The UMAP algorithm, reimagined

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UMAP: "Uniform Manifold Approximation and Projection for Dimension Reduction"

Healy-McInnes, 2018 [4],
Healy-McInnes-Melville, 2020 [5]

Outline

- Fuzzy sets, Fuzzy sets in $[0,\infty]$
- ep-metric spaces, coefficients in $[0,\infty]$
- TDA in $[0,\infty]$
- UMAP complex, excision
- Dimension reduction
- Optimisation: cross entropy

$[0,1]$

$[0,1]$ is a locale.

A **locale** L is a poset ($\leq$) with infinite joins (least upper bounds) and finite meets (greatest lower bounds), in which finite meets distribute over all joins.

Facts: L has a terminal object (empty meet), an initial object (empty join), and infinite meets ($\vee_{x \,\leq\, \text{all}\ a_i}\, x$).

Examples: 1) $[a,b]$ and $[a,b]^{op}$ are locales, so $[0,1]^{op}$ is a locale.

2) $[0,\infty]$ and $[0,\infty]^{op}$ are locales.

3) op_X (open subsets of a space X) is a locale.

Locales are topos-theoretic abstractions of topological spaces.

4) $[0,a) \leftrightarrows a$ defines an isomorphism of poset of half-open intervals in $[0,1]$ with elements of $[0,1]$.

Fuzzy sets

Classical definition: A **fuzzy set** is a function $p : X \to [0,1]$,

aka. a "fuzzy subset" of a universal set X.

Barr, 1974 [1]: A **fuzzy set** with coefficients in a locale L is a function $\phi : X \to L$.

A **morphism** $\phi \to \psi$ of fuzzy sets is a function $f : X \to Y$ such that in the picture

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & L & \end{tikzcd}$$

$\phi(x) \leq \psi(f(x))$ for all $x \in X$. i.e. the diagram homotopy commutes.

There is a category Fuzz(L) of fuzzy sets with coefficients in L.

Cases to care about: $L = [0,\infty], [0,\infty]^{op}, [0,1], [0,1]^{op}$.

Fuzzy sets and sheaves (Barr)

Every fuzzy set $\phi : X \to [0, \infty]$ defines a system of subsets $\phi^{-1}[0, a]$ for $a \in [0, \infty]$.

NB: If $a \leq b$ then $\phi^{-1}[0, a] \to \phi^{-1}[a, b]$ is an **inclusion**.

The assignment

$$T(\phi)(a) = \begin{cases} \phi^{-1}[0, a] & a \in [0, \infty] \\ * & a = + \end{cases}$$

defines a **sheaf** $T(\phi)$ on $[0, \infty]^{op}_{+}$, where $[0, \infty]^{op}_{+} = [0, \infty]^{op} \sqcup \{+\}$ and $+$ is new initial object (or new point at infinity for $[0, \infty]$).

Theorem 1 (Barr).

The functor $\phi \mapsto T(\phi)$ determines an equivalence of categories

$$\mathrm{Fuzz}([0, \infty]^{op}) \simeq \mathrm{Mon}([0, \infty]^{op}_{+})$$

with sheaves of monomorphisms on the locale $[0, \infty]^{op}_{+}$.

Classical fuzzy sets

Barr's theorem: there is an equivalence of categories

$$\mathrm{Fuzz}([0,1]^{op}) \simeq \mathrm{Mon}([0,1]^{op}_{+}).$$

Start with a sheaf F of monomorphisms on $[0,1]^{op}_{+}$, $F(s) \subset F(1)$ ("generic fibre"):

Given $x \in F(1)$ there is a minimum s_x such that $x \in F(s_x)$.

The assignment $x \mapsto s_x =: \phi_F(x)$ defines a function (fuzzy set) $\phi_F : F(1) \to [0,1]$.

