Categories, symmetric groups and spheres

So what's a category?

A *category* C is a set $\mathrm{Ob}(C)$ of objects (or states) tother with a set $\mathrm{Mor}(C)$ of morphisms (aka. arrows) which define ways $\alpha : x \to y$ of getting from one object to another. So far that's just a graph — one also requires a law of composition

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[dr, "\beta\cdot\alpha"'] & y \arrow[d, "\beta"] \\ & z \end{tikzcd}$$

which is associative and has two-sided identities.

Examples:

1) Look at the set $\{0, 1, \ldots, n\}$ of counting numbers between 0 and n and say that there is a morphism $i \to j$ if and only if $i \leq j$. I write $\mathbf{n}$ for the corresponding category. We've really taken the order relation in a partially ordered set and turned it into a category here, and you can do the same for any poset.

2) Suppose that G is a group. Take a one-point set $\{*\}$ and identify the elements $g \in G$ with morphisms $* \xrightarrow{g} *$. In this way, you get a one-object category in which every morphism is invertible,

and you can characterize all groups as such. More generally, a groupoid is a category in which every morphism is invertible.

A functor $f : C \to D$ between categories consists of functions $f : \mathrm{Ob}(C) \to \mathrm{Ob}(D)$ and $f : \mathrm{Mor}(C) \to \mathrm{Mor}(D)$ which respect all structure: f associates to each morphism $\alpha : x \to y$ of C the morphism $f(\alpha) : f(x) \to f(y)$ in such a way that f preserves identities and respects composition laws.

Examples:

1) Every increasing function (aka. poset morphism) $\theta : \mathbf{m} \to \mathbf{n}$ is a functor: θ is a function $i \mapsto \theta(i)$ such that $\theta(i) \leq \theta(j)$ if $i \leq j$.

There are various forms of functor categories: the finite posets $\mathbf{n}$, $n \geq 0$ and the poset morphisms (functors) define a category, which is called the *ordinal number category* and is usually denoted by $\mathbf{\Delta}$.

2) There is a (giant) category **Set** consisting of sets (objects) and all functions between them (morphisms). Suppose that I is some small category, and let $X : I \to \mathbf{Set}$ be a functor. Then X consists of the assignment $i \mapsto X(i)$ of a set $X(i)$ to

each object $i \in \mathrm{Ob}(I)$ and X associates a function

$$\alpha_* = X(\alpha) : X(i) \to X(j)$$

to each morphism $\alpha : i \to j$ of I in such a way that identities and the composition laws are preserved.

Here's a new category $E_I X$ that you can make out of such a functor: the objects of $E_I X$ are the pairs (i, x) with $i \in \mathrm{Ob}(I)$ and $x \in X(i)$, and a morphism

$$\alpha : (i, x) \to (j, y)$$

is a morphism $\alpha : i \to j$ of I such that $\alpha_*(x) = y \in X(j)$. $E_I X$ is called either the *translation category* or the *category of elements* for the functor X, depending on what you are reading. It's one of the standard sources for the theory of homotopy colimits.

3) I use the notation **Mon** to denote the category of all finite sets

$$\underline{n} = \{1, 2, \ldots, n\}$$

and all injective functions (monomorphisms) $\underline{m} \to \underline{n}$ between them. This category includes the empty set $\emptyset$ which is the smallest possible object in unique way: there's only one function $\emptyset \to \underline{n}$.

A *pointed set* X is a set with a choice of distinguished element $x \in X$, which is usually called a *base point*, and a pointed function is a function $f : X \to Y$ between pointed sets which preserves base points. These things form a category, for which I write $\mathbf{Set}_*$.

Every pointed set X determines a functor

$$P_X : \mathbf{Mon} \to \mathbf{Sets}_*$$

such that

$$\underline{n} \mapsto X^n = \hom(\underline{n}, X).$$

Every monomorphism $\theta : \underline{m} \to \underline{n}$ determines a pointed function

$$\theta_* : X^m \to X^n$$

which takes a function $f : \mathbf{m} \to X$ and extends it to a function $\theta_*(f) : \underline{n} \to X$ which takes $\underline{n} - \mathrm{im}(\theta)$ to the base point of X.

