Galois cohomological descent

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Étale site of a field

k a field, with $k \hookrightarrow \overline{k}$.

Consider all $k \subset L \subset \overline{k}$ (defined by elements), L/k finite, separable, ie. $L = k(\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \overline{k}$, defined by separable polynomials.

Fin_k: all finite separable extensions L/k, k-alg. morphisms $L \to L'$.

A **presheaf** (of sets) on Fin_k is a covariant functor $E : \mathrm{Fin}_k \to \mathbf{Set}$.

E is an **étale sheaf** if, for every finite Galois extension N/L in Fin_k with Galois group H, there is a bijection

$$E(L) \xrightarrow{\cong} E(N)^H.$$

$$\sqcup_H \ \mathrm{Sp}(N) \cong \mathrm{Sp}(N) \times_{\mathrm{Sp}(k)} \mathrm{Sp}(N) \rightrightarrows \mathrm{Sp}(N) \xrightarrow{\pi} \mathrm{Sp}(L) \qquad (1)$$

def. by action of H on N, sheaf coequalizer for étale cover π.

Stalks

Sheaves and presheaves E have stalks at geo. point $k \subset \overline{k}$.

The **stalk** $E(\overline{k})$ of a presheaf E is the filtered colimit

$$E(\overline{k}) = E(k_{sep}) = \varinjlim_{L \subset \overline{k} \text{ fin. sep.}} E(L).$$

If E is a presheaf, set

$$\tilde{E}(L) = E(\overline{k})^{G_L} = \varinjlim_{N/L \text{ fin. Gal.}} E(N)^{G(N/L)}.$$

Pro-group $G_L = \{\ G(N/L)\ \}$ is the **absolute Galois group** of L.

$\eta : E(L) \to \tilde{E}(L)$ define the **associated sheaf map** $\eta : E \to \tilde{E}$.

η induces an isom. of stalks $E(\overline{k}) \xrightarrow{\cong} \tilde{E}(\overline{k})$, since (1) pulls back to (split) presheaf coequalizer over $\mathrm{Sp}(N)$.

Schemes and sheaves

Every k-scheme S represents a sheaf on Fin_k:

$$S(L) = \hom_k(\mathrm{Sp}(L), S)$$

$S(L)$ is the set of L-**points** of S.

$S(N)$ def. by solutions $\alpha_1, \dots, \alpha_n \in N$ of $f(x_1, \dots, x_n) = 0$.

If $H = Gal(N/L)$, then H acts on the α_i (hence on the N-points of S), and fixes the α_i if and only if they live in L.

Thus

$$S(L) \cong S(N)^H,$$

and S is a sheaf.

Examples

1) Gl_n is the algebraic group of $n \times n$ invertible matrices over k, defined as a scheme by

$$Gl_n = \mathrm{Sp}(k[x_{i,j}]_{det}).$$

$Gl_n(L)$ is the group of $(n \times n)$ invertible matrices in L.

The stalk is $Gl_n(k_{sep})$.

2) If N/k is a finite Galois extension, then $\mathrm{Sp}(N)$ represents a sheaf, with

$$\mathrm{Sp}(N)(L) = \hom_k(N, L).$$

The stalk

$$\mathrm{Sp}(N)(\overline{k}) = \hom(N, \overline{k}) \cong Gal(N/k)$$

3) X a set. The **constant presheaf** $\Gamma^* X$ has $\Gamma^* X(L) = X$.

Simplicial presheaves and sheaves

A **simplicial presheaf** X on Fin_k is a covariant functor $\mathrm{Fin}_k \to s\mathbf{Set}$.

A **simplicial sheaf** is a simplicial presheaf X such that each presheaf X_n is a sheaf.

Examples:

1) Gl_n is a sheaf of groups. BGl_n is a simplicial sheaf, with p-simplices $Gl_n^{\times p}$.

1.5) If H is a sheaf of groups (or groupoids, or categories), then BH is a simplicial sheaf.

2) $BGl = \varinjlim_n BGl_n$ is a simplicial presheaf.

3) The terminal simplicial presheaf $*$ on Fin_k, rep. by $\mathrm{Sp}(k)$.

$*(L)$ is a one-point set: there is only one k-alg. morphism $k \to L$.

A trick

$f : E \to F$ a function.

Groupoid E/f: objects E, and a unique map $x \to y$ if and only if $f(x) = f(y)$.

