Simplicial sheaves, cocycles and torsors
J.F. Jardine

Simplicial sheaves

Suppose that $\mathcal{C}$ is a small Grothendieck site, which typically gives rise to one of the standard topologies for a (nice) scheme S.
This includes standard issue things like

1) the Zariski site $Zar|_S$ of open subsets $U \subset S$.
2) the étale site $et|_S$ of étale maps $U \to S$, with the étale topology, in which coverings are étale maps $V \to U$ which are surjective on points (faithfully flat), or the Nisnevich topology, in which the coverings are étale maps $V \to U$ such that all maps $\mathrm{Sp}(K) \to U$ lift to V if K is a field.
2) the finite étale site $fet|_S$ with finite étale maps $U \to S$ and finite étale covers $V \to U$ — this produces a variant of Galois theory.
4) "big" sites whose objects are scheme homomorphisms $T \to S$ which are locally of finite type, but *subject to a cardinality bound* on both the points of T and sections of the corresponding sheaf of rings $\mathcal{O}_T$. These sites can be inflicted

with any of the standard geometric topologies, so that one has the big Zariski site $(Sch|)_{Zar}$ the big étale site $(Sch|_S)_{et}$ or the big Nisnevich site $(Sm|_S)_{Nis}$. This last thing is usually not too useful, and one usually restricts to smooth S-schemes, where S is very nice, like the spectrum $\mathrm{Sp}(k)$ of a perfect field k.

A *sheaf* on a site $\mathcal{C}$ is a contravariant functor $\mathcal{C}^{op} \to$ **Set** which satisfies a patching condition determined by the topology on $\mathcal{C}$. A *presheaf* on $\mathcal{C}$ is just a contravariant functor. In many cases the representable functors $\hom(\ ,U)$ are sheaves (eg. this is a consequence of the theorem of faithfully flat descent, for the étale topology), they are always presheaves.

A *simplicial sheaf* (resp. *simplicial presheaf*) X is a simplicial object in the category of sheaves (resp. presheaves). In other words, X is a contravariant functor $\mathbf{\Delta}^{op} \to \mathbf{Shv}(\mathcal{C})$, where $\mathbf{\Delta}$ is the category of finite ordinal numbers $\mathbf{n} = \{0, 1, \ldots, n\}$ and the order preserving maps between them.

Examples:

1) Every simplicial set K determines a "constant" simplicial presheaf K with $K(U) = K$ for all

$U \in \mathcal{C}$. The associated sheaf $\tilde{K}$ is the constant simplicial sheaf, also denoted by $\Gamma^*(K)$, or by K. The functor Γ^* is left adjoint to the global sections functor.

All standard simplices $\Delta^n = \hom(\ ,\mathbf{n})$ have associated constant simplicial sheaves $\Gamma^*\Delta^n$.

2) Any small category A has a nerve BA, which is a simplicial set with

$$BA_n = \hom(\mathbf{n}, A),$$

which is the set of strings of arrows

$$a_0 \to a_1 \to \cdots \to a_n$$

of length n in A. Examples include the nerve (classifying space) BG of a group G, thought of as a groupoid with one object, and the Borel construction

$$EG \times_G X = B(E_G X)$$

for a group action $G \times X \to X$. $E_G X$ is a groupoid whose objects are the members of X and the morphisms $x \to y$ are the group elements g such that $g \cdot x = y$.

Elements of BG_n can be identified with elements of the product $G^{\times n}$ if G is a group.

These constructions are functorial, so that any sheaf (resp. presheaf) of categories A determines a simplicial sheaf (resp. simplicial presheaf) BA. An algebraic group H represents (usually) a sheaf of groups on a geometric site, and the classifying "space" BH is a simplicial sheaf. If $H \times F \to F$ is an action on a sheaf (usually scheme) F, then the Borel construction $EH \times_H F$ is a simplicial sheaf.

3) Suppose that $p : V \to U$ is a function. Then there is a groupoid $G(p)$ whose objects are the elements of V and whose morphisms $x \to y$ are the pairs of elements (x, y) such that $p(x) = p(y)$. The corresponding nerve $BG(p)$ is the Čech resolution $C(p)$

$$V, V \times_U V, V \times_U V \times_U V, \dots$$

associated to the function p. There is a canonical simplicial set map $C(V) \to U$ (U is a constant simplicial set), which is a weak equivalence (of associated CW-complexes) if p is surjective.

