Cocycles and cocycle categories

Short exact sequences: abelian kernels

Suppose that

$$E: \quad e \to A \to G \xrightarrow{p} H \to e$$

is a short exact sequence of groups with A abelian.

Choose a section $\sigma : H \to G$ of p such that $\sigma(e) = e$. Then we find the following:

1) σ induces a homomorphism $p_* : H \to \mathrm{Aut}(A)$ (an action of H on A), defined by

$$p_*(h)(a) = \sigma(h)a\sigma(h)^{-1}.$$

The morphism (action) p_* independent of the choice of σ.

Fix a choice of action $\gamma : H \to \mathrm{Aut}(A)$: consider only those extensions such that $p_* = \gamma$.

2) $p(\sigma(xy)) = p(\sigma(x)\sigma(y))$, so that

$$c(x,y)\sigma(x,y) = \sigma(x)\sigma(y)$$

for some unique $c(x,y) \in A$. $c(x,y)$ is a morphism

$$\sigma(xy) \xrightarrow{c(x,y)} \sigma(x)\sigma(y)$$

in the translation category $E_G A$.

3) Here's a picture:

$$\begin{tikzcd}
\sigma(xyz) \arrow[r, "c(xy,z)"] \arrow[d, "c(x,yz)"'] & \sigma(xy)\sigma(z) \arrow[d, "c(x,y)"] \\
\sigma(x)\sigma(yz) \arrow[r, "\sigma(x)c(y,z)\sigma(x)^{-1}"'] & \sigma(x)\sigma(y)\sigma(z)
\end{tikzcd}$$

To put it a different way, there is an identity

$$x \cdot c(y,z) - c(xy,z) + c(x,yz) - c(x,y) = 0$$

so that the function

$$c : H \times H \to A$$

defines a 2-cocycle in the chain complex $\hom_H(EH, A)$ determined by the action γ, and hence defines an element in

$$[E] = [c] \in H^2(H, A)$$

associated to the H-module A.

4) This element $[E]$ is independent of the "class" of E. If there is a picture of group homomorphisms

$$\begin{tikzcd}
 & G \arrow[dr, "p"] \arrow[dd, "\theta"', "\cong"] & \\
A \arrow[ur] \arrow[dr] & & H \\
 & G' \arrow[ur, "p'"'] &
\end{tikzcd}$$

and σ and σ' are normalized sections for p, p', then $p_* = p'_* : H \to \mathrm{Aut}(A)$ and $\sigma'(x) = h(x)\theta\sigma(x)$ for a unique function $h : H \to A$ such that $h(e) = 0$. There is a diagram

$$\begin{tikzcd}
\theta\sigma(xy) \arrow[rr, "h(xy)"] \arrow[d, "c(x,y)"'] & & \sigma'(xy) \arrow[d, "c'(x,y)"] \\
\theta\sigma(x)\theta\sigma(y) \arrow[r, "h(x)"'] & \sigma'(x)\theta\sigma(y) \arrow[r, "x\cdot h(y)"'] & \sigma'(x)\sigma'(y)
\end{tikzcd}$$

In particular,

$$c'(x,y) = c(x,y) + x \cdot h(y) - h(xy) + h(x)$$

so that $c'(x,y) - c(x,y)$ is a coboundary. We have therefore found a function

$$\phi : \mathrm{Ext}_\gamma(H,A) \to H^2(H,A).$$

where $\mathrm{Ext}_\gamma(H,A)$ is the set of equivalence classes of those extensions with corresponding action γ.

5) The function ϕ is surjective: any cocycle $c(x,y)$ determines a group law on $A \times H$, with

$$(a,x) \cdot (b,y) = (a(x \cdot b)c(x,y), xy).$$

The projection $(a,x) \mapsto x$ defines a group homomorphism with normalized section $\sigma(x) = (e,x)$, and the 2-cocycle associated to σ is c.

6) If $h : H \to A$ is a normalized chain such that

$$c'(x,y)h(xy) = (x \cdot h(y))h(x)c(x,y)$$

(ie. if $c - c' = \delta(h)$) then the assignment

$$(a, x) \mapsto (ah(x), x)$$

defines a homomorphism $A \times H \to A \times H$ from the group structure determined by c' to the group structure determined by c which respects inclusion of A and projection onto H. In effect,

$$\begin{aligned}(ah(x), x)(bh(y), y) &= (ah(x)(x \cdot (bh(y))c(x, y), xy) \\ &= (a(x \cdot b)h(x)(x \cdot h(y))c(x, y), xy) \\ &= (a(x \cdot b)c'(x, y)h(xy), xy).\end{aligned}$$

We have proved

Theorem: There is a bijection

$$H^2(H, A) \cong \mathrm{Ext}_\gamma(H, A)$$

Some comments:

1) $H^2(H, A)$ is H-equivariant homotopy classes of maps $[*, K(A, 2)]$ so that this bijection gives a homotopy classification of $\mathrm{Ext}_\gamma(H, A)$.

