitions for a proper closed simplicial model category.*

The proof is the usual thing.

Suspensions and shifts work in $\mathrm{Spt}_T(\mathcal{C})$ just like for ordinary spectra:

- Given a pointed simplicial presheaf K, the *suspension spectrum* $\Sigma^\infty_T K$ is the T-spectrum
$$K, T \wedge K, T^2 \wedge K, \dots$$
with $T^n = T \wedge \cdots \wedge T$ (n-fold smash power). The functor $K \mapsto \Sigma^\infty_T K$ is left adjoint to the 0-level functor $X \mapsto X^0$.

The suspension spectrum $\Sigma_T^\infty S^0$ is also denoted by S_T and is called the *T-sphere spectrum.*

- Given a T-spectrum X, $n \in \mathbb{Z}$,

$$X[n]^k = \begin{cases} X^{n+k} & n+k \geq 0 \\ * & n+k < 0 \end{cases}$$

Lemma 25.2. *Suppose given the diagram*

$$\begin{array}{ccc} A \cap X & \longrightarrow & X \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i]{} & Y \end{array}$$

in spectra, where j is a cofibration and i is a levelwise cofibration. Then the induced map j_* : $A \cap X \to A$ is a cofibration.

Proof. The proof is set theoretic. $\square$

What now follows is a general set of tricks that applies to any set S of cofibrations $i : A \to B$ of $\mathrm{Spt}_T(\mathcal{C})$.

Suppose that α is a cardinal such that $\alpha > |\operatorname{Mor}(\mathcal{C})|$. Suppose also that $\alpha > |B|$ for all morphisms i : $A \to B$ appearing in the set S and that $\alpha > |S|$. Choose a cardinal λ such that $\lambda > 2^\alpha$.

Suppose that $f : X \to Y$ is a morphism of $\mathrm{Spt}_T(\mathcal{C})$. Define a functorial system of factorizations

$$\begin{array}{ccc} X & \xrightarrow{i_s} & E_s(f) \\ & \searrow^{f} & \downarrow f_s \\ & & Y \end{array}$$

of the map f indexed on all ordinal numbers $s < \lambda$ as follows:

1) Given the factorization (f_s, i_s) define the factorization (f_{s+1}, i_{s+1}) by requiring that the diagram

$$\begin{array}{ccc} \bigvee_{\mathbf{D}} A & \xrightarrow{(\alpha_{\mathbf{D}})} & E_s(f) \\ {\scriptstyle \vee i}\downarrow & & \downarrow \\ \bigvee_{\mathbf{D}} B & \to & E_{s+1}(f) \end{array}$$

is a pushout, where the wedge is indexed over all diagrams $\mathbf{D}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{\alpha_{\mathbf{D}}} & E_s(f) \\ {\scriptstyle i}\downarrow & & \downarrow {\scriptstyle f_s} \\ B & \xrightarrow[\beta_{\mathbf{D}}]{} & Y \end{array}$$

with $i : A \to B$ in the set S. Then the map i_{s+1} is the composite

$$X \xrightarrow{i_s} E_s(f) \xrightarrow{g_*} E_{s+1}(f)$$

2) If s is a limit ordinal, set $E_s(f) = \varinjlim_{t<s} E_s(f)$.

Set $E_\lambda(f) = \varinjlim_{s<\lambda} E_s(f)$. Then there is an induced factorization

$$\begin{array}{ccc} X & \xrightarrow{i_\lambda} & E_\lambda(f) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f_\lambda} \\ & & Y \end{array}$$

of the map f. Then i_λ is a cofibration. The map f_λ has the right lifting property with respect to the cofibrations $i : A \to B$ in S by a standard argument, since any map $\alpha : A \to E_\lambda(f)$ must factor through some $E_s(f)$ by the choice of cardinal λ.