Again, this function is functorial, and so any sheaf epimorphism (ie. covering) $U \to F$ determines a simplicial sheaf map $C(U) \to F$

which is a *local weak equivalence* in the sense that it induces a weak equivalence

$$C(U_x) = C(U)_x \to F_x$$

in all stalks. This how Čech resolutions always arise.

Example 1. Suppose that L/k is a finite Galois extension with Galois group G. Then $\mathrm{Sp}(L) \to \mathrm{Sp}(k)$ is covering for the étale topology, and one can show that there is an isomorphism of simplicial sheaves

$$C(\mathrm{Sp}(L)) \cong EG \times_G \mathrm{Sp}(L) = B(E_G \,\mathrm{Sp}(L))$$

for the étale topology. There is a canonical map

$$EG \times_G \mathrm{Sp}(L) \to BG.$$

I say that the sheaf of groupoids $E_G \,\mathrm{Sp}(L)$ is the *Galois groupoid* for L/k.
The map

$$EG \times_G \mathrm{Sp}(L) \to *$$

is a local weak equivalence for simplicial sheaves on the étale site $et|_K$, because $\mathrm{Sp}(k)$ represents the point on that site.

I'll repeat this: a simplicial sheaf (or presheaf) map $X \to Y$ is a *local weak equivalence* if it induces weak equivalences of simplicial sets

$$X_x \to Y_x$$

in all stalks. That is, if you have a theory of stalks — you may not have this, eg. flat topology, in which case there's a "fat point" construction, called Boolean localization, which you use instead. Local weak equivalences can also be described internally by sheaves of homotopy groups, with a little care.

There are a multitude of Quillen model structures on categories of simplicial sheaves $s\mathbf{Shv}(\mathcal{C})$ or simplicial presheaves $s\operatorname{Pre}(\mathcal{C})$ for which the weak equivalences are the local weak equivalences. I won't describe these, except to say that the *injective structures* have all *monomorphisms as cofibrations*, and then the *injective fibrations* are what they are, which is always a bit mysterious.

There is a time honoured method, due to Quillen, of constructing a homotopy category from a model structure, which amounts to formally inverting the local weak equivalences. Example: morphisms

$$[X, Y]$$

in the original homotopy category of topological spaces are ordinary homotopy classes of maps $X \to Y$ if X and Y are CW-complexes.

Local weak equivalences are non-abelian analogs of quasi-isomorphisms, and the homotopy categories arising from simplicial sheaves (or presheaves) are *non-abelian derived categories*. I use the notation $[X, Y]$ for morphisms from X to Y in any homotopy category.

Cocycles

I want to give you a take on constructing elements of $[X, Y]$ for simplicial sheaves X and Y

(or presheaves — there's no difference, because a simplicial presheaf is locally weakly equivalent, even locally isomorphic, to its associated sheaf).

Suppose I give you a picture (cocycle)

$$X \underset{\simeq}{\xleftarrow{g}} U \xrightarrow{f} Y$$

where g is a local weak equivalence. Then g is inverted in the homotopy category, and so the assignment $(g, f) \mapsto f \cdot g^{-1}$ defines a function from pictures to elements of $[X, Y]$. If there is a commutative diagram

$$\begin{tikzcd}
 & U \arrow[dl, "g"', "\simeq"] \arrow[dr, "f"] \arrow[dd, "\theta"] & \\
X & & Y \\
 & U' \arrow[ul, "g'", "\simeq"'] \arrow[ur, "f'"'] &
\end{tikzcd} \tag{1}$$

then θ is a local weak equivalence and

$$f \cdot g^{-1} = f \cdot \theta^{-1} g'^{-1} = f' \cdot g'^{-1}$$

in the set $[X, Y]$. The cocycles (g, f) are the objects, and the diagrams (1) are the morphisms of a category $h(X, Y)$, called the *cocycle category*, or the category of cocycles from X to Y.

