Galois descent and pro objects

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Suppose that k is a field, and let $et|_k$ be its **finite étale site**. This category consists of all schemes

$$U = \mathrm{Sp}(L_1) \sqcup \cdots \sqcup \mathrm{Sp}(L_n)$$

where each L_i/k is a finite separable extension, and all k-scheme (equivalently k-algebra) maps between them.

A **simplicial presheaf** $X : et|_k^{op} \to s\mathbf{Set}$ on $et|_k$ is a contravariant functor defined on $et|_k$ which takes values in simplicial sets, and a **morphism of simplicial presheaves** is a natural transformation of such functors. I usually write

$$s\,\mathrm{Pre}(et|_k)$$

for the category of simplicial presheaves on this site.

Example: the mod ℓ algebraic K-theory presheaf K/ℓ^0, which is defined by $U \mapsto K/\ell(U)^0$, where ℓ is a prime distinct from the characteristic of k. This is the "space" at level 0 of the mod ℓ algebraic K-theory presheaf of spectra K/ℓ, which is defined in the stable category by the cofibre sequence

$$K \xrightarrow{\times \ell} K \to K/\ell$$

The homotopy groups $\pi_j K/\ell^0(U)$ are the mod ℓ K-groups $K_j(U, \mathbb{Z}/\ell)$ of the scheme U. For the record, there is a weak equivalence

$$K^0(U) \simeq \Omega BQ\mathcal{P}(U),$$

where $BQ\mathcal{P}(U)$ is Quillen's algebraic K-theory space which is associated to the exact category $\mathcal{P}(U)$ of vector bundles on U.

There is an **injective model structure** for simplicial presheaves on the étale site $et|_k$, for which the cofibrations are the monomorphisms, the (local) weak equivalences are defined stalkwise, and the fibrations, the injective fibrations, are defined by a right lifting property. Here, a map $X \to Y$ is a **local weak equivalence** if the induced map

$$\varinjlim_{L/k} X(L) \to \varinjlim_{L/k} Y(L)$$

is a weak equivalence of simplicial sets, where L varies within the finite Galois extensions L/k in a fixed algebraic closure (geometric point) of k. The local weak equivalences very much depend on the étale topology.

Injective fibrant objects Z, while difficult to "see", have the magic propery that the homotopy groups

$\pi_k Z(k)$ in global sections (indeed, in all sections) can be recovered from étale cohomology via an étale cohomological descent spectral sequence.

Every simplicial presheaf X on $et|_k$ has an **injective fibrant model** $j : X \to GX$, which is a local weak equivalence such that GX is injective fibrant. We say that X **satisfies étale descent** if the map j is a sectionwise weak equivalence in the sense that all maps

$$X(U) \to GX(U)$$

are weak equivalences of simplicial sets.

What we *really* want to know, in applications, is the extent to which the map

$$X(k) \to GX(k)$$

in global sections is a weak equivalence.

This question seems more naive than the full descent question, but typically answers about the map in global sections depend on conditions on the Galois cohomological dimension of k, which are the same across the full étale site $et|_k$. Thus, the global sections question is, in practice, equivalent to a full descent problem.

In this context, here's a reformulation of the Lichtenbaum-Quillen conjecture for algebraic K-theory:

Conjecture: Suppose that k is a field and that $\ell > 2$ is a prime which is distinct from the characteristic of k. Suppose that k has finite Galois cohomological dimension d with respect to ℓ-torsion sheaves. Then the map

$$K_i(k, \mathbb{Z}/\ell) = \pi_i K/\ell^0(k) \to \pi_i GK/\ell^0(k) =: K_i^{et}(k, \mathbb{Z}/\ell)$$

is an isomorphism if $i \geq d - 1$.

Remark: This conjecture is known to be a consequence of the Bloch-Kato conjecture [6], which is "proved" with motivic techniques. This was not the original formulation of the conjecture: étale K-theory and the comparison map were originally defined by quite different models (étale homotopy theory of Friedlander, Dwyer-Friedlander [2]), and the map was originally thought to be an isomorphism in high but unspecified degrees. The bound $d - 1$ was introduced much later.

