**Lecture 14** (April 6, 2009)

## 27 Stable homotopy theory of $T$-spectra

Here is how the basic applications of the localization result Theorem 25.12 arise.

In what follows, $\alpha$ is an infinite cardinal which is greater than the cardinality of the set of morphisms of the underlying site $\mathcal{C}$.

Write $J$ for the set of maps

$$\Sigma_T^\infty C[-n] \to \Sigma_T^\infty D[-n],\ n \geq 0, \tag{27.1}$$

which are induced by a set of cofibrations containing the $\alpha$-bounded trivial cofibrations $C \to D$ of pointed simplicial presheaves.

Suppose that the set $S$ includes the set $J$ together with the set of cofibrant replacements of the maps

$$\Sigma_T^\infty T[-1-n] \to S_T[-n]. \tag{27.2}$$

Finally, suppose that $S$ satisfies the closure property that $S$ includes all cofibrations

$$(A \wedge D) \cup (B \wedge C) \to B \wedge D$$

induced by all $A \to B$ in $S$ and all $\alpha$-bounded pointed cofibrations $C \to D$ of simplicial presheaves.

In general, such a set of cofibrations $S$ satisfies the list of Basic Assumptions of Lecture 12, and the model structure on $T$-spectra given by Theorem 25.12 is a *stable* model structure for some localization of the category of pointed simplicial presheaves. The cofibrations are the cofibrations of $T$-spectra, the weak equivalences are the *stable equivalences* (or $L$-equivalences) and the fibrations are the *stable fibrations* (or $L$-fibrations).

In general, the word "stable" means that we are formally inverting the maps (27.2).

For example, the maps in $J$ could be determined by a generating set of trivial cofibrations for some $f$-local theory for some cofibration $f : A \to B$, as in Theorem 22.2, in which case Theorem 25.12 produces the $f$-local stable theory.

If $J$ consists of nothing but the maps induced by the $\alpha$-bounded trivial cofibrations of pointed simplicial presheaves, we are producing a "bare" stable model structure on $\mathrm{Spt}_T(\mathcal{C})$.

The examples to keep in mind are the $T$-spectrum objects $\mathrm{Spt}_T(Sm|_S)_{Nis}$ on the smooth Nisnevich site for a Noetherian scheme $S$ of finite dimension, where $T$ is either the simplicial circle $S^1$ or the

Tate object $S^1 \wedge \mathbb{G}_m$, and $f : * \to \mathbb{A}^1$ is a section of the affine line over $S$. All choices of such maps $f$ give the same theory, but $f$ is usually chosen to be the 0-section.

• The "bare" theory for $T = S^1$ is the stable structure for presheaves of spectra on $(Sm|_S)_{Nis}$ that we've already discussed.

• The stable $f$-local structure on $\mathrm{Spt}_T(Sm|_S)_{Nis}$ for $T = S^1 \wedge \mathbb{G}_m$ is the Morel-Voevodsky motivic stable model structure [2]. The stable $f$-local structure on the ordinary category $\mathrm{Spt}_{S^1}(Sm|_S)_{Nis}$ (aka. motivic $S^1$-spectra) is also important, in that it is a technical device fore analyzing the former. By the same techniques, there is a motivic stable model structure for $T$-spectrum objects, for *all* pointed simplicial presheaves $T$ (compare with [1]), in particular for $\mathbb{G}_m$-spectra.

Here's the first thing that's special about these stable homotopy theories:

**Lemma 27.1.** *A map $p : X \to Y$ is an injective fibration if and only if $p$ is a strict fibration and all diagrams of pointed simplicial presheaf* *maps*

$$\begin{array}{ccc} X^n & \xrightarrow{\sigma_*} & \Omega_T X^{n+1} \\ \downarrow & & \downarrow \\ Y^n & \xrightarrow[\sigma_*]{} & \Omega_T Y^{n+1} \end{array}$$

*are homotopy cartesian.*

The notation $\Omega_T Y$ is shorthand for the pointed *internal function complex* $\mathbf{Hom}_*(T,Y)$, which is the simplicial presheaf defined by

$$\mathbf{Hom}_*(T,Y)(U) = \mathbf{hom}_*(T|_U, Y|_U)$$

for $U \in \mathcal{C}$. This is the internal pointed function complex, also defined by the adjunction isomorphism

$$\hom(A \wedge T, Y) \cong \hom(A, \mathbf{Hom}_*(T,Y)).$$

**Examples**: 1) If $K$ is a pointed simplicial set, then

$$\mathbf{Hom}_*(\Gamma^* K, Y)(U) = \mathbf{hom}_*(K, Y(U))$$

by adjointness.

