Cosimplicial spaces and cocycles

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The finite sets

$$\mathbf{n} = \{0, 1, \dots, n\}, \ \ n \geq 0,$$

and the order-preserving functions $\theta : \mathbf{m} \to \mathbf{n}$ define a category $\mathbf{\Delta}$, called the **ordinal number category**.

A functor $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$ (contravariant on $\mathbf{\Delta}$) is called a **simplicial set**.

"Space" = "simplicial set".

A cosimplicial object in $\mathcal{C}$ is a covariant functor $\mathbf{\Delta} \to \mathcal{C}$. A **cosimplicial space** is a cosimplicial object in spaces (simplicial sets).

Examples

1) If X and Y are simplicial sets then

$$\mathbf{n} \mapsto \hom(X_n, Y)$$

defines a cosimplicial space, used to analyze the function complex $\mathbf{hom}(X, Y)$.

2) If X is a simplicial scheme and Y is a simplicial sheaf, then

$$\mathbf{n} \mapsto \hom(X_n, Y) = Y(X_n)$$

is a cosimplicial space, used to analyze function complexes in simplicial sheaves.

Example: if $U \to *$ is a hypercover then whether or not the map

$$Y(*) \to \mathbf{hom}(U, Y)$$

is a weak equivalence is a type of descent question for Y.

Examples 2

3) Given an adjoint pair of functors

$$F : s\mathbf{Set} \leftrightarrows s\mathbf{Set} : G,$$

then iterates of the composite GF and the canonical adjunction maps define a cosimplicial space

$$GF(X) \underset{\leftarrow}{\rightrightarrows} GFGF(X) \dots$$

which is often a resolution of X along the unit map $\eta : X \to GF(X)$. The same applies for sheaf or presheaf categories: the *Godement resolution* is an example.

4) If $X : I \to s\mathbf{Set}$ is a (small) diagram of spaces then the *homotopy inverse limit* is usually constructed from a cosimplicial space with

$$\mathbf{n} \mapsto \prod_{i_0 \to \cdots \to i_n} X(i_n).$$

Examples 3

5) The finite ordinal numbers $\mathbf{n}$ are posets, hence categories, and there is a cosimplicial space Δ with

$$\mathbf{n} \mapsto B(\mathbf{n}) = \Delta^n.$$

This is a fat point in cosimplicial spaces: all Δ^n are contractible.

Cosimplicial spaces were introduced by Bousfield and Kan [1] in the early 1970s, as a technical device in their construction of homology completions of spaces. The homotopy theory of cosimplicial spaces was was an early application of Quillen model structures.

Model structures

We will discuss two model structures on cosimplicial spaces:

1) **Bousfield-Kan structure**: the weak equivalences are maps $X \to Y$ such that $X^n \to Y^n$ is a weak equivalence of spaces for all $n \geq 0$; the cofibrations $A \to B$ are levelwise monomorphisms which induce isomorphisms $\varprojlim_n A^n \cong \varprojlim_n B^n$.

2) **injective structure**: the weak equivalences are defined levelwise; the cofibrations are levelwise monomorphisms.

Fibrations for both structures are defined by a right lifting property with respect to trivial cofibrations. The fibrations for the Bousfield-Kan structure are the **Bousfield-Kan fibrations** and the fibrations for the injective structure are the **injective fibrations**.

Fact: Every injective fibration is a Bousfield-Kan fibration.

History

Injective model structures for diagram categories (like cosimplicial spaces) were introduced by Alex Heller before the advent of simplicial sheaf homotopy theory (Joyal's letter to Grothendieck and the "Simplicial presheaves" paper), but most applications have been in simplicial sheaves and presheaves.

Injective structures are essentially sheaf theoretic.

Suppose that A is a cosimplicial space.

Bousfield and Kan write

$$A^{-1} = \varprojlim_n A^n,$$

and call it the *maximal augmentation* of A.

From a sheaf-theoretic point of view this object is the space $\Gamma_* A$ of *global sections* of A.

Fact: The inverse limit $\varprojlim_n A^n$ is the equalizer of the structure maps

$$d^0, d^1 : A^0 \rightrightarrows A^1.$$

These maps are induced by ordinal number maps $d^0, d^1 : \mathbf{0} \to \mathbf{1}$ which pick out the numbers 1 and 0 respectively.

