Homotopy theories of dynamical systems

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Dynamical systems

A *dynamical system* (or S*-dynamical system*, or S*-space*) is a map of simplicial sets

$$\phi : X \times S \to X,$$

giving an action of a *parameter space* S on a *state space* X.

Equivalently, a dynamical system is a map

$$\phi_* : S \to \mathbf{hom}(X, X)$$

into the topological monoid of endomorphisms of X.

$s \mapsto \phi_*(s) : X \to X$ is continuous in $s \in S$.

If S has a monoidal structure, then ϕ_* is required to be a homomorphism.

Most often, X is a manifold, and S is a time parameter which is a submanifold of the real numbers.

Examples: Discrete dynamical systems

If $S = *$ is a one-point space, then a dynamical system parameterized by S is just a map $X \to X$.

The free monoid on the one-point space is a copy of $\mathbb{N}$, and so there is an associated monoid map

$$f_* : \mathbb{N} \to \mathbf{hom}(X, X)$$

Cellular automata: $X = (\mathbb{Z}^n)^k$ consists of points in an integral lattice, each of which can be in a set of k states.

Category of S-spaces

A *morphism* $f : X \to Y$ of S-spaces is a map $f : X \to Y$ which preserves the respective S-actions. Morphisms are also called *S-equivariant maps*.

$S - s\mathbf{Set}$ is the category of S-spaces and their morphisms.

Question: (Carlsson) What could be meant by a homotopy theory of dynamical systems, or S-spaces?

Naive Definition: A map $X \to Y$ of S-spaces is a *weak equivalence* if and only if the underlying map of simplicial sets (spaces) is a weak equivalence.

This is analogous to the traditional naive definition of G-equivariant weak equivalence for spaces equipped with an action by a group G.

Varying the parameter space

It should mean something in the homotopy theory of dynamical systems if the parameter space S is contractible.

We need a category of dynamical systems which contains the S-space categories for all parameter spaces S, and for which we can vary S.

A *map* $(\theta, f) : X \to Y$ consists of maps $\theta : S \to T$ and $f : X \to Y$ such that the following commutes:

$$\begin{array}{ccc} S \times X & \to & X \\ {\scriptstyle \theta\times f}\downarrow & & \downarrow {\scriptstyle f} \\ T \times Y & \to & Y \end{array}$$

There is a homotopy theory for this category, but the weak equivalences are more difficult to describe.

Feel good fact: if θ and f are weak equivalences, then (θ, f) is a weak equivalence for this theory (whatever it is).

Quillen model categories

A *closed model category* is a category $\mathcal{M}$ equipped with *weak equivalences*, *fibrations* and *cofibrations* s.t. the following hold:

CM1: $\mathcal{M}$ has all limits and colimits.
CM2: If any two of f, g, $g \cdot f$ is a weak equivalence, so is the third.
CM3: Weak equivalences, cofibrations and fibrations are closed under retraction.
CM4: Given a cofibration i, a fibration p and diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

then the lift exists if either i or p is a weak equivalence (trivial).
CM5: Every f has $f = p \cdot j = q \cdot i$, where p is a fibration, j is a triv. cofibration, q is a triv. fibration, j is a cofibration.

Examples: ordinary homotopy theory

$s\mathbf{Set}$ = simplicial sets, and $\mathbf{Top}$ = topological spaces.

Fibrations for $\mathbf{Top}$ are Serre fibrations, and weak equivalences are weak homotopy equivalences. CW-complexes are cofibrant objects.

There are adjoint functors

$$| \ | : s\mathbf{Set} \leftrightarrows \mathbf{Top} : S$$

The weak equivalences $X \to Y$ of $s\mathbf{Set}$ are those maps which induce weak equivalences $|X| \to |Y|$, and the cofibrations are monomorphisms. Fibrations are Kan fibrations.

The adjoint functors form a "Quillen equivalence", and induce an adjoint equivalence of homotopy categories

$$| \ | : \mathrm{Ho}(s\mathbf{Set}) \leftrightarrows \mathrm{Ho}(\mathbf{Top}) : S$$

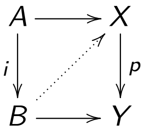
Homotopy theory of S-spaces, 1

A map $f : X \to Y$ of S-spaces is a

1) *weak equivalence* if f is a weak equivalence of simplicial sets
2) *cofibration* if f is a monomorphism
3) *projective fibration* if f is a Kan fibration.

An *injective fibration* is a map which has the right lifting property (RLP) with respect to all trivial cofibrations.

A *projective cofibration* is a map which has the left lifting property (LLP) with respect to all trivial projective fibrations.



Homotopy theory of S-spaces, 2

Theorem

Suppose that S is a fixed choice of parameter space.