Meets, joins: $A, B \subset F$ in the sheaf category. Then $A \cup B$ and $A \cap B$ (sheaf theoretic) are subobjects of F, and

$$\begin{aligned} \phi_{A\cup B}(x) &= \max\{\phi_A(x), \phi_B(x)\} \quad \text{"}t\text{-conorm"} \\ \phi_{A\cap B}(x) &= \min\{\phi_A(x), \phi_B(x)\} \quad \text{"}t\text{-norm"} \end{aligned}$$

We have

$$\begin{aligned} &\phi_A(x) \cdot \phi_B(x) \leq \phi_{A\cap B}(x), \\ &\phi_{A\cup B}(x) \leq \phi_A(x) + \phi_B(x) - \phi_A(x) \cdot \phi_B(x). \end{aligned}$$

Complement

For $s, t \in [0,1]$,

$$1-(1-s)(1-t) = s+t-s\cdot t$$

whereas $(A^c \cap B^c)^c = A \cup B$ for subobjects $A, B \subset X$.

The function $s \mapsto 1-s =: e(s)$ defines a poset isomorphism (idempotent) $e : [0,1] \to [0,1]^{op}$ that I call the **dual**.

Classical: If $p : X \to [0,1]$ is a fuzzy subset of X, its **complement** p^c is defined by $p^c(x) = 1-p(x)$.

Alternatively, $p^c = e \cdot p$.

Definition: p^c is the **dual** of p.

Duality in the large

We have poset isomorphisms

$$\phi : [0,1] \overset{\cong}{\leftrightarrows} [0,\infty]^{op} : \psi$$

with $\phi(s) = -\log(s)$ and $\psi(t) = e^{-t}$ (order reversing).

ϕ is Shannon's **information function** [6].

The commutative diagram of poset isomorphisms

$$\begin{tikzcd} {[0,\infty]} \arrow[r, "\psi"] \arrow[d, dotted, "e"'] & {[0,1]^{op}} \arrow[d, "e"] \\ {[0,\infty]^{op}} \arrow[r, "\psi"'] & {[0,1]} \end{tikzcd} \tag{1}$$

defines a duality isomorphism (idempotent) $e : [0,\infty] \to [0,\infty]^{op}$.

$$x^* := e(x) = -\log(1 - e^{-x}) = x - \log(e^x - 1),\ x \in [0,\infty].$$

Facts: $x \mapsto x^*$ is continuous. x and x^* are close if x is near $\log(2)$.

Duals II

X is a set: there is a poset map $s : \mathcal{P}(X) \to \text{Fuzz}([0,1])$, with

$$s(A)(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

The fuzzy subset $s(A) : X \to [0,1]$ of X is "crisp".

Any morphism $f : L \to L'$ of locales induces a functor $\text{Fuzz}(L) \to \text{Fuzz}(L')$, by composition with f.

There is a commutative diagram of poset morphisms

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[r, "s"] \arrow[d, "c"'] & \text{Fuzz}([0,1]) \arrow[r, "\phi"] \arrow[d, "e"] & \text{Fuzz}([0,\infty]^{op}) \arrow[d, "e"] \\
\mathcal{P}(X)^{op} \arrow[r, "s^{op}"'] & \text{Fuzz}([0,1]^{op}) \arrow[r, "\phi"'] & \text{Fuzz}([0,\infty])
\end{tikzcd}$$

where $c(A) = X - A$ is the complement of A in X.

Cross entropy I

Suppose that $u, v : X \to [0,1]$ are fuzzy sets. The **cross entropy** from v to u is defined by

$$C(u,v) = \sum_{x \in X} (u(x)\log(\frac{u(x)}{v(x)}) + (1-u(x))\log(\frac{1-u(x)}{1-v(x)}))$$

(Bhandari 1993 [2]).

$C(u,v)$ is a sum of elements of the form

$$F(s,t) = s \cdot \log(\frac{s}{t}) + (1-s) \cdot \log(\frac{1-s}{1-t}) \geq 0.$$

with $s, t \in [0,1]$ (Kullback-Liebler divergence, 1951).

u (hence s) is fixed. $C(u,v)$ is a (non-symmetric) measure of the distance of v from u.

Cross entropy II

Suppose $s = e^{-y}$ and $t = e^{-x}$. Then

$$F(s,t) = F(e^{-y}, e^{-x}) = e^{-y}(x-y) + e^{-y^*}(x^* - y^*).$$

$|F(e^{-y}, e^{-x})|$ depends on $|x-y|$ in $[0,\infty]$.