Homotopy inverse limits

The applications of the homotopy theory of cosimplicial spaces are all about derivations of the inverse limit, aka. global sections. Also for the homotopy theory of simplicial sheaves.

Homotopy inverse limits are easily constructed from injective model structures on diagrams.

Example: if X is a cosimplicial space, it has an injective fibrant model $j : X \to Y$ (weak equivalence with Y injective fibrant), and then the **homotopy inverse limit** of X is defined by

$$\underleftarrow{\mathrm{holim}}_{\,\mathbf{\Delta}} X = \varprojlim_{n} Y^n.$$

Classically, $\underleftarrow{\mathrm{holim}}_{\,\mathbf{\Delta}} X = \mathbf{hom}(B(\mathbf{\Delta}/?), X)$, is same: $B(\mathbf{\Delta}/?) \to *$ is weak equiv.

The *Tot* complex

X is Bousfield-Kan fibrant. Then Bousfield and Kan define

$$\mathrm{Tot}(X) := \mathbf{hom}(\Delta, X),$$

in terms of the standard diagram-theoretic function complex:

$$\mathbf{hom}(\Delta, X)_n = \mathrm{hom}(\Delta \times \Delta^n, X).$$

Let $j : X \to Y$ be an injective fibrant model for X. Then Y is Bousfield-Kan fibrant and the maps

$$\mathbf{hom}(\Delta, X) \to \mathbf{hom}(\Delta, Y) \leftarrow \mathbf{hom}(*, Y) = \varprojlim_n Y^n$$

are weak equivalences.

Lemma

There is a natural weak equivalence

$$\mathrm{Tot}(X) \simeq \underleftarrow{\mathrm{holim}}_{\mathbf{\Delta}} X$$

for Bousfield-Kan fibrant objects X.

Here's another fact which is imported from sheaf theory:

Lemma

Suppose that X is a cosimplicial set. Then the corresponding cosimplicial space is injective fibrant.

Proof: Suppose that $A \to B$ is a trivial cofibration. Then the map $\pi_0 A \to \pi_0 B$ of cosimplicial sets is an isomorphism, and the lift exists in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \pi_0 A \arrow[r] \arrow[d] & X \\ B \arrow[r] & \pi_0 B \arrow[ur, dotted] & \end{tikzcd}$$

Analog: Suppose that F is a sheaf, identified with a simplicial sheaf. Then F is injective fibrant.

Cocycles

Consequence: There are injective fibrant cosimplicial spaces with empty inverse limit, eg. $\Delta_0 = \{\Delta_0^n\}$.

This can be a bit disconcerting, but there's a workaround which involves cocycles.

A **cocycle** is a diagram

$$* \xleftarrow[\simeq]{g} U \xrightarrow{f} X$$

in cosimplicial spaces, and a **morphism of cocycles** is a diagram

$$\begin{tikzcd} & U \arrow[dl, "\simeq"'] \arrow[dr] \arrow[dd] & \\ * & & X \\ & V \arrow[ul, "\simeq"] \arrow[ur] & \end{tikzcd}$$

The corresponding category is called the **cocycle category** (from $*$ to X), and is denoted by $h(*, X)$.

Theorem

The assigment

$$[* \xleftarrow[\simeq]{g} U \xrightarrow{f} X] \mapsto fg^{-1}$$

defines a bijection

$$\pi_0 h(*, X) \cong [*, X].$$

Corollary

The space $\varprojlim_{\Delta} X$ *is non-empty if and only if* X *has a cocycle*

$$* \xleftarrow[\simeq]{g} U \xrightarrow{f} X.$$

Proof: Suppose that $j : X \to Y$ is an injective fibrant model. Then

$$\pi_0 h(*, X) \cong \pi_0 h(*, Y) \cong [*, Y],$$

which is homotopy classes of maps $* \to \varprojlim_{\mathbf{\Delta}} Y$ since Y is injective fibrant.

