

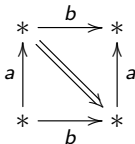
Homotopy theory and concurrency

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Basic idea (V. Pratt, 1991): represent the simultaneous execution of processors a and b as a picture (2-cell)



Simultaneous action of multiple processors is represented by higher dimensional cubes.

Restrictions on the system arising from shared resources are represented by removing cubical cells of varying dimensions, so one is left with a cubical subcomplex $K \subset \square^N$ of an N -cell, where N is the number of processors.

Higher dimensional automata

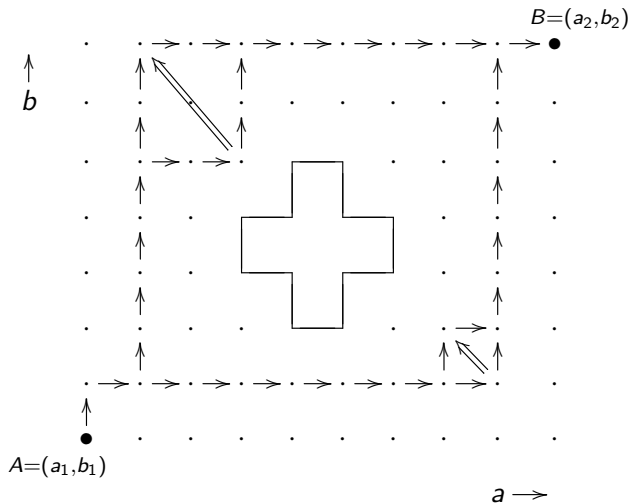
Higher dimensional automata are cubical subcomplexes $K \subset \square^N$ of the “standard” N -cell \square^N .

States are objects (vertices), and “execution paths” are morphisms of the “path category” $P(K)$.

Execution paths are equivalence classes of combinatorial paths through the complex. Executions paths between states x and y are the morphisms $P(K)(x, y)$.

Basic problem: Compute $P(K)(x, y)$.

Example: the Swiss flag



The n -**cell** \square^n is the poset

$$\square^n = \mathcal{P}(\underline{n}),$$

the set of subsets of the totally ordered set $\underline{n} = \{1, 2, \dots, n\}$.

There is a unique poset isomorphism

$$\phi : \mathcal{P}(\underline{n}) \xrightarrow{\cong} \mathbf{1}^{\times n},$$

where $\mathbf{1}$ is the 2-element poset $0 \leq 1$. Here,

$$A \xrightarrow{\phi} (\epsilon_1, \dots, \epsilon_n)$$

where $\epsilon_i = 1$ if and only if $i \in A$.

We use the ordering of \underline{n} to specify the poset isomorphism ϕ .

The box category

Suppose that $A \subset B \subset \underline{n}$. The **interval** $[A, B] \subset \mathcal{P}(\underline{n})$ is defined by

$$[A, B] = \{C \mid A \subset C \subset B\}.$$

There are canonical poset maps

$$\mathcal{P}(\underline{m}) \cong \mathcal{P}(B - A) \xrightarrow{\cong} [A, B] \subset \mathcal{P}(\underline{n}).$$

where $m = |B - A|$. These compositions are the **coface** maps $d : \square^m \subset \square^n$.

There are also **co-degeneracy** maps $s : \square^n \rightarrow \square^r$, determined by subsets $A \subset \underline{n}$, where $|A| = r$, and such that $s(B) = B \cap A$.

The cofaces and codegeneracies are the generators for the **box category** \square consisting of the posets \square^n , $n \geq 0$, subject to the cosimplicial identities.

Cubical sets

A **cubical set** is a functor $X : \square^{op} \rightarrow \mathbf{Set}$.

$\square^n \mapsto X_n$, and X_n is the set of **n -cells** of X .

The collection of all such functors and natural transformations between them is the category **cSet** of cubical sets.

Examples

1) The **standard n -cell** \square^n is the functor $\mathrm{hom}(_, \square^n)$ represented by $\square^n = \mathcal{P}(\underline{n})$ on the box category \square .

The n -cells of a cubical set X can be identified with maps $\sigma : \square^n \rightarrow X$.

2) Deleting the top cell from \square^n gives the **boundary** $\partial \square^n$.