Write $L(X) = E_\lambda(c)$ for the result of this construction when applied to the canonical map $c : X \to *$. Then we have the following:

Lemma 25.3. *1) Suppose that $t \mapsto X_t$ is a diagram of level cofibrations indexed by any cardinal $\gamma > 2^\alpha$. Then the natural map*

$$\varinjlim_{t<\gamma} L(X_t) \to L(\varinjlim_{t<\gamma} X_t)$$

is an isomorphism.

2) The functor $X \mapsto L(X)$ preserves level cofibrations.

3) Suppose that ζ is a cardinal with $\zeta > \alpha$, and let $\mathcal{F}_\zeta(X)$ denote the filtered system of *subobjects of X having cardinality less than ζ. Then the natural map*

$$\varinjlim_{Y \in \mathcal{F}_\zeta(X)} L(Y) \to L(X)$$

is an isomorphism.

4) *If $|X| < 2^\omega$ where $\omega \geq \alpha$ then $|L(X)| < 2^\omega$.*

5) *Suppose that U, V are subobjects of a presheaf of T-spectra X. Then the natural map*

$$L(U \cap V) \to L(U) \cap L(V)$$

is an isomorphism.

Proof. The argument is the same as for Lemma 22.4. □

Basic Assumptions: Suppose that S is a set of cofibrations such that

1) A is cofibrant for all $i : A \to B$ in S,

2) S includes the set I of generating maps

$$\Sigma^\infty_T C[-n] \to \Sigma^\infty_T D[-n],\ n \geq 0,$$

for the strict trivial cofibrations of $\mathrm{Spt}_T(\mathcal{C})$, which are induced by the α-bounded trivial cofibrations $C \to D$ of pointed simplicial presheaves, and

3) S includes all cofibrations

$$(A \wedge D) \cup (B \wedge C) \to B \wedge D, \ m \geq 0,$$

for $A \to B$ in S and all α-bounded pointed cofibrations $C \to D$ of simplicial presheaves.

A map $p : X \to Y$ is said to be *injective* if it has the right lifting property with respect to all maps of S. An object X is injective if the map $X \to *$ is injective. By construction, LX is injective for every object X. Every injective object is strictly fibrant.

Say that a map $f : X \to Y$ of $\mathrm{Spt}(\mathcal{C})$ is an *L-equivalence* if it induces a bijection

$$f^* : [Y, Z] \xrightarrow{\cong} [X, Z]$$

in morphisms in the strict homotopy category for every injective object Z.

Every strict equivalence $X \to Y$ is an L-equivalence.

Lemma 25.4. *Suppose that $i : A \to B$ is a cofibration with A cofibrant. Then i is an L-equivalence if*

1) i induces a trivial fibration

$$i^* : \mathbf{hom}(B, Z) \to \mathbf{hom}(A, Z)$$

for all injective Z, or

2) all injective Z have the right lifting property with respect to i and with respect to the cofibration

$$(A \wedge \Delta^1_+) \cup (B \wedge \partial\Delta^1_+) \to B \wedge \Delta^1_+.$$

Proof. The first claim is trivial.

The second claim is almost as easy: we must show that the induced function

$$i^* : \pi(B, Z) \to \pi(A, Z)$$

in naive homotopy classes is a bijection for all injective Z. This suffices, because A and B are cofibrant and Z is strictly fibrant.

Every morphism $A \to Z$ extends to a morphism $B \to Z$ because $Z \to *$ has the right lifting property with respect to i. It follows that i^* is surjective.

Given $f, g : B \to Z$, if there is a homotopy $h : A \wedge \Delta^1_+ \to Z$ from $f|_A$ to $g|_A$, then there is a diagram

$$\begin{array}{ccc} (B \wedge \partial\Delta^1_+) \cup (A \wedge \Delta^1_+) & \xrightarrow{((f,g),h)} & Z \\ \downarrow & \nearrow & \\ B \wedge \Delta^1_+ & & \end{array}$$

where the indicated lifting exists because Z is injective and the vertical map is a member of S. But then f and g are homotopic, so that i^* is injective. $\square$

Corollary 25.5. *All cofibrations appearing in the set S are L-equivalences.*

Proof. Every cofibration $i : A \to B$ appearing in the set S induces a trivial fibration

$$i^* : \mathbf{hom}(B, Z) \to \mathbf{hom}(A, Z)$$

by construction. $\square$

Note that a map $f : Z \to W$ between injective objects is an L-equivalence if and only if it is a strict equivalence. In effect, the requirement that f is an L-equivalence forces f to be an isomorphism in the strict homotopy category, and hence a strict equivalence.

