Cocycle categories J.F. Jardine

Cocycles

I will be using the injective model structure on the category $s\,\mathrm{Pre}(\mathcal{C})$ of simplicial presheaves on a small Grothendieck site $\mathcal{C}$. You can think in terms of simplicial sheaves if you like — both approaches have technical advantages in particular circumstances.

I remind you that a cofibration for this theory is a monomorphism, and a weak equivalence is a map $f : X \to Y$ which induces weak equivalences $X_x \to Y_x$ in all stalks (if there are stalks). Equivalently, f is a weak equivalence if and only if it induces isomorphisms

$$\tilde{\pi}_0 X \xrightarrow{\cong} \tilde{\pi}_0 Y$$

in path component sheaves, and the diagrams

$$\begin{tikzcd} \pi_n X \arrow[r] \arrow[d] & \pi_n Y \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

induce pullback diagrams of associated sheaves for all $n \geq 1$. These are the local weak equivalences. The injective fibrations are those maps which have the right lifting property with respect to all trivial cofibrations.

The function complex $\mathbf{hom}(X,Y)$ is the simplicial set with n-simplices consisting of all maps $X \times \Delta^n \to Y$.

Theorem 1. *With these definitions, the category $s\,\mathrm{Pre}(\mathcal{C})$ has the structure of a proper closed simplicial model category. This model structure is cofibrantly generated, and weak equivalences are closed under finite products.*

Suppose that X, Y are simplicial presheaves.

$h(X,Y)$ = category whose objects are all pairs of maps (f,g)

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

where f is a weak equivalence. A morphism $\alpha : (f,g) \to (f',g')$ of $h(X,Y)$ is a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\alpha"] & \\ X & & Y \\ & Z' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

$h(X,Y)$ is the **category of cocycles** from X to Y.

Example 2. Suppose that k is a field and L/k is a finite Galois extension with Galois group G. Suppose that H is an algebraic group over k. Then

$\mathrm{Sp}(L) \to \mathrm{Sp}(k)$ is an étale covering map and represents a sheaf epi $\mathrm{Sp}(L) \to *$ on the étale site $et|_k$. There is an isomorphism of simplicial sheaves

$$C(L) \cong EG \times_G \mathrm{Sp}(L)$$

where $C(L)$ is the Čech resolution for the covering $\mathrm{Sp}(L) \to *$. Then the picture of simplicial sheaf maps

$$* \xleftarrow{\cong} EG \times_G \mathrm{Sp}(L) \to BH$$

is a cocycle on the Galois group G in the algebraic group H in the traditional sense.

Example 3. Suppose that H is a sheaf of groups an F is a sheaf with an action $H \times F \to F$. Then F is an H**-torsor** if and only if the canonical map $EH \times_H F \to *$ is a local weak equivalence. The picture

$$* \xleftarrow{\simeq} EH \times_H F \to BH$$

is the **canonical cocycle** associated to the torsor F.

Of course H is an H-torsor since

$$EH \times_H H \cong EH \simeq *,$$

and the standard map

$$* \xleftarrow{\simeq} EH \to BH$$

is the canonical cocycle associated to the trivial H-torsor H.

$\pi_0 h(X,Y) =$ class of path components of $h(X,Y)$. There is a function

$$\phi : \pi_0 h(X,Y) \to [X,Y] \quad (f,g) \mapsto g \cdot f^{-1}$$

Theorem 4. *The canonical map*

$$\phi : \pi_0 h(X,Y) \to [X,Y]$$

is a bijection for all simplicial presheaves X and Y.

Cocycles can be defined for all model categories $\mathcal{M}$ and the theorem holds if 1) $\mathcal{M}$ is right proper, 2) weak equivs. are closed under finite products in $\mathcal{M}$.

Examples: spaces, simplicial sets, presheaves of simplicial sets, spectra, presheaves of spectra, any good localizations including motivic homotopy theories.

Remark 5. The cocycle category $h(X,Y)$ has appeared before, in the context of the Dwyer-Kan theory of “hammock localizations”, but all identifications in that theory involve the assumption that Y is fibrant. The interesting applications of the Theorem involve objects Y which are not fibrant.

