J.F. Jardine: **Cocycles and pro-objects**

Cocycles

Suppose that X and Y are simplicial presheaves. A **cocycle** from X to Y is a picture

$$X \xleftarrow[\simeq]{g} U \xrightarrow{f} Y,$$

where g is a local (ie. stalkwise) weak equivalence. A morphism of cocycles is a commutative diagram

$$\begin{array}{ccccc} & & U & & \\ & {}^{g}\swarrow\simeq & & \searrow^{f} & \\ X & & \downarrow \theta & & Y \\ & {}_{g'}\nwarrow\simeq & & \nearrow_{f'} & \\ & & U' & & \end{array} \tag{1}$$

The corresponding cocycle category is denoted by $h(X, Y)$.

This discussion is all with respect to a model structure for the category of simplicial presheaves, for which the weak equivalences are the local weak equivalences and the cofibrations are monomorphisms, just like in simplicial sets. The fibrations for the theory are defined by a lifting property — they are called **injective fibrations**, and this is the **injective model structure** for simplicial presheaves.

I write $[X, Y]$ to denote morphisms from X to Y in the associated homotopy category (one formally inverts the local weak equivalences).

Every cocycle $X \xleftarrow[\simeq]{g} U \xrightarrow{f} Y$ defines a morphism fg^{-1} in $[X, Y]$, and if there is a morphism $\theta : (g, f) \to (g'f,)$ in $h(X, Y)$, then $fg^{-1} = f'(g')^{-1}$ in $[X, Y]$. We therefore have a well defined function

$$\phi : \pi_0 h(X, Y) \to [X, Y].$$

Theorem 1. *The function ϕ is a bijection.*

To prove this result, one shows that any local weak equivalences $X \to X'$ and $Y \to Y'$ induces weak equivalences

$$Bh(X', Y) \xleftarrow{\simeq} Bh(X, Y) \xrightarrow{\simeq} Bh(X, Y'),$$

and that the function

$$\phi : \pi_0 h(X, Z) \to [X, Z]$$

is bijective if Z is injective fibrant. In this last case $[X, Z]$ is simplicial homotopy classes of maps $\pi(X, Z)$ and one constructs an inverse for ϕ.

Remark 2. Theorem 1 is a souped up version of the Verdier hypercovering theorem (we'll describe this later). Cocycle categories are now part of the basic foundational structure of local homotopy theory [7].

There's another way to think about $h(X,Y)$:

Write we/X for the category of local weak equivalences $U \xrightarrow{\simeq} X$. The morphisms of we/X are the diagrams

$$\begin{array}{ccc} U & & \\ \downarrow & \searrow^{\simeq} & \\ & & X \\ \downarrow & \nearrow_{\simeq} & \\ U' & & \end{array}$$

Notice that we/X has a terminal object, namely $1_X : X \to X$, and is therefore contractible in the sense that $B(we/X)$ (and $B(we/X)^{op}$) is a contractible simplicial set.

There is a (contravariant) functor

$$\hom(\ ,Y) : (we/X)^{op} \to \mathbf{Set}$$

which takes a weak equivalence $U \xrightarrow{\simeq} X$ to the set $\hom(U,Y)$.

Set-valued functors have translation categories. The translation category for the functor $\hom(\ ,Y)$ has objects consisting of pairs

$$(X \underset{\simeq}{\xleftarrow{g}} U, U \xrightarrow{f} Y),$$

and then a morphism

$$\theta : (g, f) \to (g', f')$$

is a morphism $\theta : U \to U'$ which makes the diagram (1) commute.

We're just starting to play: there is a simplicial set-valued functor

$$\mathbf{hom}(\ , Y) : (we/X)^{op} \to s\mathbf{Set}$$

which takes a local weak equivalence $U \xrightarrow{\simeq} X$ to the function complex $\mathbf{hom}(U, Y)$.