2) If $V$ is an $S$-scheme, then

$$\mathbf{Hom}(V_+, Y)(U) \cong Y(V \times U)$$

on $Sch|_S$ for all $S$-schemes $U$.

Lemma 27.1 is a generalization of Lemma 24.2, and has the same proof. It follows from the fact that the map $p$ is an injective fibration if and only if $p$ induces trivial fibrations of simplicial presheaves

$$\mathbf{Hom}_*(D,X) \to \mathbf{Hom}_*(C,X) \times_{\mathbf{Hom}_*(C,Y)} \mathbf{Hom}_*(C,X)$$

for all generators $C \to D$ of the set $S$. Apply this criterion to the two classes of generators for $S$.

**Corollary 27.2.** *A $T$-spectrum $X$ is stably fibrant if and only if all simplicial presheaves $X^n$ are injective fibrant and all maps $X^n \to \Omega_T X^{n+1}$ are local weak equivalences.*

To go further, we need to make assumptions on the suspending object $T$.

Say that $T$ is *compact up to equivalence* if for any filtered diagram $i \mapsto X_i$ of fibrant pointed simplical presheaves the map

$$\varinjlim_i \Omega_T X_i \to \Omega_T F(\varinjlim_i X_i)$$

is a weak equivalence, where $j : Y \to F(Y)$ is a natural choice of fibrant model for $Y$.

**Examples**

1) Suppose that $K$ is a pointed finite simplicial set. Then $K$ is compact up to equivalence for the

injective model structure on $s\,\mathrm{Pre}(\mathcal{C})$. In effect, the functor $\Omega_K$ commutes with filtered colimits, so one only has to show that $\Omega_K$ preserves local weak equivalences between presheaves of Kan complexes. But it's easy to see that $\Omega_K$ preserves local trivial fibrations, so the standard trick does the job.

2) Suppose that $S$ is a decent scheme (ie. Noetherian and of finite dimension) and that the category $Sm|_S$ has the Nisnevich topology.

The pointed simplicial presheaf $V_+$ on $(Sm|_S)_{Nis}$ associated to a smooth $S$-scheme $V$ is compact up to equivalence. All $Y_i$ take distinguished squares to homotopy cartesian diagrams, so that $\varinjlim_i Y_i$ also has this property. The Nisnevich descent theorem (Theorem 26.10) implies that the injective fibrant model

$$j : \varinjlim_i X_i \to F(\varinjlim_i X_i)$$

is a weak equivalence in each section, so that all maps

$$\varinjlim_i X_i(V \times U) \to F(\varinjlim_i X_i)(V \times U)$$

are weak equivalences. In particular, the map

$$\varinjlim_i \Omega_{V_+} X_i \to \Omega_{V_+} F(\varinjlim_i X_i)$$

is a sectionwise weak equivalence.

If the smooth $S$-scheme $U$ has a base point $* \to U$, then there is a natural sectionwise fibre sequence

$$\Omega_U X \to \Omega_{U_+} X \to X$$

for presheaves of Kan complexes $X$, and the comparison of fibre sequences

$$\begin{array}{ccccc}
\varinjlim_i \Omega_U X_i & \longrightarrow & \varinjlim_i \Omega_{U_+} X_i & \longrightarrow & \varinjlim_i X_i \\
\downarrow & & \downarrow \simeq & & \downarrow \simeq \\
\Omega_U F(\varinjlim_i X_i) & \longrightarrow & \Omega_{U_+} F(\varinjlim_i X_i) & \longrightarrow & F(\varinjlim_i X_i)
\end{array}$$

in which the indicated maps are sectionwise equivalences shows that the map

$$\varinjlim_i \Omega_U X_i \to \Omega_U F(\lim_i X_i)$$

is a sectionwise weak equivalence.

3) The same arguments imply that all finite simplicial sets $K$ and all schemes $V$ are compact up to equivalence in the motivic model structure on $s\operatorname{Pre}(Sm|_S)_*$.

