Diagrams and torsors J.F. Jardine

Cocycles

Suppose that X, Y are spaces.

$H(X,Y)$ = category whose objects are all pairs of maps (f,g)

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

where f is a weak equivalence. A morphism $\alpha : (f,g) \to (f',g')$ of $H(X,Y)$ is a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\alpha"] & \\ X & & Y \\ & Z' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

$H(X,Y)$ is the **category of cocycles** from X to Y.

"Example": $V_0 \to *$ is a sheaf epi (arising from a covering) and G is a sheaf of groups. Cocycles on V_0 with coefficients in G are simp. presheaf maps

$$* \xleftarrow{\simeq} C(V_0) \to BG$$

where $C(V_0)$ = Čech resolution for the cover. The present definition is an expansion of this idea.

$\pi_0 H(X,Y) =$ class of path components of $H(X,Y)$. There is a function

$$\phi : \pi_0 H(X,Y) \to [X,Y] \quad (f,g) \mapsto g \cdot f^{-1}$$

Theorem: The canonical map $\phi : \pi_0 H(X,Y) \to [X,Y]$ is a bijection for all X and Y.

Lest you think that I've done away with the homotopy theory in this statement, suppose that $f \simeq g : X \to Y$. Then there is a picture

$$\begin{tikzcd}
 & X \arrow[dl, "1"'] \arrow[d, "d_0"'] \arrow[dr, "f"] & \\
X & X \times I \arrow[l, "pr"'] \arrow[r, "h"] & Y \\
 & X \arrow[ul, "1"] \arrow[u, "d_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

where h is the homotopy. Then

$$(i_X, f) \sim (pr, h) \sim (1_X, g)$$

Thus $f \mapsto [(1_X, f)]$ defines a function

$$\psi : \pi(X,Y) \to \pi_0 H(X,Y)$$

If X has the good manners to be cofibrant, then the function ψ is inverse to ϕ. More generally, there are a couple of things to prove:

Lemma 1: Suppose $\alpha : X \to X'$ and $\beta : Y \to Y'$ are weak equivalences. Then

$$(\alpha, \beta)_* : \pi_0 H(X, Y) \to \pi_0 H(X', Y')$$

is a bijection.

Lemma 2: Suppose that Y is fibrant and X is cofibrant. Then the canonical map

$$\phi : \pi_0 H(X, Y) \to [X, Y]$$

is a bijection.

The Theorem is a formal consequence. The result holds in extreme generality, specifically in any model category which is right proper (weak equivalences pull back to weak equivalences along fibrations), and such that weak equivalences are closed under finite products.

Examples: spaces, simplicial sets, presheaves of simplicial sets, spectra, presheaves of spectra, any localizations.

Proof of Lemma 1 $(f,g) \in H(X',Y')$ is a map $(f,g) : Z \to X' \times Y'$ s.t. f is a weak equivalence.

There is a factorization

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "(f,g)"'] & W \arrow[d, "(p_{X'},p_{Y'})"] \\ & X' \times Y' \end{tikzcd}$$

s.t. j is a triv. cofibration and $(p_{X'}, p_{Y'})$ is a fibration. $p_{X'}$ is a weak equivalence.

Form the pullback

$$\begin{tikzcd} W_* \arrow[r, "(\alpha\times\beta)_*"] \arrow[d, "(p_X^*,p_Y^*)"'] & W \arrow[d, "(p_{X'},p_{Y'})"] \\ X \times Y \arrow[r, "\alpha\times\beta"'] & X' \times Y' \end{tikzcd}$$

(p_X^*, p_Y^*) is a fibration and $(\alpha\times\beta)_*$ is a weak equivalence (since $\alpha \times \beta$ is a weak equivalence, and by right properness). p_X^* is also a weak equivalence.

$(f,g) \mapsto (p_X^*, p_Y^*)$ defines a function

$$\pi_0 H(X',Y') \to \pi_0 H(X,Y)$$

which is inverse to $(\alpha,\beta)_*$. $\square$

Proof of Lemma 2 $\pi(X,Y) =$ naive homotopy classes.

$\pi(X,Y) \to [X,Y]$ is a bijection since X is cofibrant and Y is fibrant.