Lest you think that I've done away with the homotopy theory in this statement, suppose that $f \simeq g : X \to Y$. Then there is a picture

$$\begin{tikzcd} & X \arrow[dl, "1"'] \arrow[d, "d_0"'] \arrow[dr, "f"] & \\ X & X \times I \arrow[l, "pr"'] \arrow[r, "h"] & Y \\ & X \arrow[ul, "1"] \arrow[u, "d_1"] \arrow[ur, "g"'] & \end{tikzcd}$$

where h is the homotopy. Then

$$(1_X, f) \sim (pr, h) \sim (1_X, g)$$

Thus $f \mapsto [(1_X, f)]$ defines a function

$$\psi : \pi(X, Y) \to \pi_0 h(X, Y)$$

If Y is fibrant, then the function ψ is inverse to ϕ. More generally, there are a couple of things to prove.

The following result reduces the proof Theorem 4 to the case where Y is fibrant:

Lemma 6. *Suppose that $X \to X'$, $Y \to Y'$ are weak equivalences. Then the functor $h(X, Y) \to h(X', Y')$ induces a weak equivalence*

$$Bh(X, Y) \cong Bh(X', Y').$$

Proof. $(f, g) \in h(X', Y')$ is a map $(f, g) : Z \to X' \times Y'$ s.t. f is a weak equivalence.

There is a functorial factorization

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "(f,g)"'] & W \arrow[d, "(p_{X'},p_{Y'})"] \\ & X' \times Y' \end{tikzcd}$$

s.t. j is a triv. cofibration and $(p_{X'}, p_{Y'})$ is a fibration. $p_{X'}$ is a weak equivalence.

Form the pullback

$$\begin{tikzcd} W_* \arrow[r, "(\alpha\times\beta)_*"] \arrow[d, "(p_X^*,p_Y^*)"'] & W \arrow[d, "(p_{X'},p_{Y'})"] \\ X \times Y \arrow[r, "\alpha\times\beta"'] & X' \times Y' \end{tikzcd}$$

(p_X^*, p_Y^*) is a fibration and $(\alpha \times \beta)_*$ is a weak equivalence (since $\alpha \times \beta$ is a weak equivalence, and by right properness). p_X^* is also a weak equivalence.

We have found a functor

$$h(X', Y') \to h(X, Y)$$

which is inverse to the original functor $h(X, Y) \to h(X', Y')$ up to homotopy. □

Lemma 7. *Suppose that* Y *is fibrant. Then the canonical map*

$$\phi : \pi_0 h(X, Y) \to [X, Y]$$

is a bijection.

Proof. $\pi(X,Y) =$ naive homotopy classes.
$\pi(X,Y) \to [X,Y]$ is a bijection since Y is fibrant.

We have seen that the assignment $f \mapsto [(1_X, f)]$ defines a function

$$\psi : \pi(X,Y) \to \pi_0 h(X,Y)$$

and there is a diagram

$$\begin{tikzcd} \pi(X,Y) \arrow[r, "\psi"] \arrow[dr, "\cong"'] & \pi_0 h(X,Y) \arrow[d, "\phi"] \\ & {[X,Y]} \end{tikzcd}$$

It suffices to show that ψ is surjective, or that any object $X \xleftarrow{f} Z \xrightarrow{g} Y$ is in the path component of some a pair $X \xleftarrow{1} X \xrightarrow{k} Y$ for some map k.
Form the diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, "j"] & \\ X & & Y \\ & V \arrow[ul, "p"] \arrow[ur, dotted, "\theta"'] & \end{tikzcd}$$

where j is a trivial cofibration and p is a fibration; θ exists because Y is fibrant.

X is cofibrant, so the trivial fibration p has a section σ, and so there is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "1"'] \arrow[d, "\sigma"] \arrow[dr, "\theta\sigma"] & \\ X & & Y \\ & V \arrow[ul, "p"] \arrow[ur, dotted, "\theta"'] & \end{tikzcd}$$

The composite $\theta\sigma$ is the required map k. $\square$

Homotopy classification of torsors:

The canonical cocycle functor

$$EH \times_H ? : H - \mathbf{tors} \to h(*, BH)$$

has a left adjoint, and so it induces a weak equivalence

$$B(H - \mathbf{tors}) \xrightarrow{\simeq} Bh(*, BH).$$

This implies the following:

Theorem 8. *There are bijections*

$$[*, BH] \cong \pi_0 h(*, BH) \cong \pi_0(H - \mathbf{tors}) =: H^1(\mathcal{C}, H).$$

Remark 9. There is a smallness issue here. In principle, the cocycle category $h(X, Y)$ might not be small, but it's possible to assume that all cocycles

$$X \xleftarrow{\simeq} U \to Y$$

involve objects U of bounded size (depending on the size of X, Y and $\mathcal{C}$). We'll just assume this.