Notice, for example, that $P_X(\emptyset) = X^\emptyset$ is a one-point set, which I shall denote by $*$, $P_X(\underline{1}) = X$ and the function $\emptyset \to \underline{1}$ is sent to the unique pointed function $* \to X$.

This construction $X \mapsto P_X$ is actually the central object of study of this talk, after some interpretation: $E_{\mathbf{Mon}}P_X$ can be turned into a pretty nice space.

You have to know that every category C can be used to construct a space, in two steps:

1) Look at the set BC_n of functors $\mathbf{n} \to C$. Insofar as the category $\underline{n}$ consists of all relations

$$0 \leq 1 \leq 2 \leq \cdots \leq n$$

a functor $\sigma : \mathbf{n} \to C$ can be identified with a string of morphisms

$$a_0 \to a_1 \to a_2 \to \cdots \to a_n$$

of length n in C. If $\theta : \mathbf{m} \to \mathbf{n}$ is a functor (aka. ordinal number morphism and $\sigma : \mathbf{n} \to C$ is a string of arrows of length n, then the composite functor

$$\mathbf{m} \xrightarrow{\theta} \mathbf{n} \xrightarrow{\sigma} C$$

is a string of arrows of length m. In this way, precomposition with θ defines a function $\theta^* : BC_n \to BC_m$, and this assignment is functorial in θ. The assignments $n \mapsto BC_n$ and $\theta \mapsto \theta^*$ define a "contravariant' functor

$$BC : \mathbf{\Delta}^{op} \to \mathbf{Set}$$

which is otherwise known as a *simplicial set*, called the *nerve* of C.

Simplicial sets X are contravariant functors $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$ and morphisms of simplicial sets are the natural transformations between them. The nerve construction defines a functor

$$B : \mathbf{cat} \to s\mathbf{Set}$$

from the category of small categories to the category of simplicial sets.

2) The simplicial set $B\mathbf{n}$ is otherwise known as the standard n-simplex Δ^n, and the n-simplices $X_n = X(\mathbf{n})$ of a simplicial set X can be identified with simplicial set maps $\Delta^n \to X$ because Δ^n is defined by a representable functor.

There is a *simplex category* $\mathbf{\Delta}/X$ for a simplicial set X which consists of all simplices $\Delta^n \to X$ and their incidence relations

$$\begin{tikzcd} \Delta^m \arrow[d] \arrow[dr] & \\ \Delta^n \arrow[r] & X \end{tikzcd}$$

There is finally, a nice collection of spaces $|\Delta^n|$, $n \geq 0$, (that you have seen) with

$$|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum t_i = 1 \}.$$

With a little thought, you can find functorial induced continuous maps $\theta_* : |\Delta^m| \to |\Delta^n|$ for the ordinal number morphisms $\theta : \mathbf{m} \to \mathbf{n}$. These maps and the simplex category for a simplicial set X can then be used to define a space $|X|$, called the *realization* of X, with

$$|X| = \varinjlim_{\Delta^n \to X} |\Delta^n|.$$

In other words, you take a copy of $|\Delta^n|$ for each simplex $\Delta^n \to X$ and then glue these spaces together along the incidence relations of the simplices of X to form $|X|$.

We therefore have functors

$$\mathbf{cat} \xrightarrow{B} s\mathbf{Set} \xrightarrow{|\ |} \mathbf{Top}$$

and the space $|BC|$ is the space that I said that you can build from a small category C.

The space $|BC|$ has a homotopy type that can be analyzed with some profit, but it turns out that you don't have to go that far. There is an internally defined homotopy structure on the simplicial set category which is equivalent to ordinary homotopy theory in a very strong sense.

