Homotopy classification of gerbes

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Cocycles

Suppose that X, Y are spaces.

$H(X,Y)$ = category whose objects are all pairs of maps (f,g)

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

where f is a weak equivalence. A morphism $\alpha : (f,g) \to (f',g')$ of $H(X,Y)$ is a commutative diagram

$$\begin{tikzcd}[row sep=small] & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\alpha"] & \\ X & & Y \\ & Z' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

$H(X,Y)$ is the **category of cocycles** from X to Y.

"Example": $V_0 \to *$ is a sheaf epi (arising from a covering) and G is a sheaf of groups. Cocycles on V_0 with coefficients in G are simp. presheaf maps

$$* \xleftarrow{\simeq} C(V_0) \to BG$$

where $C(V_0)$ = Čech resolution for the cover. The present definition is an expansion of this idea.

$\pi_0 H(X,Y) =$ class of path components of $H(X,Y)$. There is a function

$$\phi : \pi_0 H(X,Y) \to [X,Y] \quad (f,g) \mapsto g \cdot f^{-1}$$

Theorem: The canonical map $\phi : \pi_0 H(X,Y) \to [X,Y]$ is a bijection for all X and Y.

Lest you think that I've done away with the homotopy theory in this statement, suppose that $f \simeq g : X \to Y$. Then there is a picture

$$\begin{tikzcd} & X \arrow[dl, "1"'] \arrow[d, "d_0"'] \arrow[dr, "f"] & \\ X & X \times I \arrow[l, "pr"'] \arrow[r, "h"] & Y \\ & X \arrow[ul, "1"] \arrow[u, "d_1"] \arrow[ur, "g"'] & \end{tikzcd}$$

where h is the homotopy. Then

$$(i_X, f) \sim (pr, h) \sim (1_X, g)$$

Thus $f \mapsto [(1_X, f)]$ defines a function

$$\psi : \pi(X,Y) \to \pi_0 H(X,Y)$$

If X has the good manners to be cofibrant, then the function ψ is inverse to ϕ. More generally, there are a couple of things to prove:

Lemma 1: Suppose $\alpha : X \to X'$ and $\beta : Y \to Y'$ are weak equivalences. Then

$$(\alpha, \beta)_* : \pi_0 H(X, Y) \to \pi_0 H(X', Y')$$

is a bijection.

Lemma 2: Suppose that Y is fibrant and X is cofibrant. Then the canonical map

$$\phi : \pi_0 H(X, Y) \to [X, Y]$$

is a bijection.

The Theorem is a formal consequence. The result holds in extreme generality, specifically in any model category which is right proper (weak equivalences pull back to weak equivalences along fibrations), and such that weak equivalences are closed under finite products.

Examples: spaces, simplicial sets, presheaves of simplicial sets, spectra, presheaves of spectra, any localizations.

Proof of Lemma 1 $(f, g) \in H(X', Y')$ is a map $(f, g) : Z \to X' \times Y'$ s.t. f is a weak equivalence.

There is a factorization

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "(f,g)"'] & W \arrow[d, "(p_{X'},p_{Y'})"] \\ & X' \times Y' \end{tikzcd}$$

s.t. j is a triv. cofibration and $(p_{X'}, p_{Y'})$ is a fibration. $p_{X'}$ is a weak equivalence.

Form the pullback

$$\begin{tikzcd} W_* \arrow[r, "(\alpha\times\beta)_*"] \arrow[d, "(p_X^*,p_Y^*)"'] & W \arrow[d, "(p_{X'},p_{Y'})"] \\ X \times Y \arrow[r, "\alpha\times\beta"'] & X' \times Y' \end{tikzcd}$$

(p_X^*, p_Y^*) is a fibration and $(\alpha\times\beta)_*$ is a weak equivalence (since $\alpha \times \beta$ is a weak equivalence, and by right properness). p_X^* is also a weak equivalence.

$(f, g) \mapsto (p_X^*, p_Y^*)$ defines a function

$$\pi_0 H(X', Y') \to \pi_0 H(X, Y)$$

which is inverse to $(\alpha, \beta)_*$. $\square$

Proof of Lemma 2 $\pi(X,Y) =$ naive homotopy classes.