Theorem

Suppose that the cosimplicial space Y is a diagram of Kan complexes, let $Y \to Z$ be an injective fibrant model. There is a weak equivalence $Bh(X,Y) \simeq \mathbf{hom}(X,Z)$.

we/X is the category of weak equivalences $U \xrightarrow{\simeq} X$.
There is a fibre sequence

$$\mathbf{hom}(X,Z) \to \underrightarrow{\mathrm{holim}}_{\,U \xrightarrow{\simeq} X} \mathbf{hom}(U,Z) \to B(we/X)^{op} \simeq *.$$

The map

$$\underrightarrow{\mathrm{holim}}_{\,U \xrightarrow{\simeq} X} \mathbf{hom}(U,Y) \to \underrightarrow{\mathrm{holim}}_{\,U \xrightarrow{\simeq} X} \mathbf{hom}(U,Z)$$

is a weak equivalence, since all maps

$$Bh(X,Y) \to Bh(X, Y^{\Delta^n}) \to Bh(X, Z^{\Delta^n})$$

are weak equivalences.

Cosimplicial groupoids

Cosimplicial spaces are everywhere, as are cosimplicial groupoids: for a cosimplicial space X the assignment

$$\mathbf{n} \mapsto \pi(X^n)$$

(fundamental groupoid of X^n) defines a cosimplicial groupoid $\pi(X)$, called the **fundamental groupoid** of X.

Sheaf theory suggests that one controls the homotopy type of a cosimplicial groupoid H with the cosimplicial space BH: a map $G \to H$ is a **weak equivalence** (respectively **fibration**) if the map $BG \to BH$ is a weak equivalence (respectively injective fibration) of cosimplicial spaces.

These definitions give the category of cosimplicial groupoids a model structure.

Not every cosimplicial groupoid H has a Bousfield-Kan fibrant classifying space BH (by a calculation). Thus, not all cosimplicial groupoids are injective fibrant.

The cosimplicial groupoids H which are fibrant for the injective model structure above are special — they are called **stacks**, and an injective fibrant model $G \to H$ is called a **stack completion** of G.

Torsors

Suppose that H is a cosimplicial groupoid.

An H-**diagram** consists of diagrams $X_n : H^n \to \mathbf{Set}$, $n \geq 0$, which fit together along the cosimplicial structure maps of H.

Alternatively, X consists of a cosimplicial set map $\pi : X \to Ob(H)$ together with an H-action

$$\begin{tikzcd} Mor(H) \times_s X \arrow[r, "m"] \arrow[d] & X \arrow[d, "\pi"] \\ Mor(H) \arrow[r, "t"'] & Ob(H) \end{tikzcd}$$

which is associative and respects identities.

An H-diagram X determines a map $\underrightarrow{\text{holim}}_H X \to BH$.

The diagram X is an H-**torsor** if the map

$$\underrightarrow{\text{holim}}_H X \to *$$

is a weak equivalence.

Torsors 2

Maps of H-torsors are natural transformations of H-diagrams.

All such maps are isomorphisms since the diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \underrightarrow{\operatorname{holim}}_H X \arrow[d] \\ \mathrm{Ob}(H) \arrow[r] & BH \end{tikzcd}$$

are homotopy cartesian. $H-\textbf{tors}$ denotes the resulting groupoid.

Every H-torsor X has a **canonical cocycle**

$$* \xleftarrow{\simeq} \underrightarrow{\operatorname{holim}}_H X \to BH,$$

and there is a canonical cocycle functor $H-\textbf{tors} \to h(*, BH)$.

Fact: The canonical cocycle functor has a left adjoint.

Non-abelian H^1

Theorem

Suppose that H is a cosimplicial groupoid. There is a weak equivalence

$$B(H - \mathbf{tors}) \simeq Bh(*, BH),$$

and so there is an isomorphism

$$H^1(\mathbf{\Delta}, H) := \pi_0(H - \mathbf{tors}) \cong [*, BH].$$

Of course, BH might not have cocycles and so there might not be any H-torsors.

Theorem

There is a weak equivalence

$$s : Bh(*, BH) \to \mathbf{hom}(\Delta, BH)$$

for each cosimplicial groupoid H.

Corollary

There is an isomorphism

$$\pi_0\mathbf{hom}(\Delta, BH) \cong [*, BH]$$

for each cosimplicial groupoid H.

In particular, $[*, BH]$ can be identified with naive homotopy classes of maps $\Delta \to BH$, whether H is fibrant or not.