1) *The category $S-s\mathbf{Set}$, together with the cofibrations, weak equivalences and injective fibrations, satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*
2) *The category $S-s\mathbf{Set}$, together with the projective cofibrations, weak equivalences and projective fibrations, satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*

The proof follows a pattern that we know: p is an injective fibration if and only if it has the RLP wrt all bounded trivial cofibrations, and part 1) implies part 2).

Dynamical systems to diagrams

$F(S)$ is the free simplicial monoid associated to a space S:

$$F(S) = * \sqcup S \sqcup S^{\times 2} \sqcup S^{\times 3} \sqcup \dots$$

and an S-space $X \times S \to X$ is canonically a module over $F(S)$.

Alternatively, $F(S)$ is a simplicial category (or a category enriched in simplicial sets, with one object) and X is an $F(S)$-*diagram*.

Definition: A *simplicial category* A is a simplicial object in categories.

A consists of simplicial sets $\mathrm{Ob}(A)$ and $\mathrm{Mor}(A)$ such that all categorical structure $s, t : \mathrm{Mor}(A) \to \mathrm{Ob}(A)$, $e : \mathrm{Ob}(A) \to \mathrm{Mor}(A)$, compositions, are compatible with the simplicial structure.

Definition: A *category enriched in simplicial sets* is a simplicial category B such that $\mathrm{Ob}(B)$ is discrete (ie. generated by vertices).

Internal diagrams

A = simplicial category. An *A-diagram* in simplicial sets consists of a simplicial set map $\pi : X \to \mathrm{Ob}(A)$ and an action diagram

$$\begin{array}{ccc} X \times_s \mathrm{Mor}(A) & \xrightarrow{m} & X \\ \downarrow & & \downarrow \pi \\ \mathrm{Mor}(A) & \xrightarrow[t]{} & \mathrm{Ob}(A) \end{array} \qquad (x,\alpha) \mapsto \alpha(x)$$

such that $1(x) = x$ and $\beta(\alpha(x)) = (\beta\alpha)(x)$.

Set^A is the category of A-diagrams. A *morphism* (natural transformation) is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {}_{\pi}\searrow \quad \swarrow_{\pi} & \\ & \mathrm{Ob}(A) & \end{array}$$

which respects the multiplication.

Example: Ordinary functors

A functor $F : I \to \mathbf{Set}$ consists of sets $F(i)$, $i \in \mathrm{Ob}(I)$, and morphisms $F(\alpha) : F(i) \to F(j)$ satisfying the usual properties.

Alternatively, F consists of a function

$$\pi : F = \bigsqcup_{i \in \mathrm{Ob}(I)} F(i) \to \bigsqcup_{i \in \mathrm{Ob}(I)} * = \mathrm{Ob}(I),$$

and a morphism

$$m : F \times_s \mathrm{Mor}(I) = \bigsqcup_{\alpha : i \to j} F(i) \to \bigsqcup_{j} F(j) = F$$

A natural transformation of functors $\alpha : F \to G$ is a function

$$\bigsqcup_{i \in \mathrm{Ob}(I)} F(i) \to \bigsqcup_{i \in \mathrm{Ob}(I)} G(i)$$

which is fibred over $\mathrm{Ob}(I)$.

Homotopy theory of diagrams, 1

A = category enriched in simplicial sets (ie. $\mathrm{Ob}(A)$ is discrete).

A map $f : X \to Y$ of A-diagrams is

1) a *weak equivalence* if the map

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & \mathrm{Ob}(A) & \end{array}$$

is a weak equivalence of $s\mathbf{Set}/\mathrm{Ob}(A)$

2) a *cofibration* if the simplicial set map f is a monomorphism
3) a *projective fibration* if the simplicial set map f is a Kan fibration.

An *injective fibration* is a map which has the right lifting property with respect to all trivial cofibrations.

A *projective cofibration* is a map which has the left lifting property with respect to all trivial projective fibrations.

Theorem

Suppose that A is a category which is enriched in simplicial sets.

1) *The category* $\mathbf{Set}^A$*, together with the cofibrations, weak equivalences and injective fibrations, satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*
2) *The category* $\mathbf{Set}^A$*, together with the projective cofibrations, weak equivalences and projective fibrations, satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*

The theorem is a special case of a result which holds for diagrams of simplicial presheaves over a presheaf of simplicial categories with discrete objects.

Homotopy colimits

Suppose that $F : I \to \mathbf{Set}$ is an ordinary functor.

There is a category $E_I F$ whose objects are the pairs (x, i) with $x \in F(i)$. The morphisms $\alpha : (x, i) \to (y, j)$ are morphisms $\alpha : i \to j$ of I such that $\alpha_*(x) = y$.