A *cofibrant replacement* for a map $f : X \to Y$ is a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{j} & \tilde{Y} \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

in which the maps π_X and π_Y are trivial strict fibrations, $\tilde{X}$ is cofibrant and j is a cofibration. Any two cofibrant replacements for a fixed map f are strictly equivalent, by a standard argument. The map f is an L-equivalence if and only if it has a cofibrant replacement j which is an L-equivalence.

Note that if some cofibrant replacement j for f induces a trivial fibration

$$j^* : \mathbf{hom}(\tilde{Y}, Z) \to \mathbf{hom}(\tilde{X}, Z)$$

for all injective objects Z, then all cofibrant replacements for f have this property.

Lemma 25.6. *All cofibrations in the saturation of the set S are L-equivalences.*

Proof. The saturation of the set S is the family of cofibrations which has the left lifting property with respect to all injective maps $X \to Y$.

If the cofibration $j : C \to D$ is coproduct of members of S (hence with C and D cofibrant), then

$$j^* : \mathbf{hom}(D, Z) \to \mathbf{hom}(C, Z)$$

is a product of trivial fibrations and is therefore a trivial fibration.

Suppose given a pushout diagram

$$\begin{array}{ccc} C & \longrightarrow & C' \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ D & \longrightarrow & D' \end{array}$$

where j is a coproduct of members of S and C' is cofibrant. Then from the pullback diagram

$$\begin{array}{ccc} \mathbf{hom}(D', Z) & \longrightarrow & \mathbf{hom}(D, Z) \\ {\scriptstyle j'^*}\downarrow & & \downarrow{\scriptstyle j^*} \\ \mathbf{hom}(C', Z) & \longrightarrow & \mathbf{hom}(C, Z) \end{array}$$

we see that j'^* is a trivial fibration for all injective Z.

Suppose given a pushout diagram

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & E \\ {\scriptstyle j}\downarrow & & \downarrow \\ D & \longrightarrow & D \cup_C E \end{array}$$

with j as above and E arbitrary. Then there is a factorization

$$\begin{array}{ccc} C & \xrightarrow{i} & \tilde{E} \\ & {\scriptstyle \alpha}\searrow & \downarrow{\scriptstyle \pi} \\ & & E \end{array}$$

of α with π a strictly trivial fibration and i a cofibration, and there is an induced commutative diagram

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{j}_*} & D \cup_C \tilde{E} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_*} \\ E & \xrightarrow[j_*]{} & D \cup_C E \end{array}$$

The map π is a strict equivalence, so that π_* is a strict equivalence by properness. The map $\tilde{j}_*$ induces a trivial fibration

$$(\tilde{j}_*)^* : \mathbf{hom}(D \cup_C \tilde{E}, Z) \to \mathbf{hom}(\tilde{E}, Z)$$

for all injective Z, by the previous paragraph. It follows that some cofibrant replacement of the map

$$j_* : E \to D \cup_C E$$

induces a corresponding function complex weak equivalence.