We have just shown that there is a well defined function

$$\phi : \pi_0 h(X, Y) \to [X, Y],$$

and here is the basic result:

Theorem 2. *The map ϕ is a bijection.*

The Theorem is proved by invoking formal nonsense about the injective model structure: it is right proper, and the class of local weak equivalences is closed under finite products. There is such a result for any model category satisfying these two conditions, and such things abound in nature. The motivic model structure is an example.

If you like formal homotopy theory, this result is a special, practical outcome of the Dwyer-Kan theory of hammock localizations, but was discovered independently.

Example 3. Suppose that G is a topological group and that X is some space. Then G represents a sheaf of groups on the open subsets of X. Suppose that $U_\alpha \subset X$ is an open cover of X. Set $U = \sqcup_\alpha U_\alpha$, and then the inclusions $U_\alpha \subset X$ together induce a covering $U \to *$ of the one-point sheaf on $op|_X$. The Čech resolution $C(U)$ for this covering has the form

$$\sqcup\, U_\alpha,\ \sqcup_{\alpha,\beta}\, U_\alpha \cap U_\beta,\ \sqcup_{\alpha,\beta,\gamma}\, U_\alpha \cap U_\beta \cap U_\gamma,\ \dots$$

and a map $f : C(U) \to BG$ is a cocycle in the classical sense: it is determined by sections

$$f_{\alpha,\beta} \in G(U_\alpha \cap U_\beta)$$

such that the composition law holds for the various restrictions in $G(U_\alpha \cap U_\beta \cap U_\gamma)$, etc., because f is induced by a map of groupoids. The corresponding picture

$$* \xleftarrow{\simeq} C(U) \xrightarrow{f} BG$$

is a member of the cocycle category $h(*, BG)$.

Example 4. Suppose that H is an algebraic group defined over a field k, and that L/k is a finite Galois extension with Galois group G. A morphism

$$f : EG \times_G \mathrm{Sp}(L) \to BH$$

consists of a function $f : G \to H(L)$ which again satisfies the classical cocycle conditions

$$f(hg) = g^*(f(h))f(g).$$

In effect, the diagram

$$\begin{tikzcd} 1_L \arrow[r, "g"] \arrow[dr, "hg"'] & g \arrow[d, "g^*(h)"] \\ & hg \end{tikzcd}$$

commutes in the Galois groupoid $E_G \mathrm{Sp}(L)$. Any such map f determines a cocycle

$$* \xleftarrow{\simeq} EG \times_G \mathrm{Sp}(L) \xrightarrow{f} BH.$$

Remark 5. The cocycle category construction was originally informed by these classical examples, but it is thoroughly modern in the sense that the map g in a cocycle

$$X \underset{\simeq}{\xleftarrow{g}} U \xrightarrow{f} Y$$

can be *any* local weak equivalence. The map g does not have to be anything like a hypercover (as is the map $C(U) \to *$ because $C(U)$ is a presheaf of Kan complexes).

Remark 6. Theorem 2 is a generalized version of the Verdier hypercovering theorem, in which we compute path components of cocycle categories instead of relying on filtered colimit constructions associated to hypercovers, which constructions can be quite fussy.

Non-abelian cohomology

$G =$ sheaf of groups.

A G-torsor is a sheaf X with a free G-action which is also transitive in the sense that $X/G \cong *$ in the sheaf category.

Example 7. Suppose that G is a topological group and that Y is a topological space. Every principal G-bundle $X \to Y$ (ie. X has free G-action such that $X/G \cong Y$) represents a G-torsor on $\mathrm{op}\,|_Y =$ site of open subsets of Y, and conversely. G represents a sheaf of groups $\hom(\ ,G)$ on $\mathrm{op}\,|_Y$, and Y represents the terminal sheaf $*$ on $\mathrm{op}\,|_Y$.

Fact: $G \times X \to X$ is a free action means precisely that the canonical map $EG \times_G X \to X/G$ is a local weak equivalence.

Definition: X is a G-torsor iff $EG \times_G X \to *$ is a local weak equivalence.

Example 8. L/k finite Galois extension with Galois group G: $\mathrm{Sp}(L)$ is a G-torsor on $et|_{\mathrm{Sp}(k)}$.