This function complex $\mathbf{hom}(U, Y)$ is diagram-theoretic animal. Its n-simplices are simplicial presheaf maps $U \times \Delta^n \to Y$, or equivalently simplicial presheaf maps $U \to Y^{\Delta^n}$.

The game, in all that follows, is to think about the corresponding homotopy colimit spaces

$$\underrightarrow{\mathrm{holim}}_{\,U \xrightarrow{\simeq} X} \mathbf{hom}(U, Y) \to B(we/X)^{op} \simeq *$$

We're now going to use locally fibrant simplicial presheaves repeatedly. A **locally fibrant** simplicial presheaf Y is a simplicial presheaf which consists of Kan complexes Y_x in stalks. All presheaves of Kan complexes are locally fibrant, and every injective fibrant simplicial presheaf is a presheaf of Kan complexes ... but neither of these statements has a converse.

Lemma 3. *Suppose that $\alpha : Y \to Z$ is a local weak equivalence of locally fibrant objects. Then the induced map*

$$\underrightarrow{\text{holim}}_{\,U \xrightarrow{\simeq} X} \mathbf{hom}(U,Y) \to \underrightarrow{\text{holim}}_{\,U \xrightarrow{\simeq} X} \mathbf{hom}(U,Z)$$

is a weak equivalence of simplicial sets.

Proof. The simplicial set

$$\underrightarrow{\text{holim}}_{\,U \xrightarrow{\simeq} X} \mathbf{hom}(U,Y)_n$$

is the nerve of the translation category for the functor which takes $U \to X$ to the set $\hom(U, Y^{\Delta^n})$, aka. the nerve of the cocycle category $h(X, Y^{\Delta^n})$. Since Y and Z are locally fibrant, all maps $Y^{\Delta^n} \to Z^{\Delta^n}$ are local weak equivalences (of locally fibrant objects). □

Lemma 4. *Suppose that Y is locally fibrant. Then the map*

$$Bh(X,Y) \to \underrightarrow{\text{holim}}_{\,U \xrightarrow{\simeq} X} \mathbf{hom}(U,Y)$$

is a weak equivalence.

Proof. All spaces Y^{Δ^n} are locally weakly equivalent to Y, since Y is locally fibrant. $\square$

Lemma 5. *Suppose that Z is injective fibrant. Then the map*

$$\mathbf{hom}(X,Z) \to \underrightarrow{\mathrm{holim}}_{U \xrightarrow{\simeq} X} \mathbf{hom}(U,Z)$$

is a weak equivalence.

Proof. The functor $U \mapsto \mathbf{hom}(U,Z)$ for $U \xrightarrow{\simeq} X$ is a diagram of equivalences since Z is injective fibrant, so $\mathbf{hom}(X,Z)$ is the homotopy fibre over 1_X for the map

$$\underrightarrow{\mathrm{holim}}_{U \xrightarrow{\simeq} X} \mathbf{hom}(U,Z) \to B(we/X)^{op} \simeq *.$$

We've used Quillen's Theorem B. $\square$

Corollary 6. *Suppose that $j : Y \to Z$ is an injective fibrant model for Y (ie. a local weak equivalence with Z injective fibrant). Then there are weak equivalences*

$$\begin{array}{ccccc} Bh(X,Y) & \xrightarrow{\simeq} & Bh(X,Z) & \xrightarrow{\simeq} & \underrightarrow{\mathrm{holim}}_{U \xrightarrow{\simeq} X} \mathbf{hom}(U,Z) \\ & & & & \uparrow \simeq \\ & & & & \mathbf{hom}(X,Z) \end{array}$$

so that $Bh(X,Y)$ is a model for the derived function complex (Hammock localization).

A **hypercover** of a simplical presheaf X is a local trivial fibration $p : U \to X$, meaning that the induced maps $U_x \to Y_x$ in stalks are trivial Kan fibrations, ie. maps which are Kan fibrations and weak equivalences.

This is a direct generalization of the old definition of Artin and Mazur (which was about maps of simplicial schemes, with the étale topology) to the simplicial presheaf context.