Remark: There is an $\ell = 2$ version of this conjecture, in which cohomological dimension is replaced by virtual cohomological dimension. This was proved by Ostvaer and Rosenschon [5], as a consequence of the Milnor conjecture (mod 2 version of Bloch-Kato) which was proved by Voevodsky.

One effect of the success of the motivic approach is that the Galois cohomological approach to Lichtenbaum-Quillen (which arose from Thomason's descent theorem for Bott periodic K-theory) was completely dropped in the mid 1990s. There were also problems with the technical methods of the day that nobody could completely solve at the time. This talk addresses one of them.

The Galois cohomological approach to étale descent for fields is based on a construction of a model for the global sections of the injective fibrant model GX of a simplicial presheaf X, which I will now describe.

Suppose that L/k is a finite Galois extension of k with Galois group $G = G(L/k)$ (inside the chosen geometric point for k). Then the corresponding scheme homomorphism $\mathrm{Sp}(L) \to \mathrm{Sp}(k)$ is an étale cover for $\mathrm{Sp}(k)$, and represents a sheaf epimorphism $\mathrm{Sp}(L) \to *$ onto the terminal object in sheaves on the étale site $et|_k$.

In general, if $f : X \to Y$ is a function then there is a groupoid whose objects are the elements of

X and whose morphisms are the pairs (x, x') such that $f(x) = f(x')$. This groupoid has trivial automorphism groups and path components isomorphic to the set $f(X)$. Write $C(f)$ for the corresponding nerve. The construction is functorial, so we can apply it in all sections to the sheaf map $\mathrm{Sp}(L) \to *$ to give a simplicial sheaf $C(L)$. This is the Čech resolution for the covering (this is the way that all Čech resolutions are constructed). It's a consequence of elementary Galois theory that there is a functorial isomorphism of simplicial sheaves

$$C(L) \cong EG \times_G \mathrm{Sp}(L)$$

The simplicial sheaf map $C(L) \to *$ is a local weak equivalence, and is otherwise known as a (special type of) hypercover, because $C(L)$ is a diagram of Kan complexes and is therefore locally fibrant.

Some culture: 1) The object $C(L)$ is the nerve of a sheaf of groupoids $E_G \,\mathrm{Sp}(L)$ which is defined by the action of the Galois group G on the sheaf $\mathrm{Sp}(L)$. As such it is a homotopy theoretic model for the étale quotient stack.

2) The construction is functorial, and the assignment $L \mapsto E_{G(L)} \,\mathrm{Sp}(L)$ is a pro object in simplicial groupoids, which I call the *absolute Galois*

groupid. There are canonical functorial comparison maps

$$E_{G(L)}\,\mathrm{Sp}(L) \to \Gamma^* G(L)$$

or

$$EG(L)_{G(L)}\,\mathrm{Sp}(L) \to \Gamma^* BG(L)$$

which relate the absolute Galois groupoid to the absolute Galois group.

The Čech resolution $C(L)$ is representable by a simplicial scheme which is étale over k in all simplicial degrees, so we are entitled to a cosimplicial space $X(C(L))$ for any simplicial presheaf X, and such a thing gives rise to a homotopy type

$$\underleftarrow{\mathrm{holim}}_m\, X(C_m(L)) = \mathrm{Tot}\, Y(C(L)) \simeq \mathbf{hom}(C(L), Y),$$

(here, $X \to Y$ is a sectionwise fibrant model of X). We can let the finite Galois extension L/k vary within the geometric point — the corresponding diagram is filtering, and so there is a filtered colimit

$$\varinjlim_{L/k} \underleftarrow{\mathrm{holim}}_m\; X(C_m(L)),$$

and a canonical map

$$X(k) \to \varinjlim_{L/k} \underleftarrow{\mathrm{holim}}_m\; X(C_m(L)).$$

Just by the way it's constructed, it looks like there is a Galois cohomological descent spectral sequence for the thing on the right: it should be a colimit of Bousfield-Kan spectral sequences for cosimplicial objects. But this is a variant of the "canonical mistake", because the Bousfield-Kan spectral sequences for the objects

$$\underleftarrow{\operatorname{holim}}_m\, X(C_m(L))$$

are defined by towers of fibrations, and it's not clear at all that the homotopy inverse limit of the filtered colimit of those towers coincides with the filtered colimit of the homotopy inverse limits.