There are simplicial set maps

$$B(E/f) \xrightarrow{\simeq} f(E) \subset F.$$

If $f : E \to F$ is surjective, then $B(E/f) \to F$ is a weak equivalence.

This construction is absolutely functorial, hence applies to presheaves and sheaves.

Gives all Čech resolutions: if $\pi : U \to X$ is covering (aka. sheaf epi), then $B(U/\pi) = \check{C}(U)$, and is the simplicial sheaf

$$\dots \substack{\rightarrow\\\rightarrow\\\rightarrow} U \times_F U \rightrightarrows U \xrightarrow{\pi} V : \quad \check{C}(U) \xrightarrow{\simeq} V.$$

$G = Gal(L/k)$ finite. $\pi : \mathrm{Sp}(L) \to *$ is an étale cover (surjective in stalks).

In N-sections, objects are morphisms $L \to N$ (if they exist). Any two determine a unique picture

$$\begin{tikzcd} L \arrow[d, "\sigma \in Gal(L/k)"'] \arrow[dr] & \\ L \arrow[r] & N \end{tikzcd}$$

$E_\pi = E_G(\mathrm{Sp}(L)(N))$ is *translation category* for action of G on $\mathrm{Sp}(L)(N)$, and

$$B(E_G\,\mathrm{Sp}(L)) = EG \times_G \mathrm{Sp}(L) \to *$$

is a weak equivalence in sections for which there are maps $L \to N$.

$\{E_G\,\mathrm{Sp}(L)\}$ is the **abs. Galois groupoid**, with $E_G\,\mathrm{Sp}(L) \to G$, $EG \times_G \mathrm{Sp}(L) \to BG = \Gamma^* BG$.

Model structure on $s\,\mathrm{Pre}(\mathrm{Fin}_k)$ = category of simplicial presheaves on Fin_k:

Cofibrations are monomorphisms. The **stalkwise (or local) weak equivalences** are those $X \to Y$ which induce weak equivalences $X(\overline{k}) \xrightarrow{\simeq} Y(\overline{k})$ of stalks. The fibrations are **injective fibrations**.

There is a similarly defined model structure on $s\,\mathrm{Shv}(\mathrm{Fin}_k)$, which is Quillen equivalent:

$$L^2 : s\,\mathrm{Pre}(\mathrm{Fin}_k) \leftrightarrows s\,\mathrm{Shv}(\mathrm{Fin}_k) : u.$$

Examples: 1) $\eta : X \to \tilde{X}$ is a stalkwise isomorphism, hence a stalkwise equivalence.

2) $G = Gal(L/k)$: $EG \times_G \mathrm{Sp}(L) \to *$ is a stalkwise. equiv. (also hypercover)

Cohomology

1) $[\mathrm{Sp}(L), K(F,n)] \cong H^n_{et}(L,F)$ for finite separable extensions L/k and abelian sheaves F.

This applies to k: $H^n_{et}(k,F) \cong [*, K(F,n)]$.

2) If H is an algebraic group over k, then

$$\begin{aligned} [*, BH] &\cong \{\text{iso. classes of } H\text{-torsors}\} \\ &=: H^1_{et}(k,H). \end{aligned}$$

2.5) If L/k is a finite Galois extension with Galois group G, then $\mathrm{Sp}(L) \to *$ is a non-trivial G-torsor. There is a cocycle

$$* \xleftarrow{\simeq} EG \times_G \mathrm{Sp}(L) \to BG,$$

and $\mathrm{Sp}(L)$ does not have global sections $L \to k$.

$BG = \Gamma^* BG$ is a constant simplicial sheaf.

Function complexes

There is a function complex construction $\mathbf{hom}(X, Y)$, with

$$\mathbf{hom}(X, Y)_p = \{X \times \Delta^p \to Y\}$$

If Z is injective fibrant and $X \to Y$ a stalkwise weak equiv., then

$$\mathbf{hom}(Y, Z) \to \mathbf{hom}(X, Z)$$

is a weak equivalence of simplicial sets, by formal nonsense.

Lemma 1.

The map

$$\begin{aligned} Z(k) \cong \mathbf{hom}(*, Z) &\to \mathbf{hom}(EG \times_G \mathrm{Sp}(L), Z) \\ &\cong \underleftarrow{\mathrm{holim}}_G\ Z(L) = Z(L)^{hG} \end{aligned}$$

is a weak equivalence if Z is injective fibrant.