There are 2 maximal faces of $\partial \square^n$ for each $i \in \underline{n}$: $[\{i\}, \underline{n}]$, $[\emptyset, \{1, \dots, \hat{i}, \dots, n\}]$.

3) The **cubical horn** $\square_{(i, \epsilon)}^n$ is defined by deleting a top dim. face from $\partial \square^n$.

Higher dimensional automata

A **finite cubical complex** is a subcomplex $K \subset \square^n$. It is completely determined by cells

$$\square^r \subset K \subset \square^n$$

where the composites are cofaces.

Equivalently, K is a set of intervals $[A, B] \subset \mathcal{P}(\underline{n})$ which is closed under taking subintervals.

A cell (interval) is **maximal** if $r = |B - A|$ is maximal wrt these constraints.

Finite cubical complexes are **higher dimensional automata**.

Triangulation

There is a **triangulation functor**

$$|\cdot| : c\mathbf{Set} \rightarrow s\mathbf{Set},$$

with $|\square^n| := B(\mathcal{P}(\underline{n})) \cong B(\mathbf{1}^{\times n}) \cong (\Delta^1)^{\times n}$.

$B(C)$ is the **nerve** of a category C : $B(C)_n$ is the set

$$a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_n$$

The **triangulation** $|K|$ is defined by

$$|K| = \varinjlim_{\square^n \rightarrow K} |\square^n| = \varinjlim_{\square^n \rightarrow K} B(\mathbf{1}^{\times n}) \cong \varinjlim_{[A,B] \in K} B([A, B]).$$

NB: K, L simplicial complexes, then $K \times L$ is a simplicial complex, by finding a (compatible) total ordering on the vertices $K_0 \times L_0$.

Examples

1) $|\square^2| = B(\mathbf{1}^{\times 2}) = B(\mathcal{P}(2))$:

$$\begin{array}{ccc} (0, 1) \rightarrow (1, 1) & \{2\} \rightarrow \{1, 2\} \\ \uparrow \nearrow & \uparrow \nearrow \\ (0, 0) \rightarrow (1, 0) & \emptyset \longrightarrow \{1\} \end{array}$$

2) $|\square^1 \times \square^1|$ has 1-skeleton

$$\begin{array}{ccc} (0, 1) \rightarrow (1, 1) \\ \uparrow \nearrow \uparrow \\ (0, 0) \rightarrow (1, 0) \end{array}$$

with non-degenerate 2-cells $\mathcal{P}(\underline{2}) \rightarrow \mathcal{P}(\underline{1}) \times \mathcal{P}(\underline{1})$ given by the canonical isomorphism $\mathcal{P}(\underline{2}) \cong \mathcal{P}(\underline{1}) \times \mathcal{P}(\underline{1})$ and its twist. Thus

$$|\square^1 \times \square^1| \simeq S^2 \vee S^1$$

Standard homotopy theory

Say that a monomorphism of cubical sets is a **cofibration**.

A map $X \rightarrow Y$ of cubical sets is a **weak equivalence** if the induced map $f_* : |X| \rightarrow |Y|$ is a weak equivalence of simplicial sets.

Fibrations of cubical sets are defined by a right lifting property with respect to all trivial cofibrations.

Theorem 1.

- 1) *With these definitions the category $c\mathbf{Set}$ has the structure of a proper, closed (cubical) model category.*
- 2) *The adjoint functors*

$$| \cdot | : c\mathbf{Set} \rightleftarrows s\mathbf{Set} : S$$

define a Quillen equivalence.

The right adjoint S is the singular functor.

Path category (fundamental category)

The nerve functor $B : \mathbf{cat} \rightarrow s\mathbf{Set}$ has a left adjoint

$$P : s\mathbf{Set} \rightarrow \mathbf{cat},$$

called the **path category** functor. Most often see $\tau_1(X) = P(X)$.

$P(X)$ is the category generated by the 1-skeleton $sk_1(X)$ (a graph), subject to the relations:

- 1) $s_0(x)$ is the identity morphism for all vertices $x \in X_0$,
- 2) the triangle

$$\begin{array}{ccc} x_0 & \xrightarrow{d_2(\sigma)} & x_1 \\ & \searrow d_1(\sigma) & \downarrow d_0(\sigma) \\ & & x_2 \end{array}$$

commutes for all 2-simplices $\sigma : \Delta^2 \rightarrow X$.