$G - \mathbf{Tors}$ is the category of G-torsors and G-equivariant maps. It is a groupoid:

A map $f : X \to Y$ of G-torsors is induced on fibres by the map of local fibrations (stalkwise Kan fibrations)

$$\begin{tikzcd} EG \times_G X \arrow[rr] \arrow[dr] & & EG \times_G Y \arrow[dl] \\ & BG & \end{tikzcd}$$

Then $f : X \to Y$ is a weak equiv. of constant simplicial sheaves, hence an isomorphism.

Now, here's a basic construction:

Suppose $* \xleftarrow{\simeq} Y \xrightarrow{\alpha} BG$ is a cocycle, form pullback

$$\begin{tikzcd} \mathrm{pb}(Y) \arrow[r] \arrow[d] & Y \arrow[d, "\alpha"] \\ EG \arrow[r, "\pi"'] & BG \end{tikzcd}$$

$\mathrm{pb}(Y)$ has a G-action (from EG), and the map

$$EG \times_G \mathrm{pb}(Y) \to Y$$

is a local weak equivalence. The square is htpy cartesian, so that $\mathrm{pb}(Y) \to \tilde{\pi}_0\,\mathrm{pb}(Y)$ is a G-equiv. local weak equivalence.

[$\tilde{\pi}_0\,\mathrm{pb}(Y)$ is the sheaf of path components of the simplicial sheaf $\mathrm{pb}(Y)$, in this case identified with a constant simplicial sheaf.]

The maps

$$EG \times_G \tilde{\pi}_0\,\mathrm{pb}(Y) \leftarrow EG \times_G \mathrm{pb}(Y) \to Y \simeq *$$

are local weak equivs, so $\tilde{\pi}_0\,\mathrm{pb}(Y)$ is a G-torsor.

A functor

$$h(*, BG) \to G - \mathbf{Tors}$$

is defined by

$$(* \xleftarrow{\simeq} Y \to BG) \mapsto \tilde{\pi}_0\,\mathrm{pb}(Y).$$

A functor

$$G - \mathbf{Tors} \to h(*, BG) :$$

is defined by

$$X \mapsto (* \xleftarrow{\simeq} EG \times_G X \to BG).$$

I call this the "canonical cocycle" functor. It is right adjoint to the "pullback" functor.

Theorem 9. *These functors induce bijections*

$$[*, BG] \cong \pi_0 h(*, BG) \cong \pi_0(G{-}\mathbf{Tors}) = H^1(\mathcal{C}, G).$$

Theorem 9 holds over arbitrary small Grothendieck sites, and is 25 years old. Unlike the original proof, you have heard no references to hypercovers or pro objects.

Example 10. Suppose that k is a field with $char(k) \neq 2$. Let $\mathcal{O}_n$ be the algebraic group of isometries of the trivial bilinear form of rank n. A non-degenerate symmetric bilinear form β of rank n over k can be identified with a map $\beta : * \to BO_n$ in the homotopy category for simplicial sheaves on $et|_k$. There is an isomorphism

$$H^*_{et}(BO_n, \mathbb{Z}/2) \cong H^*_{et}(k, \mathbb{Z}/2)[HW_1, \dots, HW_n]$$

where $deg(HW_i) = i$. The form (homotopy class) β determines a map

$$\beta^* : H^*_{et}(BO_n, \mathbb{Z}/2) \to H^*_{et}(k, \mathbb{Z}/2)$$

and $\beta^*(HW_i) =: HW_i(\beta)$ are the higher Hasse-Witt invariants of β (Delzant Stiefel-Whitney classes). $HW_2(\beta)$ is the classical Hasse-Witt invariant of β, and $HW_1(\beta)$ is the determinant.

There is a corresponding result for sheaves of groupoids H. An (internal) H-diagram consists of a sheaf map $F \to \mathrm{Ob}(H)$, with an H-action

$$\begin{tikzcd} F \times_{\mathrm{Ob}(H)} \mathrm{Mor}(H) \arrow[r] \arrow[d] & F \arrow[d] \\ \mathrm{Mor}(H) \arrow[r, "t"'] & \mathrm{Ob}(H) \end{tikzcd}$$

which is associative, and has two-sided identities. This is the same thing as a system of diagrams $F(U) : H(U) \to \mathbf{Set}$ which respect restriction (and such that the presheaf $U \mapsto F(U)$ is a sheaf), and so one can form the homotopy colimit construction

$$\underrightarrow{\mathrm{holim}}_{H} F \to BH.$$

An H-*torsor* is an H diagram F in sheaves such that the map $\underrightarrow{\mathrm{holim}}_{H} F \to *$ is a local weak equivalence. The category $H - \mathbf{Tors}$ of H-torsors is a groupoid, and we have the following:

Theorem 11. *There is a natural bijection*

$$[*, BH] \cong \pi_0(H - \mathbf{Tors})$$

The proof is a generalization of the proof of Theorem 9.