Proof.

$EG \times_G \mathrm{Sp}(L) \to *$ is a stalkwise weak equivalence. $\square$

Finite descent

If L/k finite Gal. with Gal. group G and Z is injective fibrant then there is a weak equiv.

$$Z(k) \cong \mathbf{hom}(*, Z) \xrightarrow{\simeq} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), Z) \cong \underleftarrow{\mathrm{holim}}_G\ Z(L)$$

i.e.: $Z(k) =$ htpy fixed points space for the action of G on $Z(L)$.

Same statement: injective fibrant simplicial presheaves Z "satisfy finite Galois descent".

If X is a presheaf of Kan complexes, say that X **satisfies finite descent** if all induced maps

$$X(k) \cong \mathbf{hom}(*, X) \to \mathbf{hom}(EG \times_G \mathrm{Sp}(L), X)$$

are weak equivalences.

An **injective fibrant model** for X is a stalkwise weak equivalence $j : X \to Z$ with Z injective fibrant.

Say that X **satisfies descent** (or Galois cohomological descent), if some (hence any) injective fibrant model $j : X \to Z$ is a sectionwise weak equivalence, i.e. induces weak equivalences

$$X(L) \xrightarrow{\simeq} Z(L)$$

of simplicial sets for all $L \in \mathrm{Fin}_k$.

Facts: 1) A stalkwise weak equivalence $Z \to W$ of injective fibrant objects is a sectionwise equivalence.

2) Any two injective fibrant models of a fixed object X are sectionwise weakly equivalent.

Example: All injective fibrant objects satisfy descent.

Descent implies finite descent

Lemma 2.

Suppose that a presheaf of Kan complexes X on Fin_k satisfies descent. Then X satisfies finite descent.

Proof.

If X satisfies descent, and $j : X \to Z$ is a fibrant model, then there is a diagram

$$\begin{array}{ccc} X(k) & \longrightarrow & \underleftarrow{\mathrm{holim}}_G\, X(L) \\ \simeq \big\downarrow j & & j_* \big\downarrow \simeq \\ Z(k) & \xrightarrow[\simeq]{} & \underleftarrow{\mathrm{holim}}_G\, Z(L) \end{array}$$

□

Lemma 3.

Suppose that A is an abelian sheaf on Fin_k. Then

$$\begin{aligned} H^n_{et}(k,A) &\cong \varinjlim_{L/k \ Gal} H^n \hom(EG \times_G \mathrm{Sp}(L), A) \\ &= \check{H}^n(G_k, A) \ \textit{(classical Galois cohomology).} \end{aligned}$$

Sketch: • $\check{H}^*(G_k, B) = 0$ for abelian presheaves J st. $\tilde{B} = 0$, e.g. all homology sheaves of cokernel of an injective resolution $A \to I$.

• $\check{H}^*(G_k, A) = 0$ for simplicial abelian presheaves st. $\tilde{H}_k(A) = 0$ for all k and A has only finitely many non-trivial homology presheaves.

• The map

$$\hom(*, Tr(I[-p])) \to \hom(EG \times_G \mathrm{Sp}(L), Tr(I[-p]))$$

is quasi-iso, since $EG \times_G \mathrm{Sp}(L) \to *$ is stalkwise weak equiv.

Galois hypercohomology

Define a **Galois hypercohomology** space

$$\check{H}(G, E) = \varinjlim_{G = Gal(L/k)} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), E)$$

for any presheaf of Kan complexes E on Fin_k.

There is a canonical map

$$\phi : E(k) \to \check{H}(G, E),$$

induced by all $EG \times_G \mathrm{Sp}(L) \to *$.

ϕ is **not** a weak equivalence in general.

Example: $E = K(A, n)$ for abelian sheaves A s.t. $H^n_{Gal}(k, A) \neq 0$.

Comparisons

Take an injective fibrant model $j : E \to Z$. There is a diagram

$$\begin{tikzcd} E(k) \arrow[r] \arrow[d, "j_*"'] & \check{H}(G, E) \arrow[d, "j_*"] \\ Z(k) \arrow[r, "\simeq"'] & \check{H}(G, Z) \end{tikzcd}$$

The bottom map is a weak equivalence since Z is injective fibrant.