Given $u, v : X \to [0,\infty] \xrightarrow{\psi} [0,1]^{op}$, set

$$\begin{aligned} C(u,v) &= C(\psi \cdot u, \psi \cdot v) \\ &= \sum_{x \in X} [e^{-u(x)}(v(x) - u(x)) + e^{-u(x)^*}(v(x)^* - u(x)^*)]. \end{aligned}$$

For $u, v : X \to [0,\infty]$, $C(u,v)$ is the **cross entropy** from v to u.

ep-metric spaces

An extended pseudo-metric space (**ep-metric space**) (X, D) is a set X and a function $D : X \times X \to [0, \infty]$ such that

1) $D(x, x) = 0$,
2) $D(x, y) = D(y, x)$,
3) $D(x, z) \leq D(x, y) + D(y, z)$.

- Can have distinct x, y such that $D(x, y) = 0$ ("pseudo").
- Can have u, v such that $D(u, v) = \infty$ ("extended").

Example: Every metric space (Y, d) is an ep-metric space:

$$Y \times Y \xrightarrow{d} [0, \infty) \subset [0, \infty].$$

A **morphism** $f : (X, d_X) \to (Y, d_Y)$ of ep-metric spaces is a function $f : X \to Y$ such that

$d_Y(f(x), f(y)) \leq d_X(x, y)$ (compresses distance, "non-expanding").

$ep - \underline{\mathrm{Met}}$ is the category of ep-metric spaces and their morphisms.

Cocompleteness (Spivak [7])

(X, d) an ep-metric space and $p : X \to Y$ a surjective function.

For $x, y \in Y$ set

$$D(x, y) = \inf_P \sum_{i=0}^{k} d(x_i, y_i)$$

"Polygonal path" P : pairs (x_i, y_i), $0 \leq i \leq k$, in X with $x = p(x_0)$, $p(y_i) = p(x_{i+1})$, $y = p(y_k)$.

Quotient map $p : (X, d) \to (Y, D)$ satisfies universal property.

Lemma 2.

$ep - \underline{\mathrm{Met}}$ is cocomplete (has all small colimits).

Underlying set is colimit in sets: $\sqcup_i\ (X_i, d_i)$ has $D(x, y) = d_i(x, y)$ if x, y in some X_i, $D(x, y) = \infty$ otherwise.

Coequalizers (or pushouts) given by quotient construction.

Vietoris-Rips complex

(X, d_X) a finite **ep-metric space**, $d_X : X \times X \to [0, \infty]$.

$s \in [0, \infty]$:

1) If X totally ordered (has a listing), then $V_s(X)$ has n-simplices $x_0 \leq x_1 \leq \cdots \leq x_n$ with $d_X(x_i, x_j) \leq s$ for all i, j.

$V(X) : s \mapsto V_s(X)$, $s \in [0, \infty]$ is **Vietoris-Rips system** for X.

Simplicial fuzzy set: $V_*(X)$ is a simplicial sheaf of monomorphisms on $[0, \infty]_+^{op}$.

2) $P_s(X)$ is the poset of all subsets $\sigma \subset X$ such that $d_X(x, y) \leq s$ for all $x, y \in \sigma$.

$P_s(X)$ is the poset of non-degenerate simplices of $V_s(X)$.

Nerve $BP_s(X)$ is the **barycentric subdivision** of $V_s(X)$. There is a natural weak equivalence $\gamma : BP_s(X) \xrightarrow{\simeq} V_s(X)$, $s \in [0, \infty]$.

NB: Poset const. $BP_s(X)$ **does not** use a total ordering on X.

The nerve construction

The **nerve** BC of a category C is a simplicial set with n-simplices BC_n given by the set of strings of arrows

$$a_0 \to a_1 \to \cdots \to a_n$$

of length n in C, equivalently functors $\underline{n} \to C$, where

$$\underline{n} = \{0, 1, \dots, n\},$$

with the obvious poset structure.

Composition with the functors $\theta : \underline{m} \to \underline{n}$ defines the simplicial set structure of BC.

Examples: 1) $B\underline{n} = \Delta^n$, the standard n-simplex in simplicial sets.

2) $BG = K(G, 1)$ for a group G, classifies principal G-bundles in the homotopy category.

BC is also called the **classifying space** of C.

Comments

1) The systems $s \mapsto H_p BP_s(X) \cong H_p V_s(X)$ (coefficients in a field) define persistent homology for (X, d).

2) The sets $\pi_0 BP_s(X) \cong \pi_0 V_s(X)$ are clusters for (X, d).