Every hypercover is a local weak equivalence. Every trivial injective fibration is a hypercover (but not conversely). Here's a theorem: every map which is both a local fibration and a local weak equivalence is a hypercover [7].

Let's suppose again that Y is locally fibrant. There is a subcategory $h_{hyp}(X,Y) \subset h(X,Y)$ whose objects are the cocycles

$$X \underset{\simeq}{\xleftarrow{p}} U \xrightarrow{f} Y$$

such that p is a hypercover.

Lemma 7. *Suppose that Y is locally fibrant. Then the induced map*

$$Bh_{hyp}(X,Y) \to Bh(X,Y)$$

is a homotopy equivalence.

Proof. A cocycle is a map $(g, f) : U \to X \times Y$ such that g is a local weak equivalence. Construct a (natural) factorization

$$\begin{array}{ccc} U & \xrightarrow{j} & V \\ & {\scriptstyle (g,f)} \searrow & \downarrow {\scriptstyle (p_1,p_2)} \\ & & X \times Y \end{array}$$

such that j is a local weak equivalence and (p_1, p_2) is an injective fibration. Then the composite $p_1 = pr_L \cdot (p_1, p_1)$ is a local fibration since Y is locally fibrant, and is a local weak equivalence. □

Here's another category: the objects are pairs of naive simplicial homotopy classes of maps

$$X \xleftarrow{[p]} U \xrightarrow{[f]} Y$$

such that p is a hypercover. The morphisms are commutative diagrams of simplicial homotopy classes

$$\begin{array}{ccccc} & {\scriptstyle [p]} \swarrow & U & \searrow {\scriptstyle [f]} & \\ X & & \downarrow {\scriptstyle [\theta]} & & Y \\ & {\scriptstyle [p']} \nwarrow & U' & \nearrow {\scriptstyle [f']} & \end{array}$$

where p and p' are hypercovers. Write $h_\pi(X, Y)$ for this category. There is an obvious function

$$\phi' : \pi_0 h_\pi(X, Y) \to [X, Y]$$

which takes the path component of $([p],[f])$ to $f \cdot p^{-1}$. There are functions

$$\pi_0 h(X,Y) \xrightarrow{\cong} \pi_0 h_{hyp}(X,Y) \xrightarrow{\psi} \pi_0 h_\pi(X,Y) \xrightarrow{\phi'} [X,Y],$$

where the composite is the bijection $\phi : \pi_0 h(X,Y) \cong [X,Y]$. The simplicial homotopy class function ψ is surjective as well as injective, so all displayed functions are bijections. One concludes that ϕ' is a bijection.

The assertion that ϕ' is a bijection is the "classical" Verdier hypercovering theorem:

Theorem 8. *Suppose that Y is locally fibrant. Then the function*

$$\phi' : \varinjlim_{[p]:U \to X} \pi(U,Y) \to [X,Y]$$

is a bijection.

Etale homotopy types and pro objects

The old proof of the Verdier hypercovering theorem (which assumes also that X is locally fibrant) is a calculus of fractions argument.

That proof involves the assertion that the index category $Triv/X$ of simplicial homotopy classes of hypercovers $Y \to X$ is filtered if X is locally fibrant (which must be proved). There is no such assumption in the present proof — the traditional baggage of hypercovers and filtered categories just gets in the way.

This is indicative of the way that everything used to be done. The **étale homotopy type** of a scheme S is constructed, by Friedlander [3], by taking a rigidified family of hypercovers $U \to S$ for the étale topology, which family forms a pro-object H/S in simplicial schemes over S that is cofinal in all hypercovers. Then one uses the connected component functor (for schemes) to form a pro-object (a left filtered diagram) $U \mapsto \pi_0 U$ in simplicial sets. This pro-object $\pi_0(H/S)$ is the **étale homotopy type** of S.

One uses the étale homotopy type $\pi_0(H/S)$ for the scheme S, within a model structure for pro-objects in simplicial sets, to construct étale cohomology invariants.

