Stacks and higher stacks

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Stacks

The theory of higher stacks starts with stack theory, which is essentially the homotopy theory of groupoids.

Let **Gpd** be the category of (pre)sheaves of groupoids. Say that a map $f : G \to H$ is a (local) weak equivalence, respectively fibration, if the induced map $BG \to BH$ is a local weak equivalence, respectively injective fibration, of simplicial (pre)sheaves. Cofibrations are defined by a left lifting property with respect to trivial fibrations, and include all maps $\pi(A) \to \pi(B)$ of fundamental groupoids induced by cofibrations (monomorphisms) $A \to B$ of simplicial presheaves.
Then, with the injective model structure on simplicial presheaves (which is cofibrantly generated) in hand, it is relatively easy to show the following:

Theorem 1. *With these definitions,* **Gpd** *has the structure of a cofibrantly generated, right proper, closed simplicial model category.*

At first blush, stacks are essentially fibrant objects for this model structure: an object G is a **stack** if it satisfies descent in the sense that any fibrant model $j : G \to H$ is a sectionwise equivalence. If G is a sheaf of groupoids, this criterion, which comes from algebraic K-theory, is equivalent to the effective descent criterion that geometers use. It's better, perhaps, to say that a stack is a local homotopy type of presheaves of groupoids.

Here's how we know we're on the right track. Suppose that G is a sheaf of groups and F is a sheaf with a G-action. Say that F is a **G-torsor** if the map $EG \times_G F \to *$ is a local weak equivalence — this is equivalent to the classical definition. Given a G-torsor F, the picture

$$* \xleftarrow{\simeq} EG \times_G F \to BG$$

is the *canonical cocycle* associated to F. This cocycle construction has a left adjoint defined by pullback over $EG \to BG$, and sets up the following:

Theorem 2. *There is an isomorphism*

$$[*, BG] \cong \pi_0(G - \mathbf{tors}) = H^1(\mathcal{C}, G).$$

Remark 3. 1) Classical non-abelian H^1 is a homotopy theoretic invariant.

2) Torsors can be defined, via homotopy colim-

its, for arbitrary sheaves of groupoids G: a G-torsor is a G-diagram X in sheaves such that the map $\underrightarrow{\mathrm{holim}}_G X \to *$ is a local weak equivalence. Then an analog of the Lemma holds in this context too.

Simplicial groupoids

There are varying opinions on what a higher stack should be.

The original idea was to see higher order cohomological (and homotopical) phenomena in algebraic geometry. The early attempts by geometers (Giraud and others) to do so in the 1970s are part of the reason that topologists should have been glad not to be algebraic geometers at the time. What they created, however, was a geometric interpretation for the classification of stacks in terms of symmetries that took place in 2-groupoids. Many modern approaches to the subject, starting with Simpson, essentially throw in the towel geometrically: a higher stack for Simpson and his followers is just a simplicial presheaf.

As far as I'm concerned, there should still be symmetries.

My base category for higher stack theory is the category $s_0\mathbf{Gpd}$ of groupoids enriched in simplicial sets (or simplicial groupoids with discrete objects). These things have a history in homotopy coherence theory, and we have had homotopy theoretic structures for them for a long time.

A map $f : G \to H$ of $s_0\mathbf{Gpd}$ is the obvious thing: it's a map of simplicial groupoids. The earliest model structure is due to Dwyer and Kan: say that

1) f is a *weak equivalence* if the map $\mathrm{Mor}(G) \to \mathrm{Mor}(H)$ is a weak equivalence of simplicial sets and $f_* : \pi_0 G \to \pi_0 H$ is an isomorphism,

2) f is a *fibration* if $\mathrm{Mor}(G) \to \mathrm{Mor}(H)$ is a Kan fibration and the map $f : G_0 \to H_0$ is a fibration of groupoids.

Cofibrations are defined by a lifting property, as before.

Theorem 4 (Dwyer-Kan)**.** *With these definitions, $s_0\mathbf{Gpd}$ has the structure of a cofibrantly generated, right proper closed model category.*

Something to notice: all objects of $s_0\mathbf{Gpd}$ are fibrant, since all simplicial groups are Kan complexes.