In particular, if $i \mapsto X_i$ is a filtered diagram of motivic fibrant objects, then $\varinjlim_i X_i$ satisfied motivic descent by Theorem 26.14, and so the motivic fibrant model

$$\varinjlim_i X_i \to F(\varinjlim_i X_i)$$

is a sectionwise equivalence. The functor $\Omega_K$ commutes with filtered colimits and preserves sectionwise equivalences.

This last is a *very* tricky point, because it is not known, for example, that the ordinary loop space functor $\Omega = \Omega_{S^1}$ preserves motivic weak equivalences between presheaves of Kan complexes.

4) If $T$ and $T'$ are compact up to equivalence, then $T \wedge T'$ is compact up to equivalence.

It follows that the Tate object $S^1 \wedge \mathbb{G}_m$ is compact up to equivalence for both the Nisnevich local and motivic model structures on $Sch|_S$.

Starting with a $T$-spectrum $X$, define functors $X \mapsto Q^k X$ for $k \geq 0$ by specifying that

$$Q^0 X = FX$$

and

$$Q^{k+1} X = \Omega_T Q^k X[1]$$

(fake $T$-loops). The map $Q^k X \to Q^{k+1} X$ is the canonical map

$$Q^k X \xrightarrow{\sigma_*} \Omega_T Q^k X[1].$$

Set

$$QX = F(\varinjlim_k Q^k X).$$

Write $\eta : X \to QX$ for the natural composite

$$X \to FX = Q^0 X \to \varinjlim_k Q^k X \to F(\varinjlim_k Q^k X) = QX.$$

**Lemma 27.3.** *Suppose that $T$ is compact up to equivalence. Then $QX$ is stably fibrant.*

*Proof.* All objects $Q^k X$ are strictly fibrant. In the diagram

$$\begin{array}{ccccc}
\varinjlim_k Q^k X & & \xrightarrow{\quad j \quad} & & F(\varinjlim_k Q^k X) \\
\cong \downarrow & \searrow^{\sigma_*} & & & \downarrow \sigma_* \\
\varinjlim_k \Omega_T Q^k X[1] & \longrightarrow & \Omega_T(\varinjlim_k Q^k X)[1] & \longrightarrow & \Omega_T F(\varinjlim_k Q^k X)[1]
\end{array}$$

the indicated vertical map is an isomorphism by cofinality, and the bottom horizontal composite is a strict equivalence since $T$ is compact up to equivalence. It follows that the vertical map $\sigma_*$ is a strict equivalence. □

Now here's the theorem:

**Theorem 27.4.** *Suppose that $T$ is compact up to equivalence. Then $QX$ is stably fibrant, and the map $\eta : X \to QX$ is a stable equivalence.*

*Proof.* $QX$ is stably fibrant. In the diagram

$$\begin{array}{ccc} X & \longrightarrow & QX \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j_*} \\ LX & \longrightarrow & QLX \end{array} \tag{27.3}$$

the map $j_*$ is a strict equivalence because all pushouts $C \to D$ of generators $A \to B$ of $S$ (remember $S$?) induces strict equivalences $QC \to QD$ (since each map $C \to D$ is an equivalence above a certain level). The map $\eta : LX \to QLX$ is a strict equivalence because $LX$ is stably fibrant. It follows that $\eta : X \to QX$ is a stable equivalence. □

The *underlying model structure* on $s\,\mathrm{Pre}(\mathcal{C})$ is defined by formally inverting the set of cofibrations $C \to D$ in $s\,\mathrm{Pre}(\mathcal{C})$ which defines the set $J$ — see (27.1).

The examples to keep in mind are those theories for which the underlying model structure is the $f$-local theory, where $f : * \to A$ is a rational point of some simplicial presheaf $A$ of $s\,\mathrm{Pre}(\mathcal{C})$, and the motivic model structure on the smooth Nisnevich site $(Sm|_S)_{Nis}$ of a decent scheme $S$. Theorem 22.14 says that the underlying model structure on simplicial presheaves is proper in all such cases.