2) There is a category $\mathbf{Ext}_\gamma(H, A)$ whose objects are the short exact sequences E as above with induced morphism $\gamma : H \to \mathrm{Aut}(A)$, and whose morphisms are the commutative diagrams of group

homomorphisms

$$\begin{tikzcd}
e \arrow[r] & A \arrow[r] \arrow[d, "1"'] & G \arrow[r, "p"] \arrow[d, "\theta"] & H \arrow[r] \arrow[d, "1"] & e \\
e \arrow[r] & A \arrow[r] & G' \arrow[r, "p'"'] & H \arrow[r] & e
\end{tikzcd}$$

$\mathbf{Ext}_\gamma(H, A)$ is a groupoid, and the Theorem says that there is a bijection

$$H^2(H, A) \cong \pi_0 \mathbf{Ext}_\gamma(H, A).$$

Short exact sequences: non-abelian kernels

Much of the foregoing does not work if the kernel is non-abelian. Suppose given a short exact sequence

$$E: \quad e \to K \to G \xrightarrow{p} H \to e$$

with K non-abelian. Then first of all, $H^2(H, K)$ doesn't make any sense.

The automorphisms $\mathrm{Aut}(K)$ of the kernel obviously have to figure into the story, but we have to take a more sophisticated approach.

1) $\mathbf{Aut}(K)$ is the 2-groupoid of automorphisms of K and their homotopies.

There is one 0-cell of $\mathbf{Aut}(K)$, denoted by $*$, the

1-cells $g : * \to *$ are the automorphisms of K, and the 2-cells are the homotopies of automorphisms.

Explicitly, a 2-cell $a : g \to g'$ is an element $a \in K$ such that

$$ag(x)a^{-1} = g'(x), \text{ for all } x \in K.$$

Think of the picture

$$\begin{tikzcd} * \arrow[r, "g(x)"] \arrow[d, "a"'] & * \arrow[d, "a"] \\ * \arrow[r, "g'(x)"'] & * \end{tikzcd}$$

$\mathbf{Aut}(K)$ has a "horizontal" law of composition for 2-cells: given $b : h \to h'$, then

$$bh(a)h(g(x))h(a)^{-1}b^{-1} = bh(g'(x))b^{-1} = h'g'(x),$$

so the composite 2-cell is $bh(a) : hg \to h'g'$.

2) From the short exact sequence E

$$e \to K \to G \xrightarrow{p} H \to e$$

one finds a 2-groupoid $\tilde{p}$ with one 0-cell $*$ and a 1-cell $g : * \to *$ for every element $g \in G$. The 2-cells of $\tilde{p}$ are those pairs (g, g') such that $p(g) = p(g')$; in such a case (g, g') is a 2-cell from g to g'.

The horizontal law of composition for 2-cells in $\tilde{p}$ is the only thing that it could be.

3) We also have associated to E a canonical picture of 2-groupoid morphisms

$$\begin{tikzcd} \tilde{p} \arrow[r, "F_E"] \arrow[d, "\pi_E"', "\simeq"] & \mathbf{Aut}(K) \\ H & \end{tikzcd}$$

$\pi_E : \tilde{p} \to H$ sends 1-cells $g : * \to *$ to $p(g) : * \to *$ and sends 2-cells (g, g') to identity 2-cells $(p(g), p(g'))$. π_E is a weak equivalence since all homotopy fibres are trivial groupoids.
The 2-groupoid morphism $F_E : \tilde{p} \to \mathbf{Aut}(K)$ sends a 1-cell $g : * \to *$ to conjugation by g: $c_g : x \mapsto gxg^{-1}$, and sends the 2-cell (g, g') to the homotopy $c_{g'g^{-1}} : c_g \to c_{g'}$ defined by conjugation by $g'g^{-1} \in K$.

This picture is a canonical example of a cocycle in the category of 2-groupoids which is defined on H and takes values in the object $\mathbf{Aut}(K)$.

Cocycles

Suppose that X, Y are spaces.

$H(X,Y)$ = category whose objects are all pairs of maps (f,g)

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

where f is a weak equivalence. A morphism $\alpha : (f,g) \to (f',g')$ of $H(X,Y)$ is a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\alpha"] & \\ X & & Y \\ & Z' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

$H(X,Y)$ is the **category of cocycles** from X to Y.

$\pi_0 H(X,Y)$ = class of path components of $H(X,Y)$. There is a function

$$\phi : \pi_0 H(X,Y) \to [X,Y] \quad (f,g) \mapsto g \cdot f^{-1}$$

Theorem: The canonical map $\phi : \pi_0 H(X,Y) \to [X,Y]$ is a bijection for all X and Y.

Lest you think that I've done away with homotopy theory, suppose that $f \simeq g : X \to Y$. Then there is a picture

$$\begin{tikzcd}
& X \arrow[dl, "1"'] \arrow[d, "d_0"'] \arrow[dr, "f"] & \\
X & X \times I \arrow[l, "pr"'] \arrow[r, "h"] & Y \\
& X \arrow[ul, "1"] \arrow[u, "d_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

where h is the homotopy. Then

$$(i_X, f) \sim (pr, h) \sim (1_X, g)$$

Thus $f \mapsto [(1_X, f)]$ defines a function

$$\psi : \pi(X, Y) \to \pi_0 H(X, Y)$$

If X has the good manners to be cofibrant, then the function ψ is inverse to ϕ. More generally, there are a couple of things to prove.