One can use an enriched version of Quillen's Theorem B (first written down by Moerdijk) to show that there is a natural homotopy cartesian diagram

$$\begin{array}{ccc} \mathrm{Mor}(G) & \longrightarrow & \mathrm{Ob}(G) \\ \downarrow & & \downarrow \\ \mathrm{Ob}(G) & \longrightarrow & BG \end{array}$$

for arbitrary G in $s_0\mathbf{Gpd}$, and this can be used to show the following:

Theorem 5. *A map $G \to H$ of $s_0\mathbf{Gpd}$ is a weak equivalence if and only if the map $BG \to BH$ is a (diagonal) weak equivalence of bisimplicial sets.*

It is also a consequence of the homotopy cartesian statement that the connected components of

$$\mathrm{Mor}(G) = \bigsqcup_{x,y \in \mathrm{Ob}(G)} G(x,y)$$

are loop spaces of BG.

This is all good, but I can't see how to promote the Dwyer-Kan model structure to the sheaf theoretic context (despite the Joyal-Tierney paper). To go further, I need the Eilenberg-Mac Lane $\overline{W}$ construction, expressed in a civilized way.

Suppose that X is a simplicial set, and let x_i denote the i^{th} vertex of a simplex $x : \Delta^n \to X$. There is a graph $\Gamma_n X$ whose edges consist of $(n+1)$-simplices $x : x_0 \to x_1$.

The groupoid $G(X)_n$ is the free groupoid on $\Gamma_n(X)$, subject to requiring that $s_0(y)$ is the identity on y_0.

To define simplicial structure maps, it's convenient to imagine simplicial sets as functors on finite totally ordered posets P. Then the morphisms in $G(X)_P$ are generated by simplices $\sigma : \Delta^{\mathbf{0}*P} \to X$ with $\sigma : \sigma(0) \to \sigma(1)$, where 1 is the minimal element of P. Given a poset morphism $\theta : Q \to P$, there are poset morphisms $\tilde{\theta}, \theta_* : \mathbf{0} * Q \to \mathbf{0} * P$, which both restrict to θ on Q, and with $\tilde{\theta}(0) = 0$ and $\theta_*(0) = 1$. Then the composite

$$\sigma(0) \xrightarrow{\tilde{\theta}^*(\sigma)} \sigma(\theta(1)) \xleftarrow{\theta_*^*(\sigma)} \sigma(1)$$

is $\theta^*(\sigma)$ in $G(X)_Q$.

We therefore have a functor $G : s\mathbf{Set} \to s_0\mathbf{Gpd}$. The functor $\overline{W} : s_0\mathbf{Gpd} \to s\mathbf{Set}$ is its right adjoint:

$$\overline{W}(H)_P = \hom(G(\Delta^P), H).$$

Let's think about $G(\Delta^P)$ for a minute. There are canonical factorizations

$$\begin{array}{ccc} Q & \xrightarrow{\theta} & P \\ & {\scriptstyle \theta'} \searrow \quad \nearrow {\scriptstyle \tau} & \\ & P_{\geq \theta(0)} & \end{array}$$

and

$$\begin{array}{ccc} \mathbf{0} * Q & \xrightarrow{(v,\theta)} & P \\ & {\scriptstyle 0*\theta'} \searrow \quad \nearrow {\scriptstyle (v,\tau)} & \\ & \mathbf{0} * P_{\geq \theta(0)} & \end{array}$$

and one shows that

$$[(v, \theta)] = (\theta')^{*}[(v, \tau)]$$

in $G(\Delta^P)$. Here (v, θ) sends 0 to v and is θ on Q, and τ is the inclusion of the interval.

Any relation $v \leq w$ in P determines a canonical map

$$m(v, w) = (v, \tau) : \mathbf{0} * P_{\geq w} \to P$$

and any morphism $G(\Delta^P) \to H$ is completely determined by the images of the corresponding maps in $G(\Delta^P)$.

Lemma 6. *Suppose given $H \in s_0\mathbf{Gpd}$, objects x_i of H and morphisms $\alpha(i, j) : x_i \to x_j$ of*

$H_{P_{\geq j}}$ *such that all diagrams*

$$\begin{array}{ccc} x_i & \xrightarrow{\tau^*\alpha(i,j)} & x_j \\ & {\scriptstyle \alpha(i,k)}\searrow \quad \swarrow{\scriptstyle \alpha(j,k)} & \\ & x_k & \end{array}$$

commute. Then the assignment

$$(u, \theta) \mapsto (\theta')^*(\alpha(u, \theta(0)))$$

defines a unique morphism $G(\Delta^P) \to H$ *of* $s_0\mathbf{Gpd}$.

In this way, $\overline{W}(H)_P$ is identified with a collection of P-cocycles in H.

Remark 7. The definition of $\overline{W}(H)$ via cocycles can be extended to simplicial groupoids, and more generally to simplicial categories A, for what it's worth. One can also use the cocycle approach to define a natural map

$$dBA \to \overline{W}A$$

for all simplicial categories A. If the string

$$x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} x_n$$

in A_n is an n-simplex of dBA, then the morphisms

$$(d_0^n\alpha_1, \dots, d_0\alpha_n)$$

define the corresponding cocycle.