The ideas generalize further: torsors are defined for sheaves of categories A, and there is an identification of $[*, BA]$ with path components of the category of weak equivalences of A-torsors, suitably defined.

Example 12. Suppose that G is a sheaf of groups and that X is a sheaf with G-action. A G-torsor over X is a G-equivariant map $P \to X$ where P is a G-torsor. A map of such things is a commutative diagram of G-equivariant maps

$$\begin{tikzcd} P \arrow[rr, "\theta"] \arrow[dr] & & P' \arrow[dl] \\ & X & \end{tikzcd}$$

and of course θ is an isomorphism.

If $P \to X$ is a G-torsor over X, then the maps

$$* \xleftarrow{\simeq} EG \times_G P \to EG \times_G X \to BG$$

define a cocycle, ie. a member of $h(*, EG \times_G X)$. Given a cocycle $* \xleftarrow{\simeq} U \to EG \times_G X$, then pulling back over EG defines G-equivariant maps

$$\tilde{\pi}_0(\mathrm{pb}(U)) \to \tilde{\pi}_0 \, \mathrm{pb}(EG \times_G X) \xrightarrow{\cong} X$$

giving a G-torsor over X. These constructions are inverse to each other, giving a bijection

$$\pi_0(G\text{-torsors over } X) \cong [*, EG \times_G X].$$

Stacks and homotopy theory

The injective model structure for simplicial sheaves restricts to a model structure for sheaves of groupoids, essentially because the fundamental groupoid functor preserves local weak equivalences.

A map $G \to H$ of sheaves of groupoids is a local weak equivalence (aka. *Morita equivalence*) if the induced map $BG \to BH$ is a local weak equivalence of simplicial sheaves. The map $p : G \to H$ is a fibration if the induced map $BG \to BH$ is an injective fibration. In particular, G is fibrant if and only if BG is injective fibrant. All fibrant objects are *stacks* in that they satisfy effective descent.

All stacks H satisfy *homotopy theoretic descent*, in that if $H \to K$ is a fibrant model then all maps $BH(U) \to BK(U)$ (in sections) are weak equivalences.

Thus, every fibrant model $G \to H$ defines a stack completion for G. We can therefore, in practice, identify the homotopy type of a sheaf (or presheaf) of groupoids with the sectionwise homotopy type of its associated stack.

If $G \to H$ is a stack completion, then there is an isomorphism

$$[*, BG] \cong \pi_0 \Gamma_* H,$$

where Γ_* is global sections.

Example 13. The stack completion of the sheaf of groupoids $E_G X$ is the quotient stack X/G. Example 12 says that the classical description of the quotient stack is exactly right from the homotopy theoretic point of view.

Lemma 14. *G and H are locally weakly equivalent if and only if they are Morita equivalent.*

G and H are Morita equivalent if and only if there are maps

$$G \leftarrow K \rightarrow H$$

such that the induced maps

$$BG \leftarrow BK \rightarrow BH$$

are local trivial fibrations (aka. hypercovers, also: Morita morphisms which are essential equivalences). Such maps are in particular local weak equivalences, so G and H are locally weakly equivalent if they are Morita equivalent.

Suppose that $f : G \rightarrow H$ is a local weak equivalence. Then the cocycle

$$G \underset{\simeq}{\xleftarrow{1}} G \xrightarrow{f} H$$

determines a commutative diagram

$$\begin{tikzcd} G \arrow[r, "j"] \arrow[dr, "(1,f)"'] & K \arrow[d, "p"] \\ & G \times H \end{tikzcd}$$

where $p = (p_1, p_2)$ is a fibration and j is a weak equivalence. Both maps p_1, p_2 are local trivial fibrations, so that G and H are Morita equivalent.