The Theorem holds in extreme generality, specifically in any model category which is right proper (weak equivalences pull back to weak equivalences along fibrations), and such that weak equivalences are closed under finite products.

Examples: spaces, simplicial sets, presheaves of simplicial sets, spectra, presheaves of spectra, any localizations of such, and ... 2-groupoids

Example: Suppose that A is a non-degenerate symmetric bilinear form over $\mathbb{Q}$ (or any field of characteristic $\neq 2$). Then up to isomorphism of forms over $\mathbb{Q}$, A is diagonal, so we can assume that A is a non-singular diagonal matrix, such as

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

A form represented by a diagonal matrix is isomorphic to the trivial form over a field $L/\mathbb{Q}$ if all the entries have square roots in L. Thus there is a finite Galois extension $L/\mathbf{Q}$ with Galois group G such that there is an isomorphism $B : 1_n \to A$ of forms: in other words there is a non-singular matrix B over L such that $B^T B = A$. In the case above

$$B = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & i \end{bmatrix}$$

does the trick. The matrices $g(B)$, $g \in G$, also define isomorphisms of forms $g(B) : 1_n \to A$, since $g(B)^T g(B) = A$. The composite isomorphisms

$$1_n \xrightarrow{g(B)^{-1}} A \xrightarrow{B} 1_n$$

are therefore automorphisms of the trivial form, and are therefore in the group $O_n(L)$. The assignment

$$f(g) = Bg(B)^{-1}$$

defines a cocycle $f : G \to O_n(L)$ in a traditional sense:

- $f(e) = I_n$, and
- $f(gh) = Bgh(B)^{-1} = Bg(B)^{-1}g(B)gh(B)^{-1} = (Bg(B)^{-1})g(Bh(B)^{-1}) = f(g)g(f(h))$.

The function f can be viewed as a morphism of simplicial schemes

$$\begin{array}{ccc} EG \times_G \mathrm{Sp}(L) & \xrightarrow{f} & BO_n \\ \simeq \downarrow & & \\ \mathrm{Sp}(\mathbb{Q}) & & \end{array}$$

The Borel construction $EG \times_G \mathrm{Sp}(L)$ is the Čech resolution associated to the étale cover $\mathrm{Sp}(L) \to \mathrm{Sp}(\mathbb{Q})$, and the vertical map determines a weak equivalence of simplicial sheaves for the étale topology on $\mathrm{Sp}(\mathbb{Q})$.

Theorem: Suppose that K is a field of characteristic $\neq 2$. Then there are natural bijections

$$\begin{aligned} &\pi_0(\text{non-deg. symm. bil. forms and isomorphisms}) \\ &\cong \pi_0 H(*, BO_n) \cong [*, BO_n] \end{aligned}$$

Back to group theory;

Associating the cocycle

$$\begin{tikzcd} \tilde{p} \arrow[r, "F_E"] \arrow[d, "\pi_E"', "\simeq"] & \mathbf{Aut}(K) \\ H \end{tikzcd}$$

to a short exact sequence

$$e \to K \to G \xrightarrow{p} H \to e$$

defines a functor

$$\mathbf{Ext}(H, K) \to H_*(H, \mathbf{Aut}(K))$$

(NB: $\theta(g)x\theta(g)^{-1} = \theta(gxg^{-1}) = gxg^{-1}$ for $x \in K$) where the cocycles live in pointed 2-groupoids.

Theorem: This functor induces a bijection

$$\begin{aligned} \pi_0\mathbf{Ext}(H, K) &\cong \pi_0 H_*(H, \mathbf{Aut}(K)) \\ &\cong [H, \mathbf{Aut}(K)]_* \\ &\cong [BH, B\mathbf{Aut}(K)]_*. \end{aligned}$$

Remark: A cocycle

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "\pi"', "\simeq"] & \mathbf{Aut}(K) \\ H \end{tikzcd}$$

determines a unique homomorphism ("band") $H \to \mathrm{Out}(K)$, which does not vary with refinement of pointed cocycles.

Sketch Proof: Given a pointed cocycle

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "\pi"', "\simeq"] & \mathbf{Aut}(K), \\ H \end{tikzcd}$$

the base point $x \in A_0$ determines a 2-groupoid homomorphism $A(x,x) \to H$ which is surjective on 1-cells. In fact, the 2-cells of $A(x,x)$ are the pairs $g, h : x \to x$ such that $\pi(g) = \pi(h)$. The cocycle F can therefore be canonically replaced by its restriction to $A(x,x)$ at the base point x, and the 2-groupoid $A(x,x)$ can be canonically identified with a 2-groupoid p_* arising from a surjective group homomorphism $p : L \to H$ with 2-cells consisting of pairs (g,h) such that $p(g) = p(h)$. L is the group of 1-cells of $A(x,x)$.