This category has a nerve $B(E_I F)$, whose n-simplices are strings

$$(x_0, i_0) \xrightarrow{\alpha_1} (x_1, i_1) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} (x_n, i_n)$$

of length n. All that matters here is x_0 and the string in I:

$$\underrightarrow{\text{holim}}_I F_n = B(E_I F)_n = \bigsqcup_{i_0 \to \cdots \to i_n} F(i_0).$$

This is the *homotopy colimit* for the functor F. It is the space of *finite trajectories* associated to the functor F, or the *space of dynamics* for the functor F.

Homotopy colimits and pullbacks

A = simplicial category. Every A-diagram $\pi : X \to \mathrm{Ob}(A)$ determines a bisimplicial set map

$$\underrightarrow{\mathrm{holim}}_A\, X \to BA,$$

by taking homotopy colimit in each simp. degree, giving a functor

$$\underrightarrow{\mathrm{holim}}_A : \mathbf{Set}^A \to s^2\mathbf{Set}/BA.$$

The pullback functor

$$\mathrm{pb} : s^2\mathbf{Set}/BA \to \mathbf{Set}^A,$$

is defined by taking diagonals of the pullbacks

$$\begin{array}{ccc} \mathrm{pb}(Y)_i & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B(A/i) & \longrightarrow & BA \end{array}$$

in all simplicial degrees.

Homotopy colimit structure

A map $f : X \to Y$ of A-diagrams is a *cofibration* if the underlying simplicial set map is a monomorphism.

$f : X \to Y$ is a *weak equivalence* if the induced map

$$\underrightarrow{\text{holim}}_A X \to \underrightarrow{\text{holim}}_A Y$$

is a *diagonal weak equivalence* of bisimplicial sets.

Theorem

With these definitions, the category $\mathbf{Set}^A$ *satisfies the axioms for a proper closed model category.*

This is the *homotopy colimit model structure* for the category of A-diagrams. Schlichtkrull uses a special case in his proof of the Barratt-Kahn-Priddy-Quillen Theorem ($QS^0_0 \simeq (B\Sigma_\infty)_+$).

Relation with diagonal model structure

Theorem

The functors $\underrightarrow{\text{holim}}_A$ *and* pb *induce an equivalence of categories*

$$\text{Ho}(\mathbf{Set}^A) \simeq \text{Ho}(s^2\mathbf{Set}/BA)$$

for the homotopy colimit structure on $\mathbf{Set}^A$ *and the diagonal structure on* $s^2\mathbf{Set}/BA$.

A bisimplicial set map $f : X \to Y$ is a *cofibration* if it is a monomorphism. $f : X \to Y$ is a *diagonal weak equivalence* if the simplicial set map $f_* : d(X) \to d(Y)$ is a weak equivalence.

Theorem

There is a model structure on the category $s^2\mathbf{Set}$ *for which the cofibrations are the monomorphisms and the weak equivalences are those map* $X \to Y$ *which induce a weak equivalence* $d(X) \to d(Y)$ *of associated diagonal simplicial sets.*

Homotopy types of categories

A functor $f : C \to D$ is a fibration (respectively weak equivalence) if the induced map $BC \to BD$ is an sd^2-fibration (respectively weak equivalence).

Theorem (Thomason)

1) *With these definitions the category* **Cat** *of small categories has the structure of a proper closed model category.*
2) *The adjunction*

$$P : s\mathbf{Set} \leftrightarrows \mathbf{Cat} : B$$

is a Quillen equivalence, for the sd^2*-structure on simplicial sets.*

P is the *path category* functor.

$PBC \cong C$ for small categories C.

Homotopy types of categories, 2

Crux of the proof: if $K \to L$ is an inclusion of finite simplicial complexes, then all pushout diagrams

$$\begin{array}{ccc} NBN(K) & \to & C \\ \downarrow & & \downarrow \\ NBN(L) & \to & D \end{array}$$

induce homotopy cocartesian diagrams of simplicial sets.

$N(K)$ is the poset of non-degenerate simplices of a simplicial set K: $\sigma \leq \tau$ if $\sigma \in \langle \tau \rangle$.

$BN(K) = \text{sd}(K)$ (*order complex* of NK) is the *barycentric subdivision* of K if K is a simplicial complex.

Note that $BNBN(K) \cong \text{sd}^2(K)$ for simplicial complexes K.

The induced functor $NBN(K) \to NBN(L)$ is a "Dwyer map".

Subdivisions

The *subdivision*

$$\mathrm{sd}(X) = \varinjlim_{\Delta^n \to X} BN\Delta^n.$$

is a colimit of barycentric subdivisions of simplices.

$$Ex(X)_n = \hom(\mathrm{sd}(\Delta^n), X)$$

for simplicial sets X.