**Theorem 27.5.** *Suppose that $T$ is compact up to equivalence. Suppose that the underlying model structure on $s\,\mathrm{Pre}(\mathcal{C})$ is proper. Then the stable model structure on the category of $T$-spectra is proper.*

*Proof.* Suppose given a pullback diagram

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{f_*} & X \\ \downarrow & & \downarrow p \\ Z & \xrightarrow[f]{} & Y \end{array}$$

of $T$-spectra such that $p$ is a strict fibration and $f$ is a stable equivalence. The induced diagram

$$\begin{array}{ccc} Q(Z \times_Y X) & \longrightarrow & QX \\ \downarrow & & \downarrow \\ QZ & \longrightarrow & QY \end{array}$$

is strictly homotopy cartesian, and the map $QZ \to QY$ is a strict equivalence. The properness of the underlying model structure implies that the map $Q(Z \times_Y X) \to QX$ is a strict equivalence. □

Here's the recognition principle for stable fibrations. It is a generalization Theorem 24.1, and a corollary of Theorem 25.17 and Theorem 27.5.

**Theorem 27.6.** *Suppose that $T$ is compact up to equivalence and that the underlying model* *structure on* $s\,\mathrm{Pre}(\mathcal{C})$ *is proper. Suppose that* $p : X \to Y$ *is a strict fibration. Then* $p$ *is a stable fibration if and only if the diagram*

$$\begin{array}{ccc} X & \xrightarrow{j} & LX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Lp} \\ Y & \xrightarrow[j]{} & LY \end{array} \tag{27.4}$$

*is strictly homotopy cartesian.*

It is a basic property of ordinary stable homotopy theory that the suspension and loop are inverse to each other in that they form a Quillen equivalence

$$\Sigma : \mathrm{Spt} \leftrightarrows \mathrm{Spt} : \Omega.$$

There are various proofs of this in the literature, but secretly it depends on the fact that the cyclic permutation $(3, 2, 1)$ induces a map

$$(3, 2, 1) : S^3 \to S^3$$

(3-fold smash products) by permuting smash factors, and that this map is pointed homotopic to the identity. The latter is so because the map has degree 0.

Here's the general result:

**Theorem 27.7.** *Suppose that* $T$ *is compact up to equivalence. Suppose that* $X$ *is a* $T$*-spectrum,* *and let* $j : X \wedge T \to L(X \wedge T)$ *be the natural stable fibrant model for* $X \wedge T$*. Suppose that the map*

$$(3,2,1) : T^3 \to T^3$$

*represents the identity in the (*$f$*-local) pointed homotopy category. Then the composite*

$$X \xrightarrow{\eta} \Omega_T(X \wedge T) \xrightarrow{\Omega_T j} \Omega_T L(X \wedge T)$$

*is a stable equivalence.*

In the statement of the Theorem, the functor $X \mapsto X \wedge T$ is the normal (not fake) suspension, and $Y \mapsto \Omega_T Y$ is its right adjoint (the "real" $T$-loops functor) and $\eta$ is the unit of the adjunction.

A layer filtration argument reduces the proof of Theorem 27.7 (see also [1, Theorem 3.11]) to the case $X = \Sigma_T^\infty K$ for a pointed simplicial presheaf $K$.

The $n^{th}$ *layer* $L_n X$ of a $T$-spectrum $X$ is the $T$-spectrum

$$X^0, \dots, X^n, T \wedge X^n, T^2 \wedge X^n, \dots$$

There are natural maps

$$\Sigma_T^\infty X^0 = L_0 X \to L_1 X \to \cdots \to X$$

and an induced isomorphism

$$\varinjlim_n L_n X \cong X.$$

There is also a natural pushout diagram

$$\begin{array}{ccc}
\Sigma_T^\infty(T \wedge X^n)[-(n+1)] & \longrightarrow & L_n X \\
\downarrow & & \downarrow \\
\Sigma_T^\infty X^{n+1}[-(n+1)] & \xrightarrow[\simeq]{} & L_{n+1} X
\end{array}$$

in which the indicated map is a stable equivalence.

One uses the fact that a filtered colimit of stable equivalences is a stable equivalence (this is an exercise, which can be interesting), together with the fact that the functor $X \mapsto \Omega_T L(X \wedge T)$ preserves stable equivalences and respects shifts to reduce to the suspension $T$-spectrum case.