The equivalences of the simplicial set homotopy theory are those simplicial set maps $X \to Y$ such that the induced map $|X| \to |Y|$ is a homotopy equivalence of CW-complexes, or equivalently induces isomorphisms in all possible homotopy groups. There is a theory of cofibrations for simplicial sets, and these are just the simplicial set monomorphisms, and finally, the realization functor has a right adjoint

$$S : \mathbf{Top} \to s\mathbf{Set}$$

called the *singular functor* (this is the gadget underlying singular homology and cohomology theory) such that the adjoint pair

$$|\ | : s\mathbf{Set} \leftrightarrows \mathbf{Top} : S$$

forms what's called a Quillen equivalence between the standard homotopy theory for topological spaces which is defined by the usual variational principle and the homotopy structure which I have described very quickly for simplicial sets.

The existence of an internal (discrete, combinatorial) homotopy theory for simplicial sets (Kan, Quillen), and the equivalence with the classical variational homotopy theory for spaces (Quillen, and others) is one of the great results of twentieth century Mathematics.

Set

$$\Gamma^+(X) = B(E_{\mathbf{Mon}}P_X).$$

I remind you that the objects of the category $E_{\mathbf{Mon}}P_X$ are the pairs $(\underline{n}, f)$ with $f : \underline{n} \to X$ a function, equivalently an element of X^n. A morphism

$$\theta : (\underline{m}, f) \to (\underline{n}, g)$$

consists of an injection $\theta : \underline{m} \to \underline{n}$ such that $\theta_*(f) = g$. In other words when you extend f to $\mathbf{n}$ by adding base points you get g.

Write $\mathbf{1}_+$ for the 2-element set $\{0, 1\}$, pointed by 0. In other words, $\mathbf{1}_+$ has two colours, on 1 and off 0.

Lemma: There is an equivalence

$$\Gamma^+(\mathbf{1}_+) \simeq \coprod_{n \geq 0} B\Sigma_n.$$

Here Σ_n is the symmetric group on n-letters and $B\Sigma_n = K(\Sigma_n, 1)$ is its nerve or classifying space.

Proof: The proof of the Lemma consists of an analysis of the category $E_{\mathbf{Mon}}P_{\mathbf{1}_+}$.

There are special elements $(\underline{n}, 1)$ in the translation category, where $1 : \underline{n} \to \mathbf{1}_+$ is the function which takes all elements of $\underline{n}$ to 1. A symmetric group element $\sigma : \underline{n} \to \underline{n}$ induces a morphism

$$\sigma_* : (\underline{n}, 1) \to (\underline{n}, 1)$$

functorially in σ. There are therefore functors $\Sigma_n \to E_{\mathbf{Mon}} P_{\mathbf{1}_+}$ with $* \mapsto (\underline{n}, 1)$, and collecting them together defines a functor

$$\coprod_{n \geq 0} \Sigma_n \to E_{\mathbf{Mon}} P_{\mathbf{1}_+}.$$

The idea is to show that this functor is an equivalence of categories.

For every object $(\underline{n}, f)$, the set of elements $i \in \underline{n}$ such that $f(i) = 1$ defines a monomorphism $\alpha_{(\underline{n},f)} : \underline{k} \to \underline{n}$ and a corresponding morphism $\alpha_{(\underline{n},f)} : (\underline{k}, 1) \to (\underline{n}, f)$. Pick one of these for each object $(\underline{n}, f)$ (such that $\alpha_{(\underline{n},1)} = 1$), and observe that each morphism $\gamma : (\underline{n}, f) \to (\underline{m}, g)$ determines a unique symmetric group element $\sigma_\gamma \in \Sigma_k$ such that the diagram

$$\begin{tikzcd}
(\underline{k}, 1) \arrow[r, "\alpha_{(\underline{n},f)}"] \arrow[d, "\sigma_\gamma"'] & (\underline{m}, f) \arrow[d, "\gamma"] \\
(\underline{k}, 1) \arrow[r, "\alpha_{(\underline{m},g)}"'] & (\underline{m}, g)
\end{tikzcd}$$

commutes.