$\pi(X,Y) \to [X,Y]$ is a bijection since X is cofibrant and Y is fibrant.

We have seen that the assignment $f \mapsto [(1_X, f)]$ defines a function

$$\psi : \pi(X,Y) \to \pi_0 H(X,Y)$$

and there is a diagram

$$\begin{tikzcd} \pi(X,Y) \arrow[r, "\psi"] \arrow[dr, "\cong"'] & \pi_0 H(X,Y) \arrow[d, "\phi"] \\ & {[X,Y]} \end{tikzcd}$$

It suffices to show that ψ is surjective, or that any object $X \xleftarrow{f} Z \xrightarrow{g} Y$ is in the path component of some a pair $X \xleftarrow{1} X \xrightarrow{k} Y$ for some map k.

Form the diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "j"] & \\ X & & Y \\ & V \arrow[ul, "p"] \arrow[ur, dotted, "\theta"'] & \end{tikzcd}$$

where j is a triv. cofibration and p is a fibration; θ exists because Y is fibrant.

X is cofibrant, so the trivial fibration p has a section σ, and so there is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "1"'] \arrow[dr, "\theta\sigma"] \arrow[dd, "\sigma"] & \\ X & & Y \\ & V \arrow[ul, "p"] \arrow[ur, dotted, "\theta"'] & \end{tikzcd}$$

The composite $\theta\sigma$ is the required map k. $\square$

Proof of Theorem There are weak equivalences $\pi : X' \to X$ and $j : Y \to Y'$ such that X' and Y' are cofibrant and fibrant, respectively.

$$\begin{tikzcd} \pi_0 H(X,Y) \arrow[r, "\phi"] \arrow[d, "(1,j)_*"', "\cong"] & {[X,Y]} \arrow[d, "\cong"', "j_*"] \\ \pi_0 H(X,Y') \arrow[r, "\phi"] & {[X,Y']} \arrow[d, "\cong"', "\pi^*"] \\ \pi_0 H(X',Y') \arrow[u, "(\pi,1)_*", "\cong"'] \arrow[r, "\cong", "\phi"'] & {[X',Y']} \end{tikzcd}$$

$(1,j)_*$ and $(\pi,1)_*$ are bijections by the first Lemma, and ϕ is a bijection by the second. $\square$

Non-abelian cohomology

G = sheaf of groups (on some small Grothendieck site).

A G-torsor is a sheaf X with a free G-action such that $X/G \cong *$ in the sheaf category.

Recall that the Borel construction $EG \times_G X$ is the nerve $B(E_G X)$ of the translation category: objects are elements $x \in X$ and the morphisms $g : x \to y$ are group elements such that $g \cdot x = y$.

Fact: $G \times X \to X$ is a free action means precisely that the canonical map $EG \times_G X \to X/G$ is a local weak equivalence.

Thus, a sheaf X with G-action is a G-torsor iff $EG \times_G X \to *$ is a local weak equivalence.

$G -$ **Tors** is the category of G-torsors and G-equivariant maps. It is a groupoid:

A map $f : X \to Y$ of G-torsors is induced on fibres by the map of local fibrations

$$\begin{tikzcd} EG \times_G X \arrow[rr] \arrow[dr] & & EG \times_G Y \arrow[dl] \\ & BG & \end{tikzcd}$$

Then $f : X \to Y$ is a weak equiv. of constant simplicial sheaves, hence an isomorphism.

A construction:

Suppose $* \xleftarrow{\simeq} Y \xrightarrow{\alpha} BG$ is a cocycle, form pullback

$$\begin{tikzcd} \mathrm{pb}(Y) \arrow[r] \arrow[d] & Y \arrow[d, "\alpha"] \\ EG \arrow[r, "\pi"'] & BG \end{tikzcd}$$