Now here's what you can show:

Lemma 8. *1) The map $dBH \to \overline{W}H$ is a weak equivalence for all $H \in s_0\mathbf{Gpd}$.*

2) The functor $\overline{W}$ takes fibrations of $s_0\mathbf{Gpd}$ to Kan fibrations.

3) The adjunction maps $\eta : A \to \overline{W}G(A)$ and $\epsilon : G(\overline{W}H) \to H$ are weak equivalences.

This result allows you to define a new model structure on $s_0\mathbf{Gpd}$, for which the weak equivalences are the ones we know, and a fibration is a map $p : G \to H$ such that the induced map $\overline{W}G \to \overline{W}H$ is a Kan fibration.

If $i : A \to B$ is a cofibration (resp. trivial cofibration) of simplicial sets, then $i_* : G(A) \to G(B)$ is a cofibration (resp. trivial cofibration) for the Dwyer-Kan structure, on account of the Lemma. Also, if i is a trivial cofibration and

$$\begin{array}{ccc} G(A) & \longrightarrow & G \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i'} \\ G(B) & \longrightarrow & H \end{array}$$

is a pushout in $s_0\mathbf{Gpd}$, then i' has the left lifting property with respect to all Dwyer-Kan fibrations, and is therefore a trivial cofibration. This is enough to prove the following with standard small object arguments:

Theorem 9. *With these definitions, the category* $s_0\mathbf{Gpd}$ *has the structure of a right proper closed model category.*

The $\overline{W}$ and Dwyer-Kan model structures on $s_0\mathbf{Gpd}$ are Quillen equivalent.

Presheaves

Write $\mathrm{Pre}(s_0\mathbf{Gpd})$ for the category of presheaves of groupoids enriched in simplicial sets on a site $\mathcal{C}$.

I say that a map $G \to H$ of such presheaves is a **local weak equivalence** if (equivalently) $BG \to BH$ is a diagonal local weak equivalence of bisimplicial presheaves of $\overline{W}G \to \overline{W}H$ is a local weak equivalence of simplicial presheaves.

A map $p : G \to H$ is a **fibration** if the map $\overline{W}G \to \overline{W}H$ is an injective fibration of simplicial presheaves.

Cofibrations are defined by a left lifting property with respect to trivial fibrations.

Theorem 10. *With these definitions* $\mathrm{Pre}(s_0\mathbf{Gpd})$ *has the structure of a right proper closed model category.*

Suppose given a pushout diagram

$$\begin{array}{ccc} \tilde{G}(A) & \longrightarrow & G \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i'} \\ \tilde{G}(B) & \longrightarrow & H \end{array}$$

in sheaves of groupoids enriched in simplicial sets, where $i : A \to B$ is a local trivial cofibration of simplicial sheaves. We need to show that the map i' is a local weak equivalence.

The functor $\overline{W}$ commutes with the formation of associated sheaves, so that the map $\overline{W}\tilde{G}A \to \overline{W}\tilde{G}B$ is locally weakly equivalent to $A \to B$ and is therefore a local weak equivalence, so $\tilde{G}A \to \tilde{G}B$ is a local weak equivalence.

Suppose that $\pi : \mathbf{Shv}(\mathcal{B}) \to \mathbf{Shv}(\mathcal{C})$ is a Boolean localization (or collection of all stalks). Then $\pi^*\overline{W} \cong \overline{W}\pi^*$ so that in the pushout

$$\begin{array}{ccc} \pi^*\tilde{G}(A) & \longrightarrow & \pi^*\tilde{G} \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i'} \\ \pi^*\tilde{G}(B) & \longrightarrow & \pi^*\tilde{H} \end{array}$$

the map i_* is a local weak equivalence. It is enough to show that i' is a local weak equivalence, because the map $\overline{W}G \to \overline{W}H$ of simplicial sheaves is a local weak equivalence if and only if the map

$\pi^*\overline{W}\tilde{G} \to \pi^*\overline{W}\tilde{H}$ is a local weak equivalence. It therefore suffices to assume that our diagram lives in sheaves on $\mathcal{B}$.