Even worse, it not obvious that the construction

$$\varinjlim_{L/k} \underleftarrow{\operatorname{holim}}_m\, X(C_m(L))$$

is an invariant of local weak equivalences in X.

The construction works if X has only finitely many non-trivial presheaves of homotopy groups. Here's the theorem:

Theorem: Suppose that $f : X \to Y$ is a local weak equivalence between simplicial presheaves on $et|_K$ such that X and Y have only finitely many non-trivial presheaves of homotopy groups. Then the induced map

$$\varinjlim_{L/k} \varprojlim{}_m\, X(C(L)_m) \xrightarrow{f_*} \varinjlim_{L/k} \varprojlim{}_m\, Y(C(L)_m)$$

is a weak equivalence.

Warning: When I say that X has only finitely many non-trivial presheaves of homotopy groups, I mean that the map $X \to P_n X$ to the n^{th} Postnikov section is a sectionwise equivalence for some n.

Corollary: Suppose that X has only finitely many non-trivial presheaves of homotopy groups, and that $j : X \to GX$ is an injective fibrant model for X on $et|_k$. Then the induced map

$$\varinjlim_{L/k} \underleftarrow{\mathrm{holim}}_m\, X(C(L)_m) \xrightarrow{j_*} \varinjlim_{L/k} \underleftarrow{\mathrm{holim}}_m\, GX(C(L)_m)$$

is a weak equivalence.

Proof: If $X \to P_n X$ is a sectionwise weak equivalence, then $GX \to P_n GX$ is a sectionwise weak equivalence.

We've known this for some time: it's a consequence of the fact that if $GK(A, n)$ is an injective fibrant model of the Eilenberg-Mac Lane object associated to a sheaf A, then there is an isomorphism

$$\pi_i GK(A,n)(L) \cong \begin{cases} H^{n-i}_{et}(L,A) & \text{if } 0 \leq i \leq n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Remark: Actually, the Theorem and its Corollary could have been proved twenty years ago: the proof is really just a matter of being careful with cosimplicial objects and various model structures of diagram categories. These results were known, and there were easier proofs, for presheaves of spectra which are bounded below and additive in some sense (as are Postnikov sections of algebraic K-theory). The spectrum level statement was a step in the proof of Thomason's descent theorem for Bott periodic K-theory.

Note: The Corollary for $X = BG$, where G is a sheaf of groupoids, gives yet another construction of global sections of the étale stack associated to G.

An application

Now suppose that X is a simplicial presheaf on $et|_k$ whose étale sheaves of homotopy groups are ℓ-torsion. Suppose that k has finite Galois cohomological dimension with respect to ℓ-torsion sheaves.

Look at the diagram

$$\begin{tikzcd}
\varinjlim_{L/k} \operatorname{\underleftarrow{holim}}_m P_nX(C(L)_m) \arrow[rr, "j_*", "\simeq"'] & & \varinjlim_{L/k} \operatorname{\underleftarrow{holim}}_m GP_nX(C(L)_m) \\
P_nX(k) \arrow[u, "\alpha"] \arrow[rr, "j"] & & GP_nX(k) \arrow[u, "\alpha"', "\simeq"] \\
X(k) \arrow[u, "p"] \arrow[rr, "j"'] & & GX(k) \arrow[u, "p'"']
\end{tikzcd}$$

in simplicial sets, where the indicated (filtered) colimits are indexed on the finite Galois extensions L/k.

Remarks:

1) The indicated map j_* is a weak equivalence by the Corollary.

2) The indicated map α is a weak equivalence, since GP_nX satisfies is injective fibrant: there are weak equivalences

$$\begin{aligned}
&\operatorname{\underleftarrow{holim}}_m (C(L)_m, GP_nX) \\
&\qquad \simeq \mathbf{hom}(C(L), GP_nX) \simeq \mathbf{hom}\ \ (*, GP_nX) \\
&\qquad \cong GP_nX(k).
\end{aligned}$$

3) Letting n vary gives a diagram of pro objects. The map p is a pro equivalence by definition, while the map p' is a pro equivalence since the homotopy

group sheaves of X are ℓ-torsion and k has finite Galois cohomological dimension with respect to ℓ-torsion sheaves.