$\mathrm{pb}(Y)$ has a G-action (from EG), and the map

$$EG \times_G \mathrm{pb}(Y) \to Y$$

is a weak equivalence. The square is htpy cartesian where $Y(U) \neq \emptyset$, so that $\mathrm{pb}(Y) \to \tilde{\pi}_0\, \mathrm{pb}(Y)$ is a G-equivariant weak equiv.
[$\tilde{\pi}_0\, \mathrm{pb}(Y)$ is the sheaf of path components of the simplicial sheaf $\mathrm{pb}(Y)$, in this case identified with a constant simplicial sheaf.]
The maps

$$EG \times_G \tilde{\pi}_0\, \mathrm{pb}(Y) \leftarrow EG \times_G \mathrm{pb}(Y) \to Y \simeq *$$

are weak equivs. Then $\tilde{\pi}_0\, \mathrm{pb}(Y)$ is a G-torsor.
A functor

$$H(*, BG) \to G - \mathbf{Tors}$$

is def. by $(* \xleftarrow{\simeq} Y \to BG) \mapsto \tilde{\pi}_0\, \mathrm{pb}(Y)$.
A functor

$$G - \mathbf{Tors} \to H(*, BG) :$$

is def. by $X \mapsto (* \xleftarrow{\simeq} EG \times_G X \to BG)$.

Theorem: These functors induce bijections

$$[*, BG] \cong \pi_0 H(*, BG) \cong \pi_0(G-\mathbf{Tors}) = H^1(\mathcal{C}, G).$$

$[*, BG]$ means morphisms in the homotopy category of simplicial sheaves (or presheaves). The result holds over arbitrary small Grothendieck sites, and is about 20 years old. Unlike the original proof, you have heard no references to hypercovers or pro objects — this proof is really quite simple, modulo the simplicial sheaf homotopy theory technology.

Classification of gerbes

A **gerbe** is a stack G which is locally path connected in the sense that $\tilde{\pi}_0(G) \cong *$. Stacks are really just homotopy types, so one may as well say that a gerbe is a locally connected sheaf (or presheaf) of groupoids.

"More generally", suppose that E is a sheaf. An E**-gerbe** is a map $G \to E$ of sheaves of groupoids which induces an isomorphism $\tilde{\pi}_0(G) \cong E$. One can, however, view an E-gerbe as an ordinary gerbe on the site fibred over the sheaf E, so we'll stick to the locally path connected case.

A morphism of gerbes is a weak equivalence

$$G \to H$$

of sheaves of groupoids. Write **gerbe** for the corresponding category.

Write **Grp** for the (big!) presheaf of 2-groupoids whose objects are sheaves of groups, 1-cells are isomorphisms of sheaves of groups, and whose 2-cells are the homotopies of isomorphisms of sheaves of groups. Write $H(*, \mathbf{Grp})$ for the category of cocycles

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\simeq"'] & \mathbf{Grp} \\ * \end{tikzcd}$$

where A is a sheaf of 2-groupoids.

Theorem: There is a bijection

$$\pi_0 H(*, \mathbf{Grp}) \cong \pi_0(\mathbf{gerbe}).$$

To get a cocycle from a gerbe G, write $\tilde{G}$ for the 2-groupoid whose objects and 1-cells are the objects and morphisms of G, respectively, and say that there is a unique 2-cell $\alpha \to \beta$ between any two arrows $\alpha, \beta : x \to y$.

There is a canonical equivalence $\tilde{G} \to *$, and a map $F(G) : \tilde{G} \to \mathbf{Grp}$ which associates to $x \in G(U)$ the sheaf $G(x,x)$ of automorphisms of x on $\mathcal{C}/U$, associates to $\alpha : x \to y$ the isomorphism $G(x,x) \to G(y,y)$ defined by conjugation by α, and associates to a 2-cell $\alpha \to \beta$ the homotopy defined by conjugation by $\beta\alpha^{-1}$.

Thus there is a canonical cocycle

$$\begin{tikzcd} \tilde{G} \arrow[r, "F(G)"] \arrow[d, "\simeq"'] & \mathbf{Grp} \\ * \end{tikzcd}$$

which is associated to a gerbe G.