Then the assignment $\gamma \mapsto \sigma_\gamma$ defines a functor

$$E_{\mathbf{Mon}} P_{\mathbf{1}_+} \to \coprod_{n \geq 0} \Sigma_n$$

such that the composite

$$\bigsqcup_{n \geq 0} \Sigma_n \to E_{\mathbf{Mon}} P_{\mathbf{1}_+} \to \bigsqcup_{n \geq 0} \Sigma_n$$

is the identity, and such that composing in the other direction gives a functor

$$E_{\mathbf{Mon}} P_{\mathbf{1}_+} \to E_{\mathbf{Mon}} P_{\mathbf{1}_+}$$

which has a natural transformation

$$E_{\mathbf{Mon}} P_{\mathbf{1}_+} \times \mathbf{1} \to E_{\mathbf{Mon}} P_{\mathbf{1}_+}$$

to the identity.

The nerve functor preserves products, so this last natural transformation induces a simplicial homotopy

$$\Gamma^+(\mathbf{1}_+) \times \Delta^1 \to \Gamma^+(\mathbf{1}_+),$$

which gives the desired homotopy equivalence. $\square$

What does it mean?

The assignment

$$\mathbf{n}_+ \mapsto \Gamma^*(\mathbf{n}_+) = \Gamma^+(\mathbf{n}_+)/B\mathbf{Mon} \simeq \Gamma^+(\mathbf{n}_+)$$

defines a Γ-space (models a connective spectrum). This object is a *special* Γ-space on account of a multi-coloured version of the Lemma:

Lemma: The pinch maps $\mathbf{n}_+ \to \mathbf{1}_+$ define an equivalence

$$\Gamma^+(\mathbf{n}_+) \to \prod_{i=1}^{n} \Gamma^+(\mathbf{1}_+).$$

There is a natural transformation $\mathbf{n}_+ \to \Gamma^*(\mathbf{n}_+)$ which defines a map $1 \to \Gamma^*(1)$ from the "identity" Γ-space to $\Gamma^*(1)$.

Lemma: The map $1 \to \Gamma^*(1)$ of Γ-spaces induces a stable equivalence

$$S \to \Gamma^*(S).$$

Here, S is the sphere spectrum

$$S = S^0, S^1, S^1 \wedge S^1, \ldots$$

with $S^1 = \Delta^1/\{0,1\}$, and one gets a spectrum from a Γ-space by evaluating at the sphere spectrum.

Proving this last Lemma is where most of the work is.

From what we have so far, there are equivalences

$$QS^0 \simeq \Omega(\Gamma^*(S^1)) = \Omega(B_{\oplus}(\bigsqcup_{n\geq 0} B\Sigma_n))$$

where $QS^0 = \varinjlim_n \Omega^n S^n$ has homotopy groups given by the stable homotopy groups of spheres. Then the group-completion theorem implies

Theorem: (Barratt-Priddy) The space

$$\Omega(B(\coprod_{n\geq 0} B\Sigma_n)$$

has the integral homology of the space

$$\coprod_{\mathbb{Z}} B\Sigma_\infty.$$

The Barratt-Priddy Theorem (1972) has been based, all these years, on a rather different description of the functor $X \mapsto \Gamma^+ X$, which was originally given by Barratt (1970). Barratt's description of $\Gamma^+ X$ was given by collapsing the space

$$\coprod_{n\geq 0} X^{\times n} \times E\Sigma_n$$

by a rather strange co-end-type relation that depended on an explicit Σ_n-equivariant map

$$T : E\Sigma_{n+1} \to E\Sigma_n.$$

When you look at it through a modern lens, you get the feeling that $\Gamma^+(X)$ wants to be a homotopy colimit, and that's exactly the sort of construction that I have displayed here. I think that it's much easier to understand.