On account of Theorem 27.4, the homotopy groups $\pi_r LY^n$ for a $T$-spectrum $Y$ are computed in global sections as the filtered colimit

$$[S^r, Y^n] \xrightarrow{\Sigma} [T \wedge S^r, Y^{n+1}] \xrightarrow{\Sigma} \dots,$$

where $\Sigma$ takes a map $\theta : T^k \wedge S^r \to Y^{n+k}$ to the composite

$$T \wedge T^k \wedge S^r \xrightarrow{T \wedge \theta} T \wedge Y^{n+k} \xrightarrow{\sigma} Y^{n+k+1}.$$

When $Y = \Sigma_T^\infty K$, the map $\Sigma$ is smashing with $T$ on the left. The composite

$$[T^k \wedge S^r, Y^{n+k}] \to [T^k \wedge S^r, \Omega_T L(Y \wedge T)^{n+k}] \cong [T^k \wedge S^r \wedge T, Y^{n+k} \wedge T]$$

is smashing with $T$ on the right. There is a commutative diagram

$$\begin{array}{ccccc}
[T^k \wedge S^r, T^{n+k} \wedge K] & \xrightarrow{T^2\wedge} & [T^{2+k} \wedge S^r, T^{2+n+k} \wedge K] & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \wedge T} & & \downarrow{\scriptstyle \wedge T} & & \\
[T^k \wedge S^r \wedge T, T^{n+k} \wedge K \wedge T] & \xrightarrow{T^2\wedge} & [T^{2+k} \wedge S^r \wedge T, T^{2+n+k} \wedge K \wedge T] & \longrightarrow & \cdots \\
{\scriptstyle c_t}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle c_t} & & \\
[T^{1+k} \wedge S^r, T^{1+n+k} \wedge K] & \xrightarrow[T^2\wedge]{} & [T^{3+k} \wedge S^r, T^{3+n+k} \wedge K] & \longrightarrow & \cdots
\end{array}$$

Here, $c_t$ is a conjugation isomorphism (by twisting smash factors), and the bottom square commutes by the hypotheses on $T^3$, and the vertical composites are instances of $T\wedge$. The proof of Theorem 27.7 is finished with a cofinality argument.

**Corollary 27.8.** *Under the hypotheses of Theorem 27.7, if $Y$ is stably fibrant, then the canonical (evaluation) map $\epsilon : \Omega_T Y \wedge T \to Y$ is a stable equivalence.*

*Proof.* Let $j : \Omega_T Y \wedge T \to L(\Omega_T Y \wedge T)$ be a stably fibrant model, and extend $\epsilon$ to a map $\epsilon_* :$

$L(\Omega_T Y \wedge T) \to Y$. Form the diagram

$$\begin{array}{ccccc} \Omega_T Y & \xrightarrow{\eta} & \Omega_T(\Omega_T Y \wedge T) & \xrightarrow{\Omega_T j} & \Omega_T L(\Omega_T Y \wedge T) \\ & {\scriptstyle 1} \searrow & \downarrow {\scriptstyle \Omega_T \epsilon} & \swarrow {\scriptstyle \Omega_T \epsilon_*} & \\ & & \Omega_T Y & & \end{array}$$

Then $\Omega_T \epsilon_*$ is a stable equivalence by Theorem 27.7, so that $\epsilon_*$ is a stable equivalence by a calculation. In effect, $\Omega_T \epsilon_*$ is a stable hence strict equivalence of stably fibrant objects, and the commutative diagrams

$$\begin{array}{ccc} L(\Omega_T Y \wedge T)^n & \xrightarrow{\simeq} & \Omega_T L(\Omega_T \wedge T)^{n+1} \\ {\scriptstyle \epsilon_*} \downarrow & & \downarrow {\scriptstyle \Omega_T \epsilon_*} \\ Y^n & \xrightarrow[\simeq]{} & \Omega_T Y^{n+1} \end{array}$$

force $\epsilon_*$ to be a strict equivalence. □

The Tate object $T$ in motivic homotopy theory is the most prominent non-standard example of an object which satisfies the conditions of Theorem 27.7.

**Lemma 27.9** (Voevodsky)**.** *The cyclic permutation* $(3, 2, 1) \in \Sigma_3$ *acts as the identity on* $T^3$ *in the pointed motivic homotopy category, where* $T = S^1 \wedge \mathbb{G}_m$.