The association is only functorial up to homotopy: given a morphism of gerbes $\theta : G \to H$ there is an induced morphism of 2-groupoids

$$\tilde{\theta} : \tilde{G} \to \tilde{H}$$

which is defined by θ on 0-cells and 1-cells. One uses the list of induced isomorphisms

$$G_x \xrightarrow{\theta} H_{\theta(x)}$$

(θ is a weak equivalence, and G, H are sheaves of groupoids) to construct a homotopy

$$h : \tilde{G} \times \mathbf{1} \to \mathbf{Grp}$$

from $F(G)$ to $F(H)\tilde{\theta}$. This means that there are relations

$$F(G) \sim h \sim F(H)\tilde{\theta} \sim F(H)$$

in the cocycle category, so that $F(G)$ and $F(H)$ represent the same element of $\pi_0 H(E, \mathbf{Grp})$. In particular there is a well-defined function

$$\pi_0(E - \mathbf{gerbe}) \to \pi_0 H(E, \mathbf{Grp}).$$

To go backwards, suppose given a cocycle

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "\simeq"'] & \mathbf{Grp} \\ * \end{tikzcd}$$

in 2-groupoids. Then F assigns a group G_x to each 0-cell x, homomorphism $\alpha_* : G_x \to G_y$ to each 1-cell $\alpha : x \to y$ and a homotopy $h_\theta : \alpha_* \to \beta_*$ to each 2-cell $\theta : \alpha \to \beta$. We can identify h_θ with an element of G_y, so that

$$h_\theta \alpha_* h_\theta^{-1} = \beta_*$$

We can use this data to define a "Grothendieck construction" $E_A F$ as follows: $E_A F$ is a category such that

- the objects of $E_A F$ are the 0-cells of A,

- the morphisms of $E_A F$ are the pairs

$$(\alpha : x \to y, g)$$

with α a 1-cell of A and $g \in G_y$, subject to the relation

$$(\alpha, g) \sim (\beta, k)$$

if there is a 2-cell $\theta : \alpha \to \beta$ with $kh_\theta = g$.

Composition in the category is defined by

$$[(k, \beta)][(g, \alpha)] = [(k\beta_*(g), \beta\alpha)]$$

(one checks that this is well-defined).

Then there are other observations:

- $E_A F$ is a groupoid since A is a 2-groupoid.
- There is a canonical functor

$$E_A F \xrightarrow{\pi_F} \pi_0 A,$$

where $\pi_0 A$ is the fundamental groupoid of A and π_F is the functor $[(g, \alpha)] \mapsto [\alpha]$.

- One checks that π_F induces an isomorphism in path components, so that $E_A F$ is a gerbe.
- This construction is functorial.

The moral is that $E_A F$ is a gerbe for each cocycle F, and the assignment $F \mapsto E_A F$ defines a function

$$\pi_0 H(*, \mathbf{Grp}) \to \pi_0(\mathbf{gerbe}).$$

This function is inverse to the canonical cocycle function

$$\pi_0(\mathbf{gerbe}) \to \pi_0 H(*, \mathbf{Grp}).$$

which was defined above.

There are other results of this type, which involve smaller objects and are therefore homotopy theoretic.

Suppose that G is a gerbe, and let G_* be the presheaf of 2-groupoids whose 0-cells over $U \in \mathcal{C}$ are the sheaves G_x of automorphisms of objects $x \in G(U)$. The 1-cells of G_* are the isomorphisms of sheaves of groups, and the 2-cells are the homotopies. In other words G_* is a full subobject of $\mathbf{Grp}$ which happens to be a presheaf of 2-groupoids. Note as well that the canonical cocycle

$$F(G) : \tilde{G} \to \mathbf{Grp}$$

factors through a cocycle

$$F(G) : \tilde{G} \to G_*$$

taking values in G_*. Then one can prove:

Theorem: The canonical cocycle construction induces a bijection

$\pi_0 H(*, G_*) \cong \pi_0(\text{gerbes locally equivalent to } G).$

Of course, there is also a bijection

$$[*, dBG_*] \cong \pi_0 H(*, G_*)$$

so the Theorem gives a homotopy classification of gerbes locally equivalent to G up to equivalence.

The key point in the proof is that any presheaf F of groups locally isomorphic to presheaves of groups in G determines a full presheaf of groupoids $F_* \subset$ **Grp** and containing G_*. Further, the inclusion $G_* \subset F_*$ is a weak equivalence.

One could further: the gerbes locally equivalent to G with band $L \in H^1(\mathcal{C}, \mathrm{Out}(G_*))$ are classified by cocycles in the homotopy fibre of the map

$$G_* \to \mathrm{Out}(G_*) = \pi_0(G_*).$$