Appendix:

1) **Proof of Theorem 2**

Lemma 1: Suppose $\alpha : X \to X'$ and $\beta : Y \to Y'$ are weak equivalences. Then

$$(\alpha, \beta)_* : \pi_0 H(X, Y) \to \pi_0 H(X', Y')$$

is a bijection.

Lemma 2: Suppose that Y is fibrant and X is cofibrant. Then the canonical map

$$\phi : \pi_0 H(X, Y) \to [X, Y]$$

is a bijection.

Proof of Lemma 1 $(f, g) \in H(X', Y')$ is a map $(f, g) : Z \to X' \times Y'$ s.t. f is a weak equivalence.

There is a factorization

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "(f,g)"'] & W \arrow[d, "(p_{X'}, p_{Y'})"] \\ & X' \times Y' \end{tikzcd}$$

s.t. j is a triv. cofibration and $(p_{X'}, p_{Y'})$ is a fibration. $p_{X'}$ is a weak equivalence.

Form the pullback

$$\begin{tikzcd}
W_* \arrow[r, "(\alpha\times\beta)_*"] \arrow[d, "(p_X^*, p_Y^*)"'] & W \arrow[d, "(p_{X'}, p_{Y'})"] \\
X \times Y \arrow[r, "\alpha\times\beta"'] & X' \times Y'
\end{tikzcd}$$

(p_X^*, p_Y^*) is a fibration and $(\alpha\times\beta)_*$ is a weak equivalence (since $\alpha \times \beta$ is a weak equivalence, and by right properness). p_X^* is also a weak equivalence.

$(f, g) \mapsto (p_X^*, p_Y^*)$ defines a function

$$\pi_0 H(X', Y') \to \pi_0 H(X, Y)$$

which is inverse to $(\alpha, \beta)_*$. □

Proof of Lemma 2 $\pi(X, Y) =$ naive homotopy classes.

$\pi(X, Y) \to [X, Y]$ is a bijection since X is cofibrant and Y is fibrant.

We have seen that the assignment $f \mapsto [(1_X, f)]$ defines a function

$$\psi : \pi(X, Y) \to \pi_0 H(X, Y)$$

and there is a diagram

$$\begin{tikzcd}
\pi(X, Y) \arrow[r, "\psi"] \arrow[dr, "\cong"'] & \pi_0 H(X, Y) \arrow[d, "\phi"] \\
 & {[X, Y]}
\end{tikzcd}$$

It suffices to show that ψ is surjective, or that any object $X \xleftarrow{f} Z \xrightarrow{g} Y$ is in the path component of some a pair $X \xleftarrow{1} X \xrightarrow{k} Y$ for some map k.

Form the diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "j"] & \\ X & & Y \\ & V \arrow[ul, "p"] \arrow[ur, dotted, "\theta"'] & \end{tikzcd}$$

where j is a triv. cofibration and p is a fibration; θ exists because Y is fibrant.

X is cofibrant, so the trivial fibration p has a section σ, and so there is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "1"'] \arrow[dr, "\theta\sigma"] \arrow[dd, "\sigma"] & \\ X & & Y \\ & V \arrow[ul, "p"] \arrow[ur, dotted, "\theta"'] & \end{tikzcd}$$

The composite $\theta\sigma$ is the required map k. $\square$

Proof of Theorem 1: There are weak equivalences $\pi : X' \to X$ and $j : Y \to Y'$ such that X'

and Y' are cofibrant and fibrant, respectively.

$$\begin{tikzcd}
\pi_0 H(X,Y) \arrow[r, "\phi"] \arrow[d, "(1{,}j)_*"', "\cong"] & {[X,Y]} \arrow[d, "\cong"', "j_*"] \\
\pi_0 H(X,Y') \arrow[r, "\phi"] & {[X,Y']} \arrow[d, "\cong"', "\pi^*"] \\
\pi_0 H(X',Y') \arrow[r, "\cong", "\phi"'] \arrow[u, "(\pi{,}1)_*", "\cong"'] & {[X',Y']}
\end{tikzcd}$$

$(1,j)_*$ and $(\pi,1)_*$ are bijections by the first Lemma, and ϕ is a bijection by the second. $\square$