We have seen that the assignment $f \mapsto [(1_X, f)]$ defines a function

$$\psi : \pi(X,Y) \to \pi_0 H(X,Y)$$

and there is a diagram

$$\begin{tikzcd} \pi(X,Y) \arrow[r, "\psi"] \arrow[dr, "\cong"'] & \pi_0 H(X,Y) \arrow[d, "\phi"] \\ & {[X,Y]} \end{tikzcd}$$

It suffices to show that ψ is surjective, or that any object $X \xleftarrow{f} Z \xrightarrow{g} Y$ is in the path component of some a pair $X \xleftarrow{1} X \xrightarrow{k} Y$ for some map k.

Form the diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "j"] & \\ X & & Y \\ & V \arrow[ul, "p"] \arrow[ur, dotted, "\theta"'] & \end{tikzcd}$$

where j is a triv. cofibration and p is a fibration; θ exists because Y is fibrant.

X is cofibrant, so the trivial fibration p has a section σ, and so there is a commutative diagram

$$\begin{tikzcd}
& X \arrow[dl, "1"'] \arrow[dr, "\theta\sigma"] \arrow[dd, "\sigma"] & \\
X & & Y \\
& V \arrow[ul, "p"] \arrow[ur, dotted, "\theta"'] &
\end{tikzcd}$$

The composite $\theta\sigma$ is the required map k. $\square$

Proof of Theorem There are weak equivalences $\pi : X' \to X$ and $j : Y \to Y'$ such that X' and Y' are cofibrant and fibrant, respectively.

$$\begin{tikzcd}
\pi_0 H(X, Y) \arrow[r, "\phi"] \arrow[d, "(1,j)_*"', "\cong"] & {[X, Y]} \arrow[d, "\cong"', "j_*"] \\
\pi_0 H(X, Y') \arrow[r, "\phi"] & {[X, Y']} \arrow[d, "\cong"', "\pi^*"] \\
\pi_0 H(X', Y') \arrow[u, "(\pi,1)_*", "\cong"'] \arrow[r, "\cong", "\phi"'] & {[X', Y']}
\end{tikzcd}$$

$(1, j)_*$ and $(\pi, 1)_*$ are bijections by the first Lemma, and ϕ is a bijection by the second. $\square$

Non-abelian cohomology

$G =$ sheaf of groups (on some small Grothendieck site).

A G-torsor is a sheaf X with a free G-action such that $X/G \cong *$ in the sheaf category.

Example: Suppose that G is a topological group and that Y is a topological space. Every principal G-bundle $X \to Y$ (ie. X has free G-action such that $X/G \cong Y$) represents a G-torsor on $\mathrm{op}\,|_Y =$ site of open subsets of Y, and conversely. G represents a sheaf of groups $\hom(\ ,G)$ on $\mathrm{op}\,|_Y$, and Y represents the terminal sheaf $*$ on $\mathrm{op}\,|_Y$.

Recall that the Borel construction $EG \times_G X$ is the nerve $B(E_G X)$ of the translation category: objects are elements $x \in X$ and the morphisms $g : x \to y$ are group elements such that $g \cdot x = y$. This is a special case of the homotopy colimit construction, which will come up later. The object that we are interested in is actually a simplicial sheaf which is constructed as the nerve of a sheaf of categories section by section, as described.

In general, if I is a small category the nerve BI of I is the simplicial set whose n-simplices are strings of arrows

$$a_0 \to a_1 \to \cdots \to a_n$$

in I. The face map d_i forgets about a_i, either by composing the arrows around it or dropping it from the list altogether (for a_0 and a_n). The degeneracies are insertions of identities.

If G is a group, BG is the nerve of the category having one object and one morphism for every element of the group. Exercise: $BG = EG \times_G *$. $EG = B(G/*)$, where $G/*$ is the category of arrows $g : * \to *$ in G.

Fact: $G \times X \to X$ is a free action means precisely that the canonical map $EG \times_G X \to X/G$ is a local weak equivalence.

Thus, X is a G-torsor iff $EG \times_G X \to *$ is a local weak equivalence.