Suppose given a string of morphisms

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \dots$$

such that each f_i is an L-equivalence. Take a "cofibrant replacement"

$$\begin{array}{ccccccc} A_0 & \xrightarrow{i_1} & A_1 & \xrightarrow{i_2} & A_2 & \longrightarrow & \dots \\ {\scriptstyle \pi_0}\downarrow & & {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} & & \\ X_0 & \xrightarrow[f_1]{} & X_1 & \xrightarrow[f_2]{} & X_2 & \longrightarrow & \dots \end{array}$$

in which A_0 is cofibrant, all i_k are cofibrations and all π_j are trivial strict fibrations. Then all maps i_k induce trivial fibrations

$$i_k^* : \mathbf{hom}(A_k, Z) \to \mathbf{hom}(A_{k-1}, Z)$$

for all injective Z, so the cofibration $A_0 \to \varinjlim_i A_i$ induces a trivial fibration

$$\mathbf{hom}(\varinjlim_i A_i, Z) \to \mathbf{hom}(A_0, Z).$$

for all injective Z. The map

$$\varinjlim_i A_i \to \varinjlim_i X_i$$

is a (sectionwise) weak equivalence, and it follows that some cofibrant replacement for the map $X_0 \to \varinjlim_i X_i$ induces a trivial fibration in all function complexes taking values in injective objects Z.

It follows that every member $i : A \to B$ of the saturation of S has a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & Z \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle \pi} \\ & & B \end{array}$$

such that π is injective and j is a member of the saturation of S which is also an L-equivalence. The map i has the left lifting property with respect to all injective maps such as π, so that i is a retract of j. $\square$

Corollary 25.7. *1) The natural map $j : X \to LX$ is an L-equivalence.*

2) A map $f : X \to Y$ is an L-equivalence if and only if the induced map $Lf : LX \to LY$ is a strict equivalence.

Lemma 25.8. *Suppose that $\gamma \geq \alpha$. Suppose further that $i : X \to Y$ is a level cofibration and a strict equivalence and that $A \subset Y$ is an γ-bounded subobject. Then there is a γ-bounded subobject $B \subset Y$ with $A \subset B$ such that the level cofibration $B \cap X \to B$ is a strict equivalence.*

Proof. First of all, consider the diagram of cofibrations

$$\begin{array}{ccc} & & X^0 \\ & & \downarrow{\scriptstyle i} \\ A^0 & \longrightarrow & Y^0 \end{array}$$

Then by Lemma 11.2 (the bounded cofibration condition for simplicial presheaves) there is a subobject $B^0 \subset Y^0$ such that B^0 is γ-bounded, $A^0 \subset B^0$ and $B^0 \cap X^0 \to B^0$ is a local weak equivalence.

Form the diagram

$$\begin{array}{ccccc} T \wedge A^0 & \longrightarrow & T \wedge B^0 & \longrightarrow & T \wedge Y^0 \\ \sigma \downarrow & & & & \downarrow \sigma \\ A^1 & & \longrightarrow & & Y^1 \end{array}$$

Then the induced map

$$A^1 \cup_{T \wedge A^0} T \wedge B^0 \to Y^1$$

factors through a γ-bounded subobject $C^1 \subset Y^1$. There is a γ-bounded subobject $B^1 \subset Y^1$ such that $C^1 \subset B^1$ and $B^1 \cap X^1 \to B^1$ is a local weak equivalence. The composite

$$T \wedge B^0 \to A^1 \cup_{T \wedge A^0} T \wedge B^0 \to C^1 \subset B^1$$

is the bonding map up to level 1 for the object B.

Construct the remaining objects B^n, $n \geq 1$, inductively according to this recipe. □

Lemma 25.9. *Suppose given a cofibration $i : X \to Y$ which is an L-equivalence, and suppose that $A \subset Y$ is a 2^λ-bounded subobject, where λ is chosen as above. Then there is a 2^λ-bounded subobject $B \subset Y$ with $A \subset B$ and such that the cofibration $B \cap X \to B$ is an L-equivalence.*

Proof. Write $B_0 = A$, and set $\kappa = 2^\lambda$.