There are adjoint functors

$$\mathrm{sd}^n : s\mathbf{Set} \leftrightarrows s\mathbf{Set} : \mathrm{Ex}^n$$

and natural weak equivalences $\mathrm{sd}^n X \xrightarrow{\simeq} X$ and $Y \xrightarrow{\simeq} \mathrm{Ex}^n Y$ for all simplicial sets X and Y.

$p : X \to Y$ is an sd^n-*fibration* if the map $\mathrm{Ex}^n X \to \mathrm{Ex}^n Y$ is a Kan fibration, or if p has the RLP wrt all $\mathrm{sd}^n(\Lambda^m_k) \to \mathrm{sd}^n(\Delta^m)$.

The sd^n-*cofibrations* are those maps which have the LLP w.r.t. all maps which are sd^n-fibrations and weak equivalences.

Theorem

1) *The category* $s\mathbf{Set}$ *of simplicial sets, together with the weak equivalences,* sd^n*-fibrations, and* sd^n*-cofibrations, satisfies the axioms for a proper closed model category.*
2) *The adjoint pair of functors*

$$\mathrm{sd}^n : s\mathbf{Set} \leftrightarrows s\mathbf{Set} : \mathrm{Ex}^n$$

defines a Quillen equivalence between the standard model structure and the sd^n*-structure for simplicial sets.*

Subdivisions for bisimplicial sets

$$\operatorname{sd}^{m,n} X = \varinjlim_{\Delta^{p,q} \to X} \operatorname{sd}^m \Delta^p \tilde{\times} \operatorname{sd}^n \Delta^q.$$

$\tilde{\times}$ is external product: $\Delta^{p,q} = \Delta^p \tilde{\times} \Delta^q$.

The functor $\operatorname{sd}^{m,n}$ has a right adjoint $\operatorname{Ex}^{m,n}$. Both functors preserve diagonal homotopy types.

A map $f : X \to Y$ of bisimplicial sets is an *$sd^{m,n}$-fibration* if the induced map $\operatorname{Ex}^{m,n} X \to \operatorname{Ex}^{m,n} Y$ is a diagonal fibration.

$\operatorname{sd}^{m,n}$-*cofibrations* are defined by a left lifting property with respect to trivial fibrations.

Theorem

1) *The category* $s^2\mathbf{Set}$*, with the* $\mathrm{sd}^{m,n}$*-fibrations, diagonal weak equivalences and* $\mathrm{sd}^{m,n}$*-cofibrations, satisfies the axioms for a proper closed model category.*
2) *The adjoint functors*

$$\mathrm{sd}^{m,n} : s^2\mathbf{Set} \leftrightarrows s^2\mathbf{Set} : \mathrm{Ex}^{m,n}$$

define a Quillen equivalence between the diagonal model structure and the $\mathrm{sd}^{m,n}$*-structure for bisimplicial sets.*

Homotopy types of simplicial categories

A morphism $f : C \to D$ of simplicial categories is

a) a *fibration* if the map $BC \to BD$ is an $\mathrm{sd}^{2,0}$-fibration, and

b) a *weak equivalence* if the map $BC \to BD$ is a diagonal equivalence.

Theorem

1) *With these definitions, the category* $s\mathbf{Cat}$ *satisfies the axioms for a proper closed model category.*

2) *The adjunction*

$$P : s^2\mathbf{Set} \leftrightarrows s\mathbf{Cat} : B$$

defines a Quillen equivalence between simplicial categories and the $\mathrm{sd}^{2,0}$*-model structure for bisimplicial sets.*

Diagram homotopy types

Let $s\mathbf{Dia}$ be the category whose objects are bisimplicial set maps $X \to BC$ where C is a simplicial category. Say that a morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ BC & \xrightarrow[g]{} & BD \end{array}$$

is a weak equivalence if f and g are weak equivalences, and is a fibration if the maps $g : BC \to BD$ and

$$X \to BC \times_{BD} Y$$

are $sd^{2,0}$-fibrations. Say that the map is a cofibration if f is an $sd^{2,0}$-cofibration and g is a cofibration of simplicial categories.

Theorem

*With these definitions, the category s**Dia** satisfies the axioms for a closed model category.*

1) We regard bisimplicial set maps $Y \to BA$ as A-diagrams, but that's okay: every A-diagram Y can be recovered from the map $\underrightarrow{\text{holim}}_A Y \to BA$ up to sectionwise weak equivalence, via the pullback functor.

2) We now have a homotopy theory for *all* dynamical systems: dynamical systems and parameter spaces can be varied simultaneously.

3) All of this can be topologized. There is a model structure for presheaves of simplicial categories which is defined by an $\text{sd}^{2,0}$-model structure for bisimplicial presheaves, and a corresponding model structure on diagram objects $Y \to BA$ in bisimplicial presheaves, all over an arbitrary Grothendieck site.

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