$G -$ **Tors** is the category of G-torsors and G-equivariant maps. It is a groupoid:

A map $f : X \to Y$ of G-torsors is induced on fibres by the map of local fibrations

$$\begin{tikzcd}
EG \times_G X \arrow[rr] \arrow[dr] & & EG \times_G Y \arrow[dl] \\
& BG &
\end{tikzcd}$$

Then $f : X \to Y$ is a weak equiv. of constant simplicial sheaves, hence an isomorphism.

A construction:

Suppose $* \xleftarrow{\simeq} Y \xrightarrow{\alpha} BG$ is a cocycle, form pullback

$$\begin{tikzcd} \mathrm{pb}(Y) \arrow[r] \arrow[d] & Y \arrow[d, "\alpha"] \\ EG \arrow[r, "\pi"'] & BG \end{tikzcd}$$

$\mathrm{pb}(Y)$ has a G-action (from EG), and the map

$$EG \times_G \mathrm{pb}(Y) \to Y$$

is a weak equivalence. The square is htpy cartesian where $Y(U) \neq \emptyset$, so that $\mathrm{pb}(Y) \to \tilde{\pi}_0 \,\mathrm{pb}(Y)$ is a G-equivariant weak equiv.
[$\tilde{\pi}_0 \,\mathrm{pb}(Y)$ is the sheaf of path components of the simplicial sheaf $\mathrm{pb}(Y)$, in this case identified with a constant simplicial sheaf.]
The maps

$$EG \times_G \tilde{\pi}_0 \,\mathrm{pb}(Y) \leftarrow EG \times_G \mathrm{pb}(Y) \to Y \simeq *$$

are weak equivs. Then $\tilde{\pi}_0 \,\mathrm{pb}(Y)$ is a G-torsor.
A functor

$$H(*, BG) \to G - \mathbf{Tors}$$

is def. by $(* \xleftarrow{\simeq} Y \to BG) \mapsto \tilde{\pi}_0 \,\mathrm{pb}(Y)$.

A functor

$$G - \mathbf{Tors} \to H(*, BG):$$

is def. by $X \mapsto (* \xleftarrow{\simeq} EG \times_G X \to BG)$.

Theorem: These functors induce bijections

$[*, BG] \cong \pi_0 H(*, BG) \cong \pi_0(G - \mathbf{Tors}) = H^1(\mathcal{C}, G)$.

$[*, BG]$ means morphisms in the homotopy category of simplicial sheaves (or presheaves). The result holds over arbitrary small Grothendieck sites, and is about 20 years old. Unlike the original proof, you have heard no references to hypercovers or pro objects — this proof is really quite simple, modulo the simplicial sheaf homotopy theory technology.

Here's where it all came from:

$k =$ a field of characteristic $\neq 2$.

$$[*, BO_n] \cong H^1_{et}(k, O_n)$$

and $H^1_{et}(k, O_n)$ is isomorphism classes of non-deg. sym. bilinear forms of rank n over k. Thus, any such form α defines a morphism $[\alpha] : * \to BO_n$ in the homotopy category for simplicial sheaves on the étale site for k. There is a calculation

$$H^*_{et}(BO_n, \mathbb{Z}/2) \cong H^*_{et}(k, \mathbb{Z}/2)[HW_1, \dots, HW_n]$$

where HW_i is a polynomial generator of degree i, Any form α therefore determines a ring homomorphism

$$\alpha^* : H^*_{et}(BO_n, \mathbb{Z}/2) \to H^*_{et}(k, \mathbb{Z}/2)$$

$HW_i(\alpha) = \alpha^*(HW_i)$ defines the i^{th} Hasse-Witt class of α. These higher Hasse-Witt classes coincide with the old Delzant Stiefel-Whitney classes. In particular, $HW_1(\alpha) = det \cdot \alpha$ and $HW_2(\alpha)$ is the classical Hasse-Witt invariant.

Diagrams and torsors

Suppose that G is a group. Simplicial X sets with G-action are diagrams (ie. functors) $X : G \to s\mathbf{Set}$, where G is identified with a category (or groupoid) with one object $*$ and a morphism $g : * \to *$ for every element $g \in G$.

Now consider functors $X : I \to \mathbf{S}$ of simplicial sets defined on a fixed index category I. I usually say that such a thing is an I-diagram in simplicial sets.

X is a **diagram of equivalences** if every $i \to j$ on I induces a weak equiv. $X(i) \to X(j)$.