Execution paths

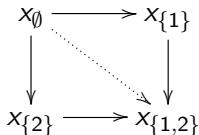
Suppose that $K \subset \square^n$ is an HDA, with states (vertices) x, y . Then

$$P(|K|)(x, y)$$

is the set of **execution paths** from x to y .

$P(K) := P(|K|)$ is the **path category** of the cubical complex K .

$P(K)$ can be defined directly for K : it is generated by the graph $sk_1(K)$, subject to the relations given by $s_0(x) = 1_x$ for vertices x , and by forcing the commutativity of



for each 2-cell $\sigma : \square^2 \subset K$ of K .

Lemma 2.

$\mathrm{sk}_2(X) \subset X$ induces $P(\mathrm{sk}_2(X)) \cong P(X)$ for simplicial sets (or cubical complexes) X .

Lemma 3.

$\epsilon : P(BC) \rightarrow C$ is an isomorphism for all small categories C .

Lemma 4.

There is an isomorphism $G(P(X)) \cong \pi(X)$ for all simplicial sets X .

$G(P(X))$ is the free groupoid on the category $P(X)$.

The path 2-category

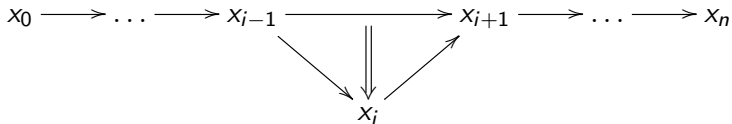
$L =$ finite simplicial complex: $P(L)$ is the path component category of a 2-category $P_2(L)$.

$P_2(L)$ consists of categories $P_2(L)(x, y)$, $x, y \in L$.

The objects (1-cells) are paths of non-deg. 1-simplices

$$x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n = y$$

of L . The morphisms of $P_2(L)(x, y)$ are composites of the pictures



where the displayed triangle bounds a non-deg. 2-simplex.

Compositions are functors

$$P_2(L)(x, y) \times P_2(L)(y, z) \rightarrow P_2(L)(x, z)$$

defined by concatenation of paths.

Theorem 5.

Suppose that L is a finite simplicial complex. Then there is an isomorphism

$$\pi_0 P_2(L) \cong P(L).$$

$\pi_0 P_2(L)$ is the **path component category** of the 2-category $P_2(L)$. Its objects are the vertices of L , and

$$\pi_0 P_2(L)(x, y) = \pi_0(BP_2(L)(x, y)).$$

Slogan: $P_2(L)$ is a “resolution” of the path category $P(L)$.

The algorithm

Here's an algorithm for computing $P(L)$ for $L \subset \Delta^N$:

- 1) Find the 2-skeleton $\text{sk}_2(L)$ of L (vertices, 1-simplices, 2-simplices).
- 2) Find all paths (strings of 1-simplices)

$$\omega : v_0 \xrightarrow{\sigma_1} v_1 \xrightarrow{\sigma_2} \dots \xrightarrow{\sigma_k} v_k$$

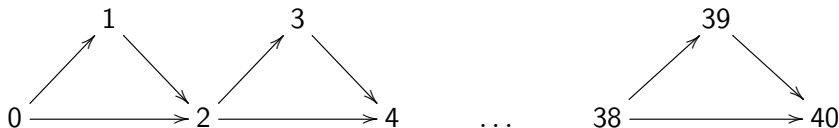
in L .

- 3) Find all morphisms in the category $P_2(L)(v, w)$ for all vertices $v < w$ in L (ordering in Δ^N).
- 4) Find the path components of all $P_2(L)(v, w)$, by approximating path components by full connected subcategories, starting with a fixed path ω .

Code: Graham Denham (Macaulay 2), Mike Misamore (C).

Example: the necklace

Let $L \subset \Delta^{40}$ be the subcomplex



This is 20 copies of the complex $\partial\Delta^2$ glued together. There are 2^{20} morphisms in $P(L)(0, 40)$.

The listing of morphisms of $P(L)$ consumes 2 GB of disk.

Moral: The size of the path category $P(L)$ can grow exponentially with L .

Complexity reduction

Suppose that $L \subset K \subset \Delta^N$.

L is a **full subcomplex** of K if the following hold:

- 1) L is path-closed in K , in the sense that, if there is a path

$$v = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_n = v'$$

in K between vertices v, v' of L , then all $v_i \in L$,

- 2) if all the vertices of a simplex $\sigma \in K$ are in L then the simplex σ is in L .