Suppose that A is an abelian group. There are isomorphisms

$$\begin{aligned}[\pi_0(H/S), K(A,n)] &\cong \varinjlim_{U\to S} \pi(\pi_0 U, K(A,n)) \\ &\cong \varinjlim_{U\to S} \pi(U, K(\Gamma^* A, n)) \\ &\cong H^n_{et}(S, \Gamma^* A),\end{aligned}$$

since the connected components functor π_0 is left adjoint to the constant sheaf functor. This means that the morphisms in the pro-homotopy category for simplicial sets pick up étale cohomology with constant coefficients (but that's all of the sheaf cohomology that they find).

Recall that a pro-object is a functor $X : I \to \mathcal{C}$ which is defined on a small left filtered index category I, If $Y : J \to \mathcal{C}$ is a second pro-object then a pro-map $X \to Y$ is a natural transformation of pro-representable functors

$$\varinjlim_j \hom(Y_j, \ ) \to \varinjlim_i \hom(X_i, \ ).$$

The model structure for pro-objects in simplicial sets has been well studied by Isaksen and others [1], [4], [2].

A few years ago, I developed a series of model structures for pro-objects in simplicial presheaves

[5]. There are two main flavours of weak equivalences:

1) the analog of the Edwards-Hastings definition, for which a pro-map $X \to Y$ is a **weak equivalence** if it induces weak equivalences

$$\underrightarrow{\text{holim}}_J \mathbf{hom}(Y, Z) \to \underrightarrow{\text{holim}}_I \mathbf{hom}(X, Z)$$

for all injective fibrant simplicial presheaves Z,

2) a pro-map $X \to Y$ is a **pro-equivalence** if map $P_*X \to P_*Y$ of Postnikov towers is an Edwards-Hastings equivalence.

The first definition is basic, and the second localizes at the Postnikov tower construction.

These model structures allow one to manipulate "étale homotopy types" directly within the simplicial presheaf category for the étale site on a scheme, or for other Grothendieck topologies. In fact, **all** sheaf cohomology can be represented within the corresponding homotopy theories of pro-objects in simplicial presheaves.

Generalized pro objects

Étale homotopy types use diagrams of weak equivalences which are constructed from hypercovers. These diagrams are also pro objects, and we have seen that there are perfectly good model structures for pro objects of simplicial presheaves, from which one can extract traditional étale homotopy theory.

Cocycle theory uses more general small diagrams of weak equivalences which are not pro objects in any sense, and yet gives an efficient description of the homotopy category of simplicial presheaves.

How far can you push it? Is there a homotopy for small diagrams of simplicial presheaves which engulfs the homotopy theory of pro-objects of simplicial presheaves?

Suppose that

$$X : I \to s\mathbf{Pre} \text{ and } Y : J \to s\mathbf{Pre}$$

are small diagrams of simplicial presheaves. It is tempting, given the fact that we want to do something homotopically correct, and (following the Grothendieck definition of pro-map discussed above) to say that a **pro-map** $X \to Y$ is a natural tranformation

$$f : \underrightarrow{\operatorname{holim}}_j \mathbf{hom}(Y_j,\ ) \to \underrightarrow{\operatorname{holim}}_i \mathbf{hom}(X_i,\ )$$

of functors defined on simplicial presheaves by function complexes.