Example: If $p : X \to BI$ is a fibration, then the diagram $i \mapsto \mathrm{pb}(X)(i)$ defined by the pullbacks

$$\begin{tikzcd} \mathrm{pb}(X)(i) \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B(I/i) \arrow[r] & BI \end{tikzcd}$$

is a diagram of equivalences.

Here I/i is the (slice) category whose objects are all arrows $j \to i$ of I.

1) There is a derived pullback functor

$$R\,\mathrm{pb} : \mathbf{S}/BI \to \mathbf{S}^I_e = \{\text{dias. of equivalences}\}$$

defined by applying pb to fibrant replacements.
2) The map $\underrightarrow{\mathrm{holim}}_I\, \mathrm{pb}(X) \to X$ is a weak equivalence, whether p is a fibration or not.

If $Y : I \to \mathbf{Set}$ is a diagram of sets, then there is a category $E_I Y$ with objects (i, x) with $x \in Y(i)$. A morphism $\alpha : (i, x) \to (j, y)$ is a morphism $\alpha : i \to j$ of I such that $\alpha_*(x) = y$.

The homotopy colimit $\underrightarrow{\mathrm{holim}}_I X$ for a diagram $X : I \to s\mathbf{Set}$ in simplicial sets is, effectively, $B(E_I X)$.

Lemma: (Quillen) $X : I \to \mathbf{S}$ a dia. of equivalences. Then

$$\begin{tikzcd} X(i) \arrow[r] \arrow[d] & \underrightarrow{\mathrm{holim}}_I\ X \arrow[d] \\ \Delta^0 \arrow[r, "i"'] & BI \end{tikzcd}$$

is homotopy cartesian, for all objects $i \in I$.

This means that, if you replace the map $\underrightarrow{\mathrm{holim}}_I X \to BI$ by a fibration, then $X(i)$ is weakly equivalent to the corresponding fibre over $i \in BI_0$.

Cor. If X is a dia. of equivs. and $Z \to BI$ is a fibrant replacement for $\underrightarrow{\mathrm{holim}}_I X \to BI$, then there are weak equivs. of I-diagrams

$$X \leftarrow \mathrm{pb}(\underrightarrow{\mathrm{holim}}_I X) \rightarrow \mathrm{pb}(Z).$$

The derived pullback $R\,\mathrm{pb}$ and homotopy colimit functors induce an equivalence

$$\underrightarrow{\mathrm{holim}} : \mathrm{Ho}(\mathbf{S}^I)_e \simeq \mathrm{Ho}(\mathbf{S}/BI) : R\,\mathrm{pb}$$

An ***I*-torsor** is a diagram of equivalences $X : I \rightarrow \mathbf{S}$ such that $\underrightarrow{\mathrm{holim}}_I\, X \rightarrow *$ is a weak equivalence. In other words a dia. of equivalences $X : I \rightarrow \mathbf{S}$ is an I-torsor if $\underrightarrow{\mathrm{holim}}_I X$ determines a cocycle

$$* \xleftarrow{\simeq} \underrightarrow{\mathrm{holim}}_I X \rightarrow BI$$

Proposition: The derived pullback $R\,\mathrm{pb}$ and homotopy colimit functors induce bijections

$$\pi_0(I - \mathbf{Tors}) \cong \pi_0 H(*, BI) \cong [*, BI]$$

Of course, you can prove by hand that $[*, BI] \cong \pi_0 I$ (path components in I), so this doesn't seem like such a big deal.

What we have, though, is a definition of I-torsor and a homotopy classification of I-torsors, and these admit wild generalization (subject to assigning meaning to the terms):

1) I = presheaf of categories on an arbitrary site (eg. torsors for sheaves of groupoids, construction of the associated stack for a sheaf of groupoids)

2) I = presheaf of categories enriched in simplicial sets on an arbitrary site("higher torsor" case)

3) I = presheaf of categories (enriched in simplicial sets) in a proper f-local structure on an arbitrary site, eg. diagrams and enriched diagrams in motivic homotopy theory (motivic torsors)

The constructions of torsors and higher torsors for sheaves of groupoids and simplicial groupoids were developed over the the last 2-3 years (Jardine, Luo). The expansion of these ideas to arbitrary diagrams is new and unexpected.