*Proof.* There is an identification

$$T^3 \simeq \mathbb{A}^3/(\mathbb{A}^3 - 0).$$

and the action of $\Sigma_3$ is the restriction of a pointed algebraic group action

$$Gl_3 \times T^3 \to T^3.$$

The permutation matrix $(3, 2, 1)$ is a product of elementary transformation matrices in $Gl_3(\mathbb{Z})$, and so there is an algebraic path

$$\omega : \mathbb{A}^1 \to Gl_3$$

from the identity matrix to $(3, 2, 1)$. The composite

$$\mathbb{A}^1 \times T^3 \to Gl_3 \times T^3 \to T^3$$

gives a pointed homotopy from $(3, 2, 1) : T^3 \to T^3$ to the identity. $\square$

## 28 $(S^1 \wedge K)$-spectra

Suppose that the pointed simplicial presheaf $K$ is compact up to equivalence. The class of pointed simplicial presheaves which are compact up to equivalence is closed under finite smash products and includes all finite pointed simplicial sets. It follows that $S^1 \wedge K$ is compact up to equivalence, so all results of the previous section apply to $(S^1 \wedge K)$-spectra.

I shall assume that $K$ is compact up to equivalence throughout this section. I shall also assume that the underlying model structure on $s\operatorname{Pre}(\mathcal{C})$ is proper.

In what follows it's best to think of the bonding maps for an $(S^1 \wedge K)$-spectrum $X$ as maps of the form

$$\sigma : S^1 \wedge X^n \wedge K \to X^{n+1}.$$

These morphisms induce maps

$$\begin{aligned} \sigma_* : & S^k \wedge X^n \wedge K^r = \\ & S^{k-1} \wedge S^1 \wedge X^n \wedge K \wedge K^{r-1} \xrightarrow{S^{k-1} \wedge \sigma \wedge K^{r-1}} S^{k-1} \wedge X^{n+1} \wedge K^{r-1} \end{aligned}$$

in the obvious way.

An $(S^1 \wedge K)$-spectrum $X$ determines a $K$-spectrum object $X^{*,*}$ in spectra, which at $K$-level $n$ is the spectrum

$$\begin{aligned} X^{n,*} :\ & X^0 \wedge K^n, X^1 \wedge K^{n-1}, \dots, \\ & X^{n-1} \wedge K, X^n, S^1 \wedge X^n, S^2 \wedge X^n, \dots \end{aligned}$$

The bonding maps for $X^{n,*}$ are the maps

$$\sigma_* : S^1 \wedge X^j \wedge K^{n-j} \to X^{j+1} \wedge K^{n-j-1}$$

up to level $n-1$, and are identities defined by smashing with $S^1$ beyond. The $K$-bonding maps

$$X^{n,*} \wedge K \to X^{n+1,*}$$

are identity maps defined by smashing with $K$ up to level $n$ and are instances of $\sigma_*$ in levels $n+1$ and above. The fact that one actually does get a map of $S^1$-spectra this way is essentially a consequence of the fact that the morphisms $\sigma_*$ respect smashing with $S^1$ and $K$.

An $(S^1 \wedge K)$-spectrum $X$ has bigraded presheaves of stable homotopy groups $\pi_{s,t}X$, defined by

$$\pi_{s,t}X(U) = \varinjlim_{n \geq 0} [S^{n+s} \wedge K^{n+t}|_U, X^n|_U]$$

where the homotopy classes of maps are computed for pointed simplicial presheaves on $\mathcal{C}/U = Sch|_U$, and the transition maps are defined by suspension in the "obvious" way: a representing map

$$\alpha : S^{n+s} \wedge K^{n+t} \to X^n$$

is sent to the composite

$$S^{n+1+s} \wedge K^{n+1+t} \xrightarrow{S^1 \wedge \alpha \wedge K} S^1 \wedge X^n \wedge K \xrightarrow{\sigma} X^{n+1}.$$

These stable homotopy group presheaves are specializations of bigraded stable homotopy groups $\pi_{s,t}Y$ which are defined for $K$-spectrum objects $Y$ in spectra by

$$\pi_{s,t}Y(U) = \varinjlim_{k,l} [S^{k+s} \wedge K^{l+t}, Y^{k,l}]_U,$$

where the notation indicates that the homotopy classes are computed over $U \in \mathcal{C}$. This means that there are natural isomorphisms

$$\pi_{s,t}Y \cong \pi_{s,t}dY$$

of presheaves for all $K$-spectrum objects $Y$, where $dY$ is the $(S^1 \wedge K)$-spectrum with $dY^n = Y^{n,n}$. It follows that there are natural isomorphisms

$$\pi_{s,t}X^{*,*} \cong \pi_{s,t}X$$

for all $(S^1 \wedge K)$-spectra $X$.