Hierarchy $\Gamma(X, d)$: a tree with vertices $(s, [x])$, $[x] \in \pi_0 BP_s(X)$, and edges $(s, [x]) \to (t, [x])$ with $s \leq t$. Typical source of hierarchical clustering algorithms.

3) $\pi BP_s X$ is **fundamental groupoid** of $BP_s(X)$.

There are isomorphisms $\pi_0 BP_s(X) \cong \pi_0 \pi BP_s(X)$ so the system $s \mapsto \pi_0 \pi BP_s(X)$ "computes" clusters and hierarchies of clusters.

4) One cares most about $\varinjlim_{s<\infty} BP_s(X)$, which is a disjoint union of **global components**, each of which consists of simplices having edges of finite length.

Each global component is contractible, and has a Vietoris-Rips filtration in the usual sense.

Stability

Theorem 3 (Rips stability).

Suppose $i : X \subset Y$ are finite ep-metric spaces such that $d_H(X,Y) < r$. There is a homotopy commutative diagram

$$\begin{array}{ccc} P_s(X) & \xrightarrow{\sigma} & P_{s+2r}(X) \\ i\downarrow & \overset{\theta}{\nearrow} & \downarrow i \\ P_s(Y) & \xrightarrow[\sigma]{} & P_{s+2r}(Y) \end{array}$$

Corollary 4 (Stability for persistence invariants).

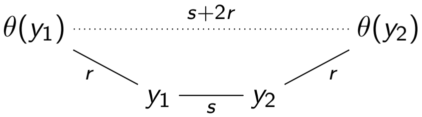
Same assumptions as Theorem 1. There are commutative diagrams

$$\begin{array}{ccc} H_k(V_s(X)) & \xrightarrow{\sigma} & H_k(V_{s+2r}(X)) \\ i\downarrow & \overset{\theta}{\nearrow} & \downarrow i \\ H_k(V_s(Y)) & \xrightarrow[\sigma]{} & H_k(V_{s+2r}(Y)) \end{array}$$

There is a corresponding statement for π_0 (clusters).

Stability: sketch proof

$y \in Y$: there is $\theta(y) \in X$ st. $d(y, \theta(y)) < r$ (from $d_H(X, Y) < r$).
$x \in X$: $\theta(x) = x$.



$\sigma = \{y_1, \dots, y_k\}$ in $P_s(Y)$, then

$$\sigma \cup \theta(\sigma) = \{y_1, \dots, y_k, \theta(y_1), \dots, \theta(y_k)\} \in P_{s+2r}(Y)$$

and there are homotopies (natural transformations)

$$\sigma \subseteq \sigma \cup \theta(\sigma) \supseteq \theta(\sigma).$$

between poset morphisms $P_s(Y) \to P_{s+2r}(Y)$.

"Fuzzy graph" for (X, d_X)

Graph $\Gamma_*(X)$ has elts of X as vertices, and edges $\{x, y\}$ with $x \neq y$ and $d_X(x, y) < \infty$. $\Gamma_s(X)$ has edges $\{x, y\}$ with $d_X(x, y) \leq s$.

$\check{C}(X)$ is **Čech groupoid**: $X =$ objects, and there is a morphism $x \to y$ iff $d_X(x, y) < \infty$. $B\check{C}(X)_s$: $d_X(x, y) \leq s$.

There are bijections

$$\pi_0 \Gamma_s(X) \xleftarrow{\cong} \pi_0 N \operatorname{sk}_1 B\check{C}(X)_s \xrightarrow{\cong} \pi_0 B\check{C}(X)_s,$$

natural in s: $s \leq t$ induces inclusion $B\check{C}(X)_s \subset B\check{C}(X)_t$.

Functor $\check{C}(X)_s \to \pi BP_s(X)$ (fund groupoid) takes a morphism $x \to y$ to the composite $\{x\} \to \{x, y\} \leftarrow \{y\}$

Theorem 5.

There are bijections

$$\pi_0 \Gamma_s(X) \xleftarrow{\cong} \pi_0 N \operatorname{sk}_1 B\check{C}(X)_s \xrightarrow{\cong} \pi_0 B\check{C}(X)_s \xrightarrow{\cong} \pi_0 \pi BP_s(X).$$

The unoriented fuzzy graph $\Gamma_*(X)$ computes clusters for (X, d_X).