Homotopy colimits are much more rigid that colimits, and the naturality implies that any such transformation f determines a functor $\alpha : J \to I$ and a natural transformation $\theta : X\alpha \to Y$ so that f is the composite

$$\underrightarrow{\operatorname{holim}}_j \mathbf{hom}(Y_j,\ ) \xrightarrow{\theta^*} \underrightarrow{\operatorname{holim}}_j \mathbf{hom}(X_{\alpha(j)},\ ) \xrightarrow{\alpha_*} \underrightarrow{\operatorname{holim}}_i \mathbf{hom}(X_i,\ ),$$

which is defined over the composite

$$BJ^{op} \xrightarrow{1} BJ^{op} \xrightarrow{\alpha} BI^{op}.$$

"Explanation": the homotopy colimit $\underrightarrow{\operatorname{holim}}_i \mathbf{hom}(X_i, Y)$ is the opposite of the nerve of a simplicial category X/Y whose objects are the morphisms $X_i \times \Delta^n \to Y$ and whose morphisms are the diagrams

$$\begin{array}{ccc} X_i \times \Delta^n & & \\ {\scriptstyle \alpha\times 1}\downarrow & \searrow & \\ & & Y \\ & \nearrow & \\ X_j \times \Delta^n & & \end{array}$$

where $\alpha : i \to j$ is a morphism of the index category I.

The pair (α, θ) is a morphism in the Grothendieck construction for a functor $\mathbf{cat} \to s\mathbf{Cat}$ which takes a small category I to the category $s\,\mathrm{Pre}^I$ of I-diagrams and their natural transformations.

Examples:

1) Every natural transformation $f : X \to Y$ of I-diagrams defines a pro-map $(1, f)$.

2) Suppose that Y is a simplicial presheaf, identified with a diagram $Y : * \to s\,\mathrm{Pre}$, and that X is an I-diagram. A pro-map $(i, f) : X \to Y$ is a simplicial presheaf map $X_i \to Y$.

3) For the same objects as in 2), a pro-map $Y \to X$ is a map $\theta : Y \to X$ of I-diagrams, where Y has been identified with a constant I-diagram.

4) Suppose that $Y : J \to s\,\mathrm{Pre}$ is a J-diagram and $\alpha : I \to J$ is a functor. Every object $Y_{\alpha(j)} \to Z$ of $Y\alpha/Z$ defines an object of Y/Z and hence a functor $Y\alpha/Z \to Y/Z$, naturally in Z and in J-diagrams Y. We therefore have a pro-map $(\alpha, 1) : Y\alpha \to Y$.

We now say that the map $(\alpha, \theta) : X \to Y$ is a **pro-equivalence** if the horizontal maps in the diagram

$$\begin{array}{ccc} \underrightarrow{\text{holim}}_{\,j}\,\mathbf{hom}(Y_j, Z) & \xrightarrow{\alpha_* \cdot \theta^*} & \underrightarrow{\text{holim}}_{\,i}\,\mathbf{hom}(X_i, Z) \\ \downarrow & & \downarrow \\ BJ^{op} & \xrightarrow[\alpha]{} & BI^{op} \end{array}$$

are weak equivalences of simplicial sets for all injective fibrant objects Z.

Examples:

1) Every sectionwise weak equivalence $f : X \to Y$ of I-diagrams determines a pro-equivalence $(1, f) : X \to Y$. In effect, the local weak equivalences $f : X_i \to Y_i$ induce weak equivalences

$$\mathbf{hom}(Y_i, Z) \xrightarrow{\simeq} \mathbf{hom}(X_i, Z)$$

for all injective fibrant simplicial presheaves Z.

2) Say that a functor $\alpha : I \to J$ is **cofinal** if all slices j/α are contractible (ie. $B(j/\alpha) \simeq *$).

If I and J are right filtered, α is cofinal in the traditional sense if all slices j/α are right filtered.

Say that α is **final** if all slices α/j are contractible. α is final if and only if $\alpha^{op} : I^{op} \to J^{op}$ is cofinal. In either case, $BI \to BJ$ is a weak equivalence by Quillen's Theorem A.

If $\alpha : J \to I$ is final and $X : I \to s\,\mathrm{Pre}$ is an I-diagram, then the pro-map $(\alpha, 1) : X\alpha \to X$ is a pro-equivalence, because all induced functors $X\alpha/Z \to X/Z$ are also final.

Question: Is there a homotopy theory for small diagrams of simplicial presheaves whose weak equivalences are the pro-equivalences?

This is work in progress.

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