Consider the diagram

$$\begin{array}{ccc} & & LX \\ & & \downarrow \\ LB_0 & \longrightarrow & LY \end{array}$$

Then the maps are level cofibrations (Lemma 25.3.2) and $LX \to LY$ is a strict equivalence by assumption. The object LB_0 is κ-bounded by Lemma 25.3.4, so there is a κ-bounded subobject $C_1 \subset LY$ with $LB_0 \subset C_1$ such that $C_1 \cap LX \to C_1$ is a strict equivalence, by Lemma 25.8. Since C_1 is κ-bounded there is a κ-bounded subobject $B_1 \subset Y$ with $B_0 \subset B_1$ such that $C_1 \subset LB_1$ (Lemma 25.3.3). Proceeding inductively we find κ-bounded subobjects

$$C_1 \subset C_2 \subset \dots$$

of LY and κ-bounded subobjects

$$B_0 \subset B_1 \subset B_2 \subset \dots$$

indexed by $i < \kappa$, such that C_s and B_s are defined at limit ordinals s by colimits, and

$$LB_i \subset C_{i+1} \subset LB_{i+1}$$

and $C_i \cap LX \to C_i$ is a level weak equivalence.

Write $B = \varinjlim_{i<\kappa} B_i$. Then B is κ-bounded, and

$$L(B) = \varinjlim_{i<\kappa} L(B_i) = \varinjlim_{i<\kappa} C_i$$

by Lemma 25.3.1 and construction. Also

$$\begin{aligned} L(B \cap X) = L(B) \cap L(X) &= \varinjlim_{i<\kappa} L(B_i) \cap L(X) \\ &\cong \varinjlim_{i<\kappa} C_i \cap L(X) \end{aligned}$$

by Lemma 25.3.1 and 25.3.5 and construction. It follows that the map

$$B \cap X \to B$$

is an L-equivalence. □

Say that a cofibration is *L-trivial* if it is an L-equivalence.

Lemma 25.10. *The set of κ-bounded L-trivial cofibrations is a generating set for the class of L-trivial cofibrations.*

Proof. Run the solution set argument of Lemma 23.5 using Lemma 25.9 for the set of κ-bounded cofibrations. Recall that the κ-bounded cofibrations generate the class of cofibrations. □

Say that a map $p : X \to Y$ is an *L-fibration* if it has the right lifting property with respect to all L-trivial cofibrations. Observe that every L-fibration is a strict fibration, since S contains a generating set for the class of strict trivial cofibrations.

Lemma 25.11. *A map $p : X \to Y$ is an L-fibration and an L-equivalence if and only if p is a trivial strict fibration.*

Proof. We need only show that p is a trivial strict fibration if it is an L-fibration and an L-equivalence, but this is the usual proof: find a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & W \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle \pi} \\ & & Y \end{array}$$

where j is a cofibration and π is a trivial strict fibration. But then j is an L-equivalence so the lifting exists in the diagram

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ W & \xrightarrow[\pi]{} & Y \end{array}$$

so that p is a retract of π. $\square$

Theorem 25.12. *Suppose that S is a set of cofibrations which satisfies the list of basic assumptions above. Let the L-equivalences and L-fibrations be defined relative to the set S. Then with these definitions the category $\mathrm{Spt}_T(\mathcal{C})$ satisfies the axioms for a closed simplicial model category.*

Proof. Every map $f : X \to Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & W \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

such that p is an L-fibration and j is a cofibration and an L-equivalence, by Lemma 25.6 and Lemma 25.10.

Every map $f : X \to Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle q} \\ & & Y \end{array}$$

such that i is a cofibration and q is a strictly trivial fibration. But then q is an L-fibration and an L-equivalence.

The rest of the closed model axioms are trivial to verify.