The bonding maps $Y^n \wedge K \to Y^{n+1}$ in a $K$-spectrum object $Y$ induce homomorphisms of stable homotopy classes of maps

$$\begin{aligned} &[S[s] \wedge K^{n+t}, Y^n]_U \to [S[s] \wedge K^{n+t+1}, Y^{n+1}]_U \\ &\to [S[s] \wedge K^{n+t+2}, Y^{n+2}]_U \to \dots \end{aligned}$$

for all $U \in \mathcal{C}$, and the filtered colimit of the system is $\pi_{s,t}Y(U)$.

Note that there are isomorphisms of presheaves

$$\pi_k QX^n(U) \cong \pi_{k-n,-n}X(U).$$

Then we have the following:

**Lemma 28.1.** *A map $f : X \to Y$ is a stable equivalence of $(S^1 \wedge K)$-spectra if and only if it* *induces isomorphisms of presheaves*

$$\pi_{s,t}X \cong \pi_{s,t}Y$$

*for all integers $s$ and $t$.*

Suppose that

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a strict fibre sequence of $(S^1 \wedge K)$-spectra. Every map $f : Z \to W$ of $K$-spectrum objects has a factorization

$$\begin{array}{ccc} Z & \xrightarrow{j} & V \\ & \searrow^{f} & \downarrow{q} \\ & & W \end{array}$$

where $q$ is a strict fibration in each $K$-level and $j$ is a cofibration and a strict weak equivalence in each $K$-level. Take such a factorization

$$\begin{array}{ccc} X^{*,*} & \xrightarrow{j} & V \\ & \searrow^{p} & \downarrow{q} \\ & & Y^{*,*} \end{array}$$

for the map induced by the $T$-spectrum map $p : X \to Y$, and let $\overline{F}$ be the fibre of $q$. Then there are induced comparisons of fibre sequences of simplicial presheaves

$$\begin{array}{ccccc} F^n & \xrightarrow{i} & X^n & \xrightarrow{p} & Y^n \\ \downarrow & & \simeq\downarrow & & \downarrow\cong \\ \overline{F}^{n,n} & \longrightarrow & Z^{n,n} & \longrightarrow & Y^{n,n} \end{array} \tag{28.1}$$

for each $n \geq 0$, and it follows (by properness for pointed simplicial presheaves) that the induced map $F \to d\overline{F}$ is a strict weak equivalence.

Suppose that

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a level strict fibre sequence of $K$-spectrum objects and that $Y$ is strictly fibrant in all $K$-levels. Then all induced sequences

$$\Omega_K^{t+n} F^n \to \Omega_K^{t+n} X^n \to \Omega_K^{t+n} Y^n$$

are strict fibre sequences of presheaves of spectra, and all spectra $\Omega_K^{t+n} Y^n$ are strictly fibrant. It follows that there is a long exact sequence in presheaves of stable homotopy groups of the form

$$\begin{aligned} \dots \to \pi_s \Omega_K^{t+n} F^n \to \pi_s \Omega_K^{t+n} X^n \to \pi_s \Omega_K^{t+n} Y^n \\ \xrightarrow{\partial} \pi_{s-1} \Omega_K^{t+n} F^n \to \dots \end{aligned}$$

There are, as well, comparisons of fibre sequences

$$\begin{array}{ccccc} \Omega_K^{t+n} F^n & \longrightarrow & \Omega_K^{t+n} X^n & \longrightarrow & \Omega_K^{t+n} Y^n \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_K^{t+n+1} F^{n+1} & \longrightarrow & \Omega_K^{t+n+1} X^{n+1} & \longrightarrow & \Omega_K^{t+n+1} Y^{n+1} \end{array}$$

induced by the respective $K$-spectrum object structures. Thus taking a filtered colimit in $n$ gives a long exact sequence

$$\dots \to \pi_{s,t}F \xrightarrow{i_*} \pi_{s,t}X \xrightarrow{p_*} \pi_{s,t}Y \xrightarrow{\partial} \pi_{s-1,t}F \to \dots \tag{28.2}$$

in presheaves of bigraded stable homotopy groups. Note that the "degree" $s$ changes while the "weight" $t$ does not.