Lemma 6.

Suppose that L is a full subcomplex of K . Then the functor $P(L) \rightarrow P(K)$ is fully faithful.

ie. if $x, y \in L$ then $P(L)(x, y) = P(K)(x, y)$.

Examples

- $\partial\Delta^2 \stackrel{d^0}{\subset} \Lambda_0^3$ and $\partial\Delta^2 \stackrel{d^3}{\subset} \Lambda_3^3$ are full subcomplexes.
- Suppose that $i \leq j$ in \mathbf{N} . $K[i, j]$ is the subcomplex of K such that $\sigma \in K[i, j]$ if and only if all vertices of σ are in the interval $[i, j]$ of vertices v such that $i \leq v \leq j$. $K[i, j]$ is a full subcomplex of K .
- Suppose that $v \leq w$ are vertices of K . Let $K(v, w)$ be the subcomplex of K consisting of simplices whose vertices appear on a path from v to w . $K(v, w)$ is a full subcomplex of K .

Construct $K(v, w)$ from $K[v, w]$ by deleting *sources* and *sinks*.

A vertex v is a **source** of K if there are no 1-simplices $u \rightarrow v$ in K . (0 is a source of Λ_0^3)

A vertex z is a **sink** if there are no 1-simplices $z \rightarrow w$ in K . (3 is a sink of Λ_3^3)

Suppose that $K \subset \square^n$ is a cubical complex. Say that a vertex x is a **corner** of K if it belongs to only one maximal cell.

Lemma 7 (Misamore).

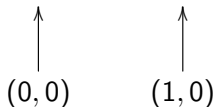
Suppose that x is a corner of K , and let K_x be the subcomplex of cells which do not have x as a vertex. Then the induced functor

$$P(K_x) \rightarrow P(K)$$

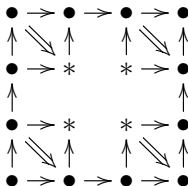
is fully faithful.

Examples

1) The cubical horn $(0, 1) \longrightarrow (1, 1)$ has a sink but no corners.



2) The Swiss flag $\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$ has 6 corners, 1 sink, 1 source.



Going beyond

- 1) The algorithm depends on having an entire HDA in storage, on a computer system that is powerful enough to analyze it.
- 2) We want local to global methods to study large (aka. “infinite”) models with patching techniques.
- 3) Nobody has any idea of what higher homotopy invariants should mean for higher dimensional automata, or even what the appropriate homotopy theory should be.

One suggestion: Joyal’s theory of quasi-categories.

$K \mapsto P(K)$ is a quasi-category invariant, but beware:

If $K \rightarrow L$ is a quasi-category weak equivalence then $P(K) \rightarrow P(L)$ is an **equivalence** of categories.

The size parameter

The assignment $A \mapsto |A|$ defines a poset map $\mathcal{P}(\underline{N}) \rightarrow \mathbb{N}$.

If $K \subset \square^N$ then the composite t

$$|K| \rightarrow B(\mathcal{P}(\underline{N})) \rightarrow B\mathbb{N}$$

$|A|$ is the number of steps the system took to reach state A , by whatever path. t is the “size parameter”.

Suppose that $m < n$ are natural numbers. $K(m, n)$ is the subcomplex of K whose cells have vertices A such that $m \leq |A| \leq n$.

Fact: $K(m, n)$ is a full subcomplex of K , so $P(K(m, n)) \rightarrow P(K)$ is fully faithful by Lemma 6.

The complexes $K(m, n)$ define a coarse “topology” on K .

Refinement

Suppose that the poset map $\alpha : \mathcal{P}(\underline{n}) \rightarrow \mathcal{P}(\underline{N})$ is a monomorphism that preserves meets and joins.

$[A, B]$ in $\mathcal{P}(\underline{n})$ determines an interval $[\alpha(A), \alpha(B)]$ in $\mathcal{P}(\underline{N})$, and α restricts to a poset monomorphism $\alpha : [A, B] \rightarrow [\alpha(A), \alpha(B)]$ (**not** a coface).

α preserves inclusion and intersections of intervals.