For the closed simplicial model structure, we need to show that if $i : A \to B$ is a cofibration and an L-equivalence, then all maps

$$i \wedge \partial\Delta^n_+ : A \wedge \partial\Delta^n_+ \to B \wedge \partial\Delta^n_+$$

are L-equivalences. By replacing by a cofibrant model if necessary, it is enough to assume that

A is cofibrant. Then one uses the usual patching argument for the category of cofibrant objects in the L-model structure for $\mathrm{Spt}_T(\mathcal{C})$ to compare pushouts of the form

$$\begin{array}{ccc} A \wedge \partial\Delta^{n-1}_+ & \longrightarrow & A \wedge \Lambda^n_{k+} \\ \downarrow & & \downarrow \\ A \wedge \Delta^{n-1}_+ & \longrightarrow & A \wedge \partial\Delta^n_+ \end{array}$$

to show inductively that the question reduces to showing that the map

$$i \vee i : A \vee A \to B \vee B$$

is an L-equivalence. But $i \vee i$ has the left lifting property with respect to all L-fibrations, and must therefore be an L-trivial cofibration. □

Lemma 25.13. *The L-structure on $\mathrm{Spt}_T(\mathcal{C})$ is left proper: given a pushout diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \xrightarrow[f_*]{} & D \end{array}$$

in which i is a cofibration, if f is an L-equivalence then f_* is an L-equivalence.

Proof. The original diagram may be replaced up to strict weak equivalence by a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f'} & C' \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \xrightarrow[f'_*]{} & D' \end{array}$$

in which f' is a cofibration and an L-equivalence. But then f'_* is also an L-trivial cofibration and is in particular an L-equivalence. □

Lemma 25.14. *Every injective object is L-fibrant, so that the L-fibrant T-spectra coincide with the injective T-spectra.*

Proof. Suppose that X is injective, and suppose given a diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & & \\ B & & \end{array}$$

where the morphism i is a cofibration and an L-equivalence. Then $\alpha = \alpha' \cdot j$ for some map $\alpha' : LA \to X$ since X is injective, and so there is a diagram

$$\begin{array}{ccccc} A & \xrightarrow{j} & LA & \xrightarrow{\alpha'} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle Li} & & \\ B & \xrightarrow[j]{} & LB & & \end{array}$$

which factorizes the original. The map Li is a strict equivalence by Corollary 25.7.

One finishes the argument in the usual way: Li has a factorization

$$\begin{array}{ccc} LA & \xrightarrow{i'} & W \\ & \searrow^{Li} & \downarrow p \\ & & LB \end{array}$$

where i' is a cofibration, p is a strict fibration and both maps are strict weak equivalences. Then X is strictly fibrant so there is a map $\sigma : W \to X$ such that $\sigma \cdot i' = \alpha'$, and there is a map $\theta : B \to W$ such that $p \cdot \theta = j$ and $\theta \cdot i = i' \cdot j$. $\square$

Now we can go further, to give a general recognition principle for L-fibrations. The most complete statement (Theorem 25.17 below) depends on right properness for the L-structure, which will be addressed in a subsequent section.

Lemma 25.15. *Suppose that $p : X \to Y$ is a strict fibration between L-fibrant T-spectra. Then p is an L-fibration.*

Proof. Suppose given a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array} \tag{25.1}$$

where i is a cofibration and an L-equivalence. Then the induced map $i_* : LA \to LB$ is a strict equivalence, as are the L-fibrant model maps $j : X \to LX$ and $j : Y \to LY$. The induced diagram

$$\begin{array}{ccc} LA & \longrightarrow & LX \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ LB & \longrightarrow & LY \end{array}$$

has a factorization

$$\begin{array}{ccccc} LA & \xrightarrow{j_A} & V_X & \xrightarrow{p_X} & LX \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle p_*} \\ LB & \xrightarrow[j_B]{} & V_Y & \xrightarrow[q_Y]{} & LY \end{array}$$

such that j_A and j_B are strict trivial cofibrations and p_X and p_Y are strict fibrations. In the pullback diagram

$$\begin{array}{ccc} V_X \times_{LX} X & \longrightarrow & X \\ {\scriptstyle j_{X*}}\downarrow & & \downarrow{\scriptstyle j_X} \\ V_X & \xrightarrow[p_X]{} & LX \end{array}$$

the map j_{X*} is a strict equivalence. The corresponding map j_{Y*} in the diagram