The number $-s$ is actually the degree and $-t$ is actually the weight. In motivic notation,

$$\pi_{s,t}Y(U) = \mathbb{H}^{-s-t}(U, Y(-t)).$$

It follows from the remarks above (specifically, the existence of diagram (28.1)) that there is a natural long exact sequence of the form (28.2) for any strict fibre sequence

$$F \to X \to Y$$

of $(S^1 \wedge K)$-spectra.

Strict fibre and cofibre sequences coincide up to natural stable equivalence in $(S^1 \wedge K)$-spectra. The proof comes in three parts:

**Lemma 28.2.** *Suppose that $p : X \to Y$ is a strict fibration of $(S^1 \wedge K)$-spectra, with fibre $F$. Then the canonical map*

$$X/F \to Y$$

*is a stable equivalence.*

*Proof.* The Lemma follows from the corresponding result for presheaves of spectra, by replacing the given fibre sequence by a fibre sequence of $K$-spectrum objects in spectra. □

**Lemma 28.3.** *Suppose that*

$$\begin{array}{ccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 \end{array}$$

*is a comparison of level cofibre sequences of $(S^1 \wedge K)$-spectra. If any two of the maps $f_1, f_2$ and $f_3$ are stable equivalences, then so is the third.*

*Proof.* It suffices to assume that all objects are cofibrant.

The comparison diagram in the statement induces a comparison of fibre sequences

$$\begin{array}{ccccc} \mathbf{hom}(B_3, Z) & \longrightarrow & \mathbf{hom}(B_2, Z) & \longrightarrow & \mathbf{hom}(B_1, Z) \\ {\scriptstyle f_3^*}\downarrow & & \downarrow{\scriptstyle f_2^*} & & \downarrow{\scriptstyle f_1^*} \\ \mathbf{hom}(A_3, Z) & \longrightarrow & \mathbf{hom}(A_2, Z) & \longrightarrow & \mathbf{hom}(A_1, Z) \end{array}$$

for all stably fibrant objects $Z$. There are stable equivalences

$$d(\Sigma_K B^{*,*}[-1] \wedge S^1 \simeq \Sigma_T B[-1] \simeq B$$

(fake suspensions) so that the comparison of fibre sequences is a comparison of fibre sequences of infinite loop spaces. Thus if any two of the vertical maps are (stable) equivalences, then so is the third. $\square$

**Lemma 28.4.** *Suppose that $i : A \to B$ is a level cofibration of $(S^1 \wedge T)$-spectra, and take a factorization*

$$\begin{array}{ccc} B & \xrightarrow{j} & Z \\ & \searrow^{\pi} & \downarrow{\scriptstyle p} \\ & & B/A \end{array}$$

*of the quotient map $\pi : B \to B/A$, where $j$ is a strict trivial cofibration and $p$ is a fibration. Let $F$ be the fibre of $p$. Then the induced map $A \to F$ is a stable equivalence.*

*Proof.* The canonical map $p_* : Z/F \to B/A$ associated to the fibration $p : Z \to B/A$ is a stable equivalence by Lemma 28.2. There is also a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & B/A \\ \downarrow & & \simeq \downarrow {\scriptstyle j} & & \downarrow {\scriptstyle j_*} \\ F & \longrightarrow & Z & \longrightarrow & Z/F \end{array}$$

But $p_* j_* = 1$ so that $j_*$ is a stable equivalence. It follows that the map $A \to F$ in the diagram (which is the map of interest) is a stable equivalence, by Lemma 28.3. □

**Corollary 28.5.** *Every level cofibre sequence*

$$A \to B \to B/A$$

*has a naturally associated long exact sequence*

$$\dots \pi_{s,t}A \to \pi_{s,t}B \to \pi_{s,t}(B/A) \xrightarrow{\partial} \pi_{s-1,t}A \to \dots.$$

*of presheaves of stable homotopy groups.*

**Corollary 28.6.** *There are natural isomorphisms*

$$\pi_{s+1,t}(Y \wedge S^1) \cong \pi_{s,t}Y$$

*for all* $(S^1 \wedge K)$*-spectra* $Y$*.*

**Corollary 28.7** (additivity)**.** *Suppose that* $X$ *and* $Y$ *are* $(S^1 \wedge K)$*-spectra. Then the canonical map*

$$c : X \vee Y \to X \times Y$$

*is a stable equivalence.*

## References


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