The subcomplex $K_\alpha \subset \mathcal{P}(\underline{N})$ is generated by the intervals $[\alpha(A), \alpha(B)]$, and there is a diagram of simp. set maps

$$\begin{array}{ccc} |K| & \xrightarrow{\alpha_*} & |K_\alpha| \\ \downarrow & & \downarrow \\ BP(\underline{n}) & \xrightarrow{\alpha} & BP(\underline{N}) \end{array}$$

K_α is a **refinement** of K .

Lemma 8.

The functor $\alpha_ : P(K) \rightarrow P(K_\alpha)$ is fully faithful.*

Frontier subcomplex

$K \subset \square^N$, finite cubical complex.

$K_0 = A \sqcup B$, with $t(v) < t(w)$ for all $v \in A$ and $w \in B$. Let A and B be the corr. full subcomplexes of K . Suppose that $P(A)$ and $P(B)$ are computed.

The **frontier subcomplex** L (of $|K|$) is generated by 1-cells and 2-cells which have vertices in both A and B .

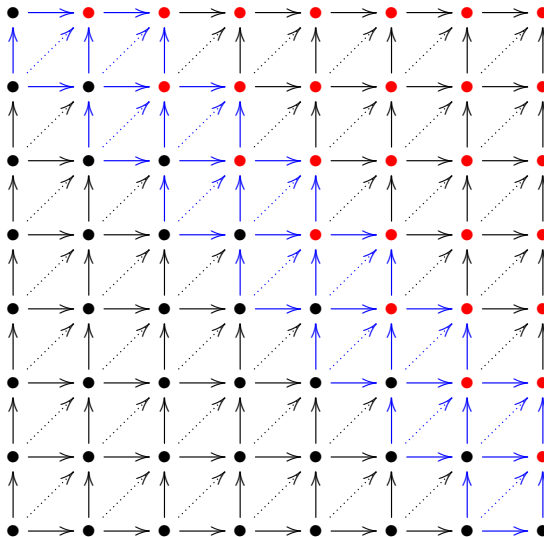
Example: if $\sigma : x \rightarrow y$ has $x \in A$ and $y \in B$, then $\sigma \in L$.

Composition with σ defines a map

$$\sigma_* : P(A)(u, x) \times P(B)(y, v) \rightarrow P(K)(u, v).$$

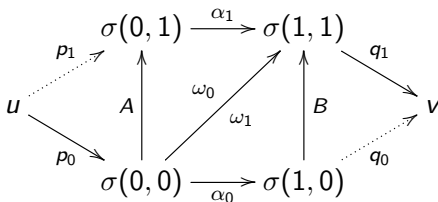
An example

Black: $t(F) \leq 7$, Red: $t(F) \geq 8$. Total t : 14.



Relations

Suppose that $\omega : \Delta^1 \times \Delta^1 \rightarrow L$ is defined by 2-simplices ω_0 and ω_1 such that $d_1\omega_0 = d_1\omega_1$ and $d_2(\omega_0) \in A$ and $d_0(\omega_1) \in B$. One of the 2-simplices ω_0 or ω_1 could be degenerate.



There are induced maps

$$\omega_0 : P(A)(u, \sigma(0,0)) \times P(B)(\sigma(1,1), v) \rightarrow P(A)(u, \sigma(0,0)) \times P(B)(\sigma(1,0), v)$$

and

$$\omega_1 : P(A)(u, \sigma(0,0)) \times P(B)(\sigma(1,1), v) \rightarrow P(A)(u, \sigma(0,1)) \times P(B)(\sigma(1,1), v)$$

ω_0, ω_1 define the parallel pair of arrows of a coequalizer diagram

$$\bigsqcup_{\omega: \Delta^1 \times \Delta^1 \rightarrow L} P(A)(u, \sigma(0, 0)) \times P(B)(\sigma(1, 1), v) \rightrightarrows \bigsqcup_{\sigma: x \rightarrow y \in L} P(A)(u, x) \times P(B)(y, v) \\ \rightarrow P(K)(u, v).$$

ie. $P(K)(u, v)$ is a set of equivalence classes for a relation on the set

$$\bigsqcup_{\sigma: x \rightarrow y \in L} P(A)(u, x) \times P(B)(y, v),$$

defined in a very specific way.

Conclusion: $P(K)$ is computable from $P(A)$ and $P(B)$, together with incidence relations defined by the frontier subcomplex L .



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