$$\begin{array}{ccccc} LA & \xrightarrow{j_A} & V_X & \xleftarrow{j_{X*}} & V_X \times_{LX} X \\ \downarrow & & \downarrow & & \downarrow \\ LB & \xrightarrow[j_B]{} & V_Y & \xleftarrow[j_{Y*}]{} & V_Y \times_{LY} Y \end{array}$$

is also a strict equivalence. It follows that the induced map

$$V_X \times_{LX} X \to V_Y \times_{LY} Y$$

is a strict equivalence, and that the diagram (25.1) has a factorization

$$\begin{array}{ccccc} A & \longrightarrow & V_X \times_{LX} X & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & V_Y \times_{LY} Y & \longrightarrow & Y \end{array}$$

in which the middle vertical map is a strict equivalence. The result follows by a standard argument: one factorizes the middle vertical map as a trivial strict cofibration followed by a trivial strict fibration. □

Proposition 25.16. *Suppose that $p : X \to Y$ is a strict fibration. Then p is an L-fibration if the diagram*

$$\begin{array}{ccc} X & \xrightarrow{i} & LX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Lp} \\ Y & \xrightarrow[i]{} & LY \end{array} \tag{25.2}$$

is strictly homotopy cartesian.

Proof. Suppose that the diagram (25.2) is strictly homotopy cartesian. There is a factorization

$$\begin{array}{ccc} LX & \xrightarrow{j} & Z \\ & \searrow^{Lp} & \downarrow q \\ & & LY \end{array}$$

of Lp such that j is a stable equivalence and q is an injective fibration. But then Z is injective, hence L-fibrant, so that j is a strict equivalence. It also follows from Lemma 25.15 that q is an L-fibration. By pulling back q along i, we see from the hypothesis that the induced map

$$X \to Y \times_{LY} Z$$

is a strict equivalence. Every trivial strict fibration is an L-fibration, and it follows that p is a retract of an L-fibration, and hence is itself an L-fibration.

□

Theorem 25.17. *Suppose that the L-structure of Theorem 25.12 is right proper. Suppose that $p : X \to Y$ is a strict fibration. Then p is an L-fibration if and only if the diagram*

$$\begin{array}{ccc} X & \xrightarrow{i} & LX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Lp} \\ Y & \xrightarrow[i]{} & LY \end{array} \qquad (25.3)$$

is strictly homotopy cartesian.

Proof. We already have Proposition 25.16.

Suppose that the map $p : X \to Y$ is a stable fibration, and take a factorization

$$\begin{array}{ccc} LX & \xrightarrow{j} & Z \\ & \searrow^{Lp} & \downarrow q \\ & & LY \end{array}$$

of the map Lp such that q is a stable fibration and j is a stably trivial cofibration. Then j is an L-equivalence between L-fibrant T-spectra, so that j is a strict equivalence on account of Lemma 25.11.

The induced map $i_* : Y \times_{LY} Z \to Z$ is an L-equivalence by the right properness assumption, so that the canonical map $\theta : X \to Y \times_{LY} Z$ is a stable equivalence, and the map

$$\begin{array}{ccc} X & \xrightarrow{\theta} & Y \times_{LY} Z \\ & {}_{p}\searrow \quad \swarrow_{q_*} & \\ & Y & \end{array}$$

is an equivalence of fibrant objects for the model structure on $\mathrm{Spt}_T(\mathcal{C})/Y$ which is induced by the

L-structure on $\mathrm{Spt}_T(\mathcal{C})$. Form the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\tilde{\theta}} & V_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ X & \xrightarrow{\theta} & Y \times_{LY} Z \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle q_*} & \\ & Y & \end{array}$$

where π_1 and π_2 are trivial strict fibrations and V_1 and V_2 are cofibrant. Then $\tilde{\theta}$ is a weak equivalence between objects of $\mathrm{Spt}_T(\mathcal{C})/T$ which are both fibrant and cofibrant, and is therefore a (fibrewise) homotopy equivalence, and hence a strict weak equivalence. □

References

[1] J.F. Jardine. *Local Homotopy Theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2015.