Relation with the function complex

Write we/X for the category whose objects are all weak equivalences $U \to X$, and whose morphisms are the commutative diagrams

$$\begin{tikzcd} U \arrow[d] \arrow[dr, "\simeq"] & \\ U' \arrow[r, "\simeq"'] & X \end{tikzcd}$$

Suppose that Y is fibrant and consider the functor

$$\mathbf{hom}(\ ,Y) : (we/X)^{op} \to s\mathbf{Set}$$

which is defined by

$$U \xrightarrow{\simeq} X \mapsto \mathbf{hom}(U,Y).$$

There is a canonical map

$$\underrightarrow{\mathrm{holim}}_{U \xrightarrow{\simeq} X} \mathbf{hom}(U,Y) \to B(we/X)^{op}.$$

Then we/X has a terminal object, namely 1_X so $B(we/X)^{op}$ is contractible, while the diagram $\mathbf{hom}(\ ,Y)$ is a diagram of equivalences since Y is fibrant. It follows (Quillen's Theorem B) that the canonical map

$$\mathbf{hom}(X,Y) \to \underrightarrow{\mathrm{holim}}_{U \xrightarrow{\simeq} X} \mathbf{hom}(U,Y)$$

is a weak equivalence. At the same time, the horizontal simplicial set

$$\underrightarrow{\mathrm{holim}}_{U \xrightarrow{\simeq} X} \mathbf{hom}(U,Y)_n$$

is the nerve of the cocycle category $h(X, Y^{\Delta^n})$ and is therefore weakly equivalent to $h(X, Y)$, for all n. This means that the canonical map

$$Bh(X, Y) \to \underrightarrow{\mathrm{holim}}_{\,U \xrightarrow{\simeq} X} \mathbf{hom}(U, Y)$$

is a weak equivalence. We have proved the following:

Theorem 10. *Suppose that Y is fibrant and that X is cofibrant. Then the canonical maps*

$$Bh(X, Y) \to \underrightarrow{\mathrm{holim}}_{\,U \xrightarrow{\simeq} X} \mathbf{hom}(U, Y) \leftarrow \mathbf{hom}(X, Y)$$

are weak equivalences.

Theorem 11. *Suppose that Y is locally fibrant and that $j : Y \to Z$ is an injective fibrant model in simplicial presheaves. Then the maps*

$$\begin{array}{ccc} Bh(X, Y) \longrightarrow \underrightarrow{\mathrm{holim}}_{\,U \xrightarrow{\simeq} X} \mathbf{hom}(U, Y) & & \\ \downarrow & & \\ \underrightarrow{\mathrm{holim}}_{\,U \xrightarrow{\simeq} X} \mathbf{hom}(U, Z) & \longleftarrow & \mathbf{hom}(X, Z) \end{array}$$

are weak equivalences.

A **local fibration** is a map $X \to Y$ such that $X_x \to Y_x$ is a Kan fibration in all stalks. Alternatively, the presheaf maps

$$\hom(\Delta^n, X) \to \hom(\Lambda^n_k, X) \times_{\hom(\Lambda^n_k, Y)} \hom(\Delta^n, Y)$$

are local epimorphisms.

Every presheaf of Kan complexes X (eg. BG for presheaves of groups G) is locally fibrant but not necessarily injective fibrant. Every injective fibrant simplicial presheaf is locally fibrant.