We can also assume that A and B are locally fibrant. In effect, there is a diagram

$$\begin{array}{ccccc}
\tilde{G}(A) & \xrightarrow{\tilde{G}\eta} & \tilde{G}\overline{W}\tilde{G}(A) & \xrightarrow{\tilde{\epsilon}} & \tilde{G}(A) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{G}(B) & \xrightarrow[\tilde{G}\eta]{} & \tilde{G}\overline{W}\tilde{G}B & \xrightarrow[\tilde{\epsilon}]{} & \tilde{G}(B)
\end{array}$$

The map $\overline{W}\tilde{G}(A) \to \overline{W}\tilde{G}(B)$ has a factorization

$$\begin{array}{ccc}
\overline{W}\tilde{G}(A) & \xrightarrow{j} & X \\
& \searrow & \downarrow p \\
& & \overline{W}\tilde{G}(B)
\end{array}$$

in simplicial sheaves, where p is a trivial injective fibration and j is a cofibration. Then the map $B \to \overline{W}\tilde{G}(B)$ lifts to X, and it follows that the map $i_* : \tilde{G}(A) \to \tilde{G}(B)$ is a retract of the map $j_* : \tilde{G}\overline{W}\tilde{G}(A) \to \tilde{G}(X)$, which is induced by the trivial cofibration $j : \overline{W}\tilde{G}(A) \to X$ of locally fibrant simplicial sheaves.

Finally, in the pushout diagram

$$\begin{array}{ccc} G(A) & \longrightarrow & G \\ {\scriptstyle i_*}\downarrow & & \downarrow \\ G(B) & \longrightarrow & H' \end{array}$$

of presheaves of groupoids enriched in simplicial sets on $\mathcal{B}$, the map $A \to B$ is a sectionwise trivial cofibration of simplicial sheaves because A and B are locally fibrant, so that $G(A) \to G(B)$ is a sectionwise trivial cofibration of presheaves of groupoids enriched in simplicial sets. It follows that the map $G \to H'$ is also a sectionwise trivial cofibration, and so the map $i' : G \to H$ of associated sheaves is a local weak equivalence.

There are adjoint functors

$$\pi : s_0\mathbf{Gpd} \leftrightarrows 2 - \mathbf{Gpd} : B$$

and these are promoted to adjoint functors

$$\pi : \mathrm{Pre}(s_0\mathbf{Gpd}) \leftrightarrows \mathrm{Pre}(2 - \mathbf{Gpd}) : B$$

on the presheaf level. An object G of $\mathrm{Pre}(s_0\mathbf{Gpd})$ consists of a simplicial presheaf map $(s,t) : \mathrm{Mor}(G) \to \mathrm{Ob}(G) \times \mathrm{Ob}(G)$ with a groupoid structure. The corresponding fundamental groupoid object is the induced groupoid morphism $(s,t) : \pi(\mathrm{Mor}(G)) \to$

$\mathrm{Ob}(G) \times \mathrm{Ob}(G)$, with the induced groupoid structure (the fundamental groupoid functor preserves finite products).

A map $f : G \to H$ of $\mathrm{Pre}(s_0\mathbf{Gpd}$ is a local weak equivalence if and only if the simplicial presheaf map $\mathrm{Mor}(G) \to \mathrm{Mor}(H)$ is a local weak equivalence and the sheaf map $\tilde{\pi}_0 G \to \tilde{\pi}_0 H$ is an isomorphism. The fundamental groupoid functor preserves local weak equivalences, so that if f is a local weak equivalence, then so is the induced map $f_* : B\pi(G) \to B\pi(H)$.

Say that a map $f : G \to H$ of $\mathrm{Pre}(2 - \mathbf{Gpd})$ is a **local weak equivalence** (respectively **fibration**) if the induced map $BG \to BH$ is a local weak equivalence (respectively fibration) of $\mathrm{Pre}(s_0\mathbf{Gpd})$. **Cofibrations** are defined by a left lifting property.

Theorem 11. *With these definitions, the category* $\mathrm{Pre}(2 - \mathbf{Gpd})$ *has the structure of a right proper closed model category.*

We have to show that if $i : A \to B$ is a trivial cofibration of simplicial presheaves and the diagram

$$\begin{array}{ccc} \pi G(A) & \longrightarrow & G \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i'} \\ \pi G(B) & \longrightarrow & H \end{array}$$

is a pushout, then the map i' is a local weak equivalence. This is easy: this diagram is obtained from the pushout diagram

$$\begin{array}{ccc} G(A) & \longrightarrow & BG \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i''} \\ G(B) & \longrightarrow & K \end{array}$$

of $\mathrm{Pre}(s_0\mathbf{Gpd})$ by applying the fundamental groupoid functor π. The map i'' is a local weak equivalence and π preserves local weak equivalences.

$\pi G(X)$ is the fundamental 2-groupoid of X.