4) It follows that the map $j : X(k) \to GX(k)$ is a pro equivalence if and only if either

$$j_* : P_*X(k) \to GP_*X(k)$$

is a pro equivalence, or (equivalently) the maps

$$P_nX(k) \to \varinjlim_{L/k} \underleftarrow{\text{holim}}_m P_nX(C(L)_m)$$

define a pro equivalence.

5) We'll see below that $j : X(k) \to GX(k)$ is a pro equivalence if and only if it is a weak equivalence of spaces.

6) Thus $j : X(k) \to GX(k)$ is a weak equivalence if and only if the map

$$P_nX(k) \to \varinjlim_{L/k} \underleftarrow{\text{holim}}_m P_nX(C(L)_m)$$

is a pro equivalence.

Homotopy theory of pro objects

The previous discussion depends on a description of homotopy types of pro objects, which now exists in vast generality.

Let $pro - s\,\mathrm{Pre}(\mathcal{C})$ be the category of pro objects in simplicial presheaves on a small Grothendieck site $\mathcal{C}$.

The *cofibrations* are monomorphisms of $pro - s\,\mathrm{Pre}(\mathcal{C})$. A pro map $f : X \to Y$ is a *(Edwards-Hastings) weak equivalence* if the induced function

$$\varinjlim_{j} [Y_j, Z] \to \varinjlim_{i} [X_i, Z]$$

is a bijection for all injective fibrant simplicial presheaves Z. *Fibrations* are defined by a right lifting property.

Examples: Pointwise local weak equivalences of I-diagrams are weak equivalences. Maps defined by cofinal functors are weak equivalences. Ordinary local weak equivalences of simplicial presheaves are weak equivalences of the pro category.

Theorem: With these definitions, the category $pro - s\,\mathrm{Pre}(\mathcal{C})$ has the structure of a proper closed simplicial model category.

1) I call this model structure the Edwards-Hastings model structure. It generalizes the model structure that they constructed for pro objects in simplicial sets. They did not describe the weak equivalences

the same way. Other people call this the "strict structure".

2) The fibrations can be understood: they are retracts of injective fibrations for contravariant diagrams on cofinite strongly directed sets (any two objects have an upper bound, and every object has only finitely many subobjects).

Suppose that X is a pro object. There is a natural map $\eta : X \to P_* X$, where $P_* X_i$ is the Postnikov tower of X_i.

Here are some facts:

A4: The functor $X \mapsto P_* X$ preserves Edwards-Hastings weak equivalences.

A5: The maps $\eta, P_*(\eta) : P_* X \to P_* P_* X$ are Edwards-Hastings equivalences.

A6: Suppose given a pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

such that p is a fibration, and the maps $P_* B \to P_* Y$ and $Y \to P_* Y$ are Edwards-Hastings equivalences. Then $P_* A \to P_* X$ is an Edwards-Hastings equivalence.

Say that a map $X \to Y$ is a *pro equivalence* if $P_*X \to P_*Y$ is an Edwards-Hastings equivalence. *Cofibrations* are monomorphisms, as before, and *pro fibrations* are defined by a right lifting property.

Theorem: The category $pro-s\,\mathrm{Pre}(\mathcal{C})$, together with the cofibrations, pro equivalences and pro fibrations, satisfies the axioms for a proper closed model category.

The proof is a matter of rolling the Bousfield-Friedlander tape. One can do the same with n-types. I learned this trick from Georg Biedermann [1].

Here's how the theory works, in part:

Lemma: If a map $f : Z \to W$ of simplicial presheaves is a pro equivalence, then it is a local weak equivalence.

Proof: The induced map

$$P_*Z \to P_*W$$

of pro-objects is a weak equivalence for the Edwards-Hastings structure, while the natural map $Z \to P_*Z$ induces an Edwards-Hastings equivalence

$$P_nZ \to P_nP_*Z.$$

The induced map

$$P_n P_* Z \to P_n P_* W$$

is an Edwards-Hastings weak equivalence for all $n \geq 0$ by subtle formal nonsense (Lemma 24 of [4]), and it follows that all maps

$$P_n Z \to P_n W$$

are Edwards-Hastings weak equivalences, and hence local weak equivalences of simplicial presheaves. It follows that $f : Z \to W$ is a local weak equivalence.

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