Explanation

Z = simplicial set.

$N \operatorname{sk}_1 Z = \operatorname{sk}_0 Z \cup (\sqcup_{\sigma:\Delta^1 \to Z,\ d_0\sigma \neq d_1\sigma}\ \Delta^1) \to Z$, π_0-isomorphism.

Boundary of a simplex $\sigma : \Delta^1 \to Z$: $\partial\sigma = \{d_0\sigma, d_1\sigma\}$.

$\Gamma(Z)$ is unoriented graph with vertices Z_0 with edges given by boundaries $\{d_0\sigma, d_1\sigma\}$ for $\sigma : \Delta^1 \to Z$ with $d_0\sigma \neq d_1\sigma$.

$$\pi_0\Gamma(Z) = Z_0/\sim,$$

where $x \sim y$ if $\{x, y\}$ is an edge, generates an equiv relation $\sim$.

There are bijections

$$\pi_0\Gamma(Z) \xleftarrow{\cong} \pi_0 N \operatorname{sk}_1 Z \xrightarrow{\cong} \pi_0 Z,$$

natural wrt inclusions $Z \subset Z'$ of simplicial sets.

Singular functor S

Spivak [7]: There is an adjoint pair of functors

$$\mathrm{Re} : s\mathrm{Set}^{[0,\infty]} \leftrightarrows ep - \mathrm{Met} : S$$

$S(Y,d)_{s,n} =$ functions $\phi : \{0,1,\dots,n\} \to Y$ with $d(\phi(i),\phi(j)) \leq s$, for (Y,d) in $ep - \mathrm{Met}$ ("bags of words").

$$S(X,d)_s = B\check{C}(X)_s \ \ (!!)$$

For (X,d) a totally ordered finite ep-metric space, there is a canonical map of systems (functors)

$$\eta : V(X,d)_s \to S(X,d)_s.$$

η takes the simplex $x_0 \leq x_1 \leq \dots \leq x_n$ to the sequence $(x_0, x_1, \dots, x_n)$.

Theorem 6 ([3]).

The map $\eta : V(X,d)_s \to S(X,d)_s$ is a weak equivalence of simplicial sets, for each s.

UMAP complex

(X, d_X) = finite data set, or finite ep-metric space ... but very big.

1) Choose **neighbourhood set** N_x, $x \in X$. Set
$U_x = \{x\} \sqcup N_x \subset X$.

e.g. $N_x = k$ nearest neighbours if X is totally ordered, has metric (if you can find them — there are algorithms).

2) Set $(U_x, D_x) = \vee_{y \in N_x} (\{x, y\}, d_y)$ in $ep - \mathrm{Met}$.
$d_y(x, y) > 0$ is a **weight**.

3) Extend to an ep-metric D_x on X by setting $D_x(y, z) = \infty$ if either y or z is outside of U_x.

4) We have inclusions $X \subset V(X, D_x)$, $x \in X$ (X discrete). Form the iterated pushout

$$V(X, N) = \vee_{x \in X}\ V(X, D_x) \simeq \vee_X\ S(X, D_x).$$

$BP(X, N) = V(X, N)$ is "the" **UMAP complex** — a simplicial presheaf on $[0, \infty]_+^{op}$, simp. fuzzy set with coefficients in $[0, \infty]^{op}$.

Excision

Given (X, d_X) and (X, D_x) as above, form the wedge sum

$$(X, D) = \vee_{x \in X} \, (X, D_x)$$

in ep-metric spaces.

(X, D_x) is the realization of $V(X, D_x)$ for each x [3], so (X, D) is the realization of the UMAP complex $V(X, N)$.

Theorem 7 (Excision for π_0).

$V(X, N) = \vee_x \, V(X, D_x) \to V(X, D)$ *induces a bijection*

$$\pi_0 V(X, N)_s \xrightarrow{\cong} \pi_0 V(X, D)_s, 0 \leq s < \infty.$$

Corollary 8.

There are bijections

$$\pi_0 \Gamma(X, D)_s \cong \pi_0 V(X, D)_s \cong \pi_0 V(X, N)_s, \; 0 \leq s < \infty.$$

UMAP algorithm: Dimension reduction I

(X, d_X) finite ep-metric space, X totally ordered, $|X| = n$.

ep-metric space (X, D) constructed from neighbours N_x as above.