$\mathbf{hom}(\Delta, BH)$ is a groupoid, weakly equivalent to global sections of the stack completion of H.

Suppose $U \simeq *$ is cosimplicial groupoid. $U^0 \neq \emptyset$ so choose vertex $v : \Delta^0 \to BU^0$. v uniquely extends to a cosimplicial groupoid map $v : \mathbf{\Delta} = \pi(\Delta) \to U$ since all groupoids U^n are trivial. Any two $v, w : \mathbf{\Delta} \to U$ are uniquely isomorphic. Pick $x_U : \mathbf{\Delta} \to U$ for all U such that $U \simeq *$.

$$\mathbf{hom}(\Delta, BH) \cong B(H^{\mathbf{\Delta}}).$$

Define $s : h(*, H) \to H^{\mathbf{\Delta}}$ by sending the cocycle

$$* \xleftarrow{\simeq} U \xrightarrow{f} H$$

to the composite

$$\mathbf{\Delta} \xrightarrow{x_U} U \xrightarrow{f} H.$$

Show that the composition

$$H - \mathbf{tors} \xrightarrow{\simeq} h(*, U) \xrightarrow{s} H^{\mathbf{\Delta}}$$

is weak equivalence of groupoids.

Abelian cohomology

Cosimplicial objects A in simplicial abelian groups are Bousfield-Kan fibrant as cosimplicial spaces.

There are isomorphisms

$$[*, A] \cong \pi_0 \mathbf{hom}(\Delta, A) \cong \pi_{ch}(N\mathbb{Z}\Delta, NA)$$

for all such A. There is an isomorphism

$$H^n(\mathbf{\Delta}, F) := [*, K(F, n)] \cong \pi_0 \mathbf{hom}(\Delta, K(F, n)) \cong H^n(F)$$

for all cosimplicial abelian groups F.

If $j : K(F, n) \to Z$ is an injective fibrant model for $K(F, n)$, then

$$\pi_j \varprojlim_{\Delta} Z \cong \begin{cases} H^{n-j}(F) & \text{if } 0 \leq j \leq n, \text{ and} \\ 0 & \text{if } j > n. \end{cases}$$

Descent spectral sequence

The descent spectral sequence for $\pi_* \underleftarrow{\text{holim}}_n X$ for a cosimplicial space X takes the form

$$E_2^{s,t} = H^s \pi_t X, \ t \geq s.$$

This is the Bousfield-Kan spectral sequence for the homotopy groups of "Tot(X)".

The descent spectral sequence for X comes from the Postnikov tower construction

$$P_1 X \leftarrow P_2 X \leftarrow \dots$$

which is applied levelwise.

Stack cohomology

There is a natural weak equivalence $P_1X \to B\pi(X)$; let $\pi(X) \to \Gamma(X)$ be a stack completion for the fundamental groupoid. Then the transition maps $P_nX \to P_{n-1}X$ are fibred over the stack completion in the sense that there is a diagram

$$\begin{tikzcd} P_nX \arrow[rr] \arrow[dr] & & P_{n-1}X \arrow[dl] \\ & B\pi(X) \arrow[d] & \\ & B\Gamma(X) & \end{tikzcd}$$

k-invariants

Form the sequence

$$P_nX \to P_{n-1}X \to P_{n-1}X/P_nX \to P_{n+1}(P_{n-1}X/P_nX)$$
$$\simeq K(\pi_nX, n+1)$$

as $\Gamma(X)$-diagrams. Then we have fibre sequences

$$P_nX \to P_{n-1}X \xrightarrow{k_n} K(\pi_nX, n+1)$$

of $\Gamma(X)$-diagrams. Taking homotopy colimits gives a homotopy cartesian diagram

$$\begin{tikzcd} P_nX \arrow[r] \arrow[d] & B\Gamma(X) \arrow[d] \\ P_{n-1}X \arrow[r, "k_{n*}"'] & \underrightarrow{\mathrm{holim}}_{\Gamma(X)} K(\pi_nX, n+1) \end{tikzcd}$$

k_n is the k-**invariant**. It defines an element in the stack coh. group

$$[P_{n-1}X, K(\pi_nX, n+1)]_{\Gamma(X)}.$$

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