Dream: Pretend that $j_* : \check{H}(G, E) \to \check{H}(G, Z)$ is a weak equivalence (eg. $E = K(A, n)$).

If E satisfies finite descent, then $j_* : E(k) \to Z(k)$ is a weak equiv.

Wakeup: It's **not clear** that the Galois hypercohomology invariant $E \mapsto \check{H}(G, E)$ preserves stalkwise equivalence.

Theorem 4.

Suppose $f : E \to F$ is a stalkwise weak equiv. on Fin_k such that E, F have only finitely many non-trivial **presheaves** *of homotopy groups. Then*

$$\check{H}(G, E) \to \check{H}(G, F)$$

is a weak equivalence.

Corollary 5.

Suppose that E has only finitely many non-trivial presheaves of homotopy groups, and that $j : E \to Z$ is an injective fibrant model on Fin_k. Then the induced map

$$\check{H}(G, E) \to \check{H}(G, Z)$$

is a weak equivalence.

Here is how Corollary 4 is proved:

If $X \to P_n X$ is a sectionwise weak equivalence and $j : X \to Z$ is an injective fibrant model, then $Z \to P_n Z$ is a sectionwise weak equivalence. Apply Theorem 3 to the map $j : X \to Z$

Reason: If $j : K(A, n) \to Z$ is an injective fibrant model, then

$$\pi_i Z(L) \cong \begin{cases} H^{n-i}_{Gal}(L, A) & \text{if } 0 \leq i \leq n, \\ 0 & \text{if } i > n. \end{cases}$$

Remark: If H is a sheaf of groupoids, then $\check{H}(G, BH)$ is global sections of the associated stack.

Suppose E is a locally connected presheaf of Kan complexes on $et|_k$ whose sheaves of homotopy groups are ℓ-primary torsion. Suppose that k has finite Galois cohomological dimension with respect to ℓ-primary torsion sheaves. (eg. $(\mathbf{K}/\ell)^n$, $n > 0$.)

Suppose that $j : E \to ZE$ is the injective fibrant model functor.

Look at the diagram of simplicial sets

$$\begin{tikzcd}
\check{H}(G, P_nE) \arrow[r, "\simeq", "j_*"'] & \check{H}(G, ZP_nE) \\
P_nE(k) \arrow[u, "\alpha"] \arrow[r, "j"] & ZP_nE(k) \arrow[u, "\simeq", "\alpha"'] \\
E(k) \arrow[u, "p"] \arrow[r, "j"'] & ZE(k) \arrow[u, "p'"']
\end{tikzcd}$$

Some observations

1) j_* is a weak equivalence by the Corollary.

2) The indicated map α is a weak equivalence, since ZP_nE is injective fibrant.

3) Letting n vary gives a diagram of pro-objects.

The map p is a pro-equivalence by definition, while the map p' is a pro-equivalence since the homotopy group sheaves of E are ℓ-torsion and k has finite Galois cohomological dimension with respect to ℓ-torsion sheaves.

4) It follows that the map $j : E(k) \to ZE(k)$ is a pro-equivalence if and only if either

$$j_* : P_*E(k) \to ZP_*E(k)$$

is a pro-equivalence, or (equivalently) the maps

$$P_nE(k) \to \check{H}(G, P_nE)$$

define a pro-equivalence.

5) One can show that $j : E(k) \to ZE(k)$ is a pro-equivalence if and only if it is a weak equivalence of spaces.

6) Thus $j : E(k) \to ZE(k)$ is a weak equivalence if and only if the map

$$P_n E(k) \to \check{H}(G, P_n E)$$

is a pro-equivalence.

This last statement is a pro-version of the finite descent property for E.

Pro-equivalences

A map $f : X \to Y$ of pro-objects is an **EH-equivalence** if

$$\varinjlim_{j} [Y_j, Z] \to \varinjlim_{i} [X_i, Z]$$

is a bijection for all fibrant Z. (EH: Edwards-Hastings).

f is a **pro-equivalence** if $P_*X \to P_*Y$ is an EH-equivalence.

Lemma 6.

Suppose that $f : X \to Y$ is a map of spaces such that $P_*X \to P_*Y$ is a pro-equivalence. Then f is a weak equivalence.

Proof.

$P_nX \to P_nP_*X$ is an EH-equivalence and P_n preserves EH-equivalences. Then $P_nX \to P_nY$ is an EH-equivalence and hence a weak equivalence for all n. $\square$

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