The Verdier hypercovering theorem

Write $h_\pi(X,Y)$ for the full subcategory of $h(X,Y)$ whose objects are the cocycles

$$X \xleftarrow{p} U \xrightarrow{f} Y$$

with p a hypercover (aka. local fibration plus local weak equivalence). Here's the modern version of the Verdier hypercovering theorem:

Theorem 12. *Suppose that Y is locally fibrant. Then the inclusion $h_\pi(X,Y) \subset h(X,Y)$ induces a weak equivalence*

$$Bh_\pi(X,Y) \xrightarrow{\simeq} Bh(X,Y).$$

Proof. Suppose that $(g,f) : U \to X \times Y$ is a cocycle (g is a weak equivalence), and find a (functorial) factorization

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "(g,f)"'] & Z \arrow[d, "(p,q)"] \\ & X \times Y \end{tikzcd}$$

with j a trivial cofibration and (p,q) an injective fibration. Then p, or the composite

$$Z \xrightarrow{(p,q)} X \times Y \xrightarrow{pr} X$$

is a weak equivalence, and is a local fibration since Y is locally fibrant. The assignment $(g,f) \mapsto (p,q)$ defines a functor $h(X,Y) \to h_\pi(X,Y)$ which is inverse to the inclusion $h_\pi(X,Y) \subset h(X,Y)$ up to homotopy specified by the cofibrations j. $\square$

Write π/X for the category whose objects are the hypercovers $p : U \to X$, and whose morphisms are the commutative diagrams

$$\begin{tikzcd} U \arrow[dr, "p"] \arrow[dd] & \\ & X \\ U' \arrow[ur, "p'"'] & \end{tikzcd}$$

Write $\pi(U,Y)$ for simplicial homotopy classes of maps $U \to Y$, and let $[f]$ be the simplicial homotopy class of a map f.

Corollary 13 (Verdier hypercovering theorem)**.** *Suppose that Y is locally fibrant. Then the canonical map*

$$\varinjlim_{[p]:U \to X} \pi(U,Y) \to [X,Y]$$

is a bijection.

Proof. The displayed colimit is the set of path components of the category $h_V(X, Y)$ whose morphisms have the form

$$\begin{tikzcd}[column sep=small, row sep=small]
& U \arrow[dl, "{[p]}"'] \arrow[dr, "{[f]}"] \arrow[dd, "{[\theta]}"] & \\
X & & Y \\
& U' \arrow[ul, "{[p']}"] \arrow[ur, "{[f']}"'] &
\end{tikzcd}$$

There is an obvious functor $h_\pi(X, Y) \to h_V(V, Y)$ which is surjective on objects. The composite of the functions

$$\pi_0 h_f(X, Y) \to \pi_0 h_V(X, Y) \to [X, Y]$$

is the canonical bijection. $\square$

There are variants of $h_V(X, Y)$, and here's one of them:

$$h'_V(X, Y) : \qquad \begin{tikzcd}[column sep=small, row sep=small]
& U \arrow[dl, "p"'] \arrow[dr, "{[f]}"] \arrow[dd, "{[\theta]}"] & \\
X & & Y \\
& U' \arrow[ul, "p'"] \arrow[ur, "{[f']}"'] &
\end{tikzcd}$$

where $[\theta]$ denotes a fibre homotopy class of maps.

Corollary 14. *The canonical function*

$$\pi_0 h'_V(X, Y) \to [X, Y]$$

is a bijection if Y is locally fibrant.

Remark 15. It has, up to now, been a big deal that there is no sort of fibrancy assumption on X in these results.

Here's the function complex version of the Verdier hypercovering theorem (Theorem 12):

Corollary 16. *Suppose that* Y *is locally fibrant and that* $j : Y \to Z$ *is an injective fibrant model of* Y*. Then there are weak equivalences*

$$\begin{tikzcd} \underrightarrow{\mathrm{holim}}_{U \xrightarrow{p} X} \mathbf{hom}(U, Y) \arrow[r, "\simeq"] & \underrightarrow{\mathrm{holim}}_{U \xrightarrow{p} X} \mathbf{hom}(U, Z) \\ & \mathbf{hom}(X, Z) \arrow[u, "\simeq"'] \end{tikzcd}$$

where the homotopy colimits are indexed by the hypercovers $p : U \to X$*.*

Proof. All functors

$$h_\pi(X, Y^{\Delta^n}) \to h_\pi(X, Z^{\Delta^n})$$

are weak equivalences by the hypercovering theorem, and the category π/X has a terminal object 1_X $\square$