Assume that (X, D) is **connected**: $D(x, y) < \infty$ for all $x, y \in X$.

- Unoriented weighted graph: $\Gamma(X) = \Gamma_*(X, D)$.

Find a node embedding $p : X \subset \mathbb{R}^d$ with d small:

- Weight matrix $W = (a_{i,j})$, with $a_{i,j} = D(x_i, x_j)$ for $i \neq j$,
- Diagonal matrix $D = (d_{i,i})$ with $d_{i,i} = \sum_{i \neq j} D(x_i, x_j)$.
- $L = W - D$ is the **weighted Laplacian** for the graph $\Gamma(X)$.

Find orthonormal basis of eigenvectors $\{w_1, \dots, w_N\}$ for L.

Lemma 9.

There is a basis for R^n *consisting of vectors* $\overline{e}_i = e_i + \delta_i w_1$*, with all* δ_i *small, such the images* $p(\overline{e}_i)$ *are distinct.*

UMAP algorithm: Dimension reduction II

Corollary 10.

Suppose that $V = \langle w_1, \dots, w_d \rangle \subset \mathbb{R}^n$. *Then the projection* $p : \mathbb{R}^n \to \mathbb{R}^d \cong V$ *onto* V *restricts to an injective function* $\overline{e}_i \mapsto p(\overline{e}_i)$.

NB: The number d is usually 2 or 3 in practice, and is bounded above by the rank of L.

Set $y_i = p(\overline{e}_i) \in \mathbb{R}^d$.

Optimisation

We have a graph (Y, E) with vertices y_i, and edges $[y_i, y_j]$ with weights $d(y_i, y_j)$ given by distance.

Adjust vertices of (Y, E) by minimizing cross entropy:

$$w_X : X \times X \to [0, \infty],\ (x, y) \mapsto D(x, y),$$
$$w_Y : X \times X \to [0, \infty],\ (x, y) \mapsto d(p(x), p(y)).$$

$$C(w_X, w_Y) = \sum_{x,y} [e^{-w_X(x,y)}(w_X(x, y) - w_Y(x, y)) + e^{-w_X(x,y)^*}(w_X(x, y)^* - w_Y(x, y)^*)]$$

Suggestion: Minimise

$$(w_X(x, y) - w_Y(x, y))^2,\ (w_X(x, y)^* - w_Y(x, y)^*)^2$$

by moving the y_i (stochastic gradient descent).

Endgame

One finishes with a graph (Y, E), $X = Y \subset \mathbb{R}^d$, which is "optimally close" to (X, D).

We're not done:

Write $(Y, E)_s$ for the subgraph of (Y, E) with vertices $Y = X \subset \mathbb{R}^d$ and edges $[x, y] \in E$ (in $\Gamma(X)$) with $d(x, y) \leq s$.

• We have clusters $\pi_0(Y, E)_s$ and a hierarchy $\Gamma(Y, E)$.

• $\Gamma(Y, E)$ is a tree with objects $(s, [x])$ with $[x] \in \pi_0(Y, E)_s$, and morphisms $(s, [x]) \to (t, [x])$ with $s \leq t$.

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Homotopies

A natural transformation h between functors $f, g : C \to D$ is a diagram of functors

$$\begin{tikzcd} C \arrow[d, "i_0"'] \arrow[dr, "f"] & \\ C \times 1 \arrow[r, "h"] & D \\ C \arrow[u, "i_1"] \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} f(a) \arrow[r, "h"] \arrow[d, "f(\alpha)"'] & g(a) \arrow[d, "g(\alpha)"] \\ f(b) \arrow[r, "h"'] & g(b) \end{tikzcd}$$

where $1 = \{0 \leq 1\}$, $i_\epsilon(a) = (a, \epsilon)$.

$$B(C \times 1) \cong BC \times B1 = BC \times \Delta^1$$

$$\begin{tikzcd} BC \arrow[d, "i_0"'] \arrow[dr, "f"] & \\ BC \times \Delta^1 \arrow[r, "h"] & BD \\ BC \arrow[u, "i_1"] \arrow[ur, "g"'] & \end{tikzcd}$$