Local higher category theory

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Path category

The nerve functor

$$B : \mathbf{cat} \to s\mathbf{Set}$$

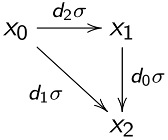
is def. by $BC_n = \hom(\mathbf{n}, C)$, where $\mathbf{n}$ is the poset

$$0 \to 1 \to \cdots \to n.$$

The **path category** functor $P : s\mathbf{Set} \to \mathbf{cat}$ is the left adjoint of the nerve:

$$P(X) = \varinjlim_{\Delta^n \to X} \mathbf{n}.$$

Alternatively, $P(X)$ is the free category on the graph $\operatorname{sk}_1 X$, subject to the relations $s_0 x = 1_x$ and $d_1(\sigma) = d_0(\sigma) d_2(\sigma)$ for each 2-simplex $\sigma : \Delta^2 \to X$.



Preliminary facts

Lemma 1.

$\mathrm{sk}_2(X) \subset X$ *induces* $P(\mathrm{sk}_2(X)) \cong P(X)$ *for all simplicial sets* X.

Lemma 2.

$\epsilon : P(BC) \to C$ *is an isomorphism for all small categories* C.

Lemma 3.

There is an isomorphism $G(P(X)) \cong \pi(X)$ *for all simplicial sets* X.

$G(P(X))$ is the free groupoid on the category $P(X)$.

Triangulation

There is a triangulation functor

$$| \ | : c\mathbf{Set} \to s\mathbf{Set}.$$

Every cubical set Y is a colimit of its cells $\square^n \to Y$, and

$$|Y| := \varinjlim_{\square^n \to Y} |\square^n| = \varinjlim_{\square^n \to Y} B(\mathbf{1}^{\times n})$$

$B(\mathbf{1}^{\times n}) = (\Delta^1)^{\times n}$ is a simplicial hypercube.

$$|\square^2| : \quad \begin{array}{ccc} (0,1) & \longrightarrow & (1,1) \\ \uparrow & \nearrow & \uparrow \\ (0,0) & \longrightarrow & (1,0) \end{array}$$

Path categories for cubical sets

1) $K \subset \square^N$ is finite dimensional cubical complex:

$$P(K) := P(|K|).$$

$$P(|K|) = \varinjlim_{\square^n \to Y} PB(\mathbf{1}^{\times n}) = \varinjlim_{\square^n \to Y} \mathbf{1}^{\times n}.$$

2) **Internal definition**:

$P(K)$ is the free category on $\mathrm{sk}_1 K$, subject to relations defined by degeneracies and 2-cells.

Geometric concurrency

Basic idea (V. Pratt, 1991): represent the simultaneous execution of processors a and b as a picture (2-cell)

$$\begin{array}{ccc} * & \xrightarrow{b} & * \\ {\scriptstyle a}\uparrow & \searrow\!\!\searrow & \uparrow{\scriptstyle a} \\ * & \xrightarrow[b]{} & * \end{array}$$

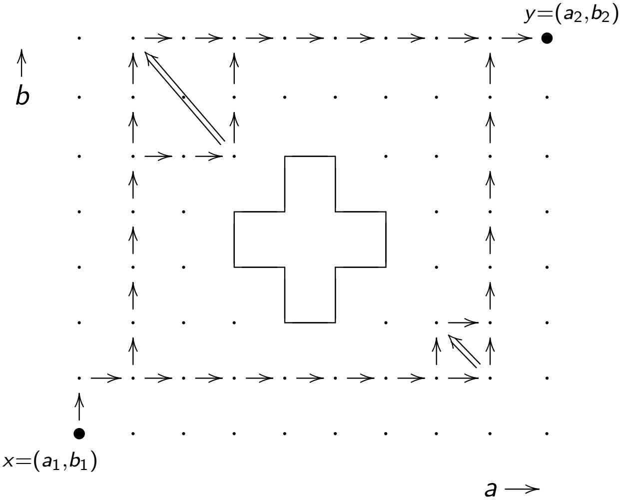
Simultaneous action of multiple processors rep. by hypercubes.

Restrictions on the system corr. to shared resources rep. by removing cubical cells, so one has a cubical subcomplex $K \subset \square^N$ of an N-cell, $N =$ number of processors.

K is a "higher dimensional automaton".

Basic problem: Compute $P(K)(x,y)$, "execution paths from state x to state y".

Example: the Swiss flag



Path 2-category

K = finite simplicial complex.

2-category "resolution" $P_2(K) \to P(K)$: $\pi_0 P_2(x,y) = P(K)(x,y)$

0-cells are vertices of K, 1-cells $x \to y$ are strings of nondegenerate 1-simplices of K

$$x = x_0 \to x_1 \to \cdots \to x_n = y,$$

and 2-cells are generated by non-degenerate 2-cells of K

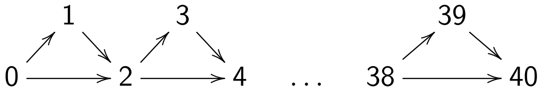
$$\begin{array}{ccccccc} x \to \dots \to x_{i-1} & & \xrightarrow{d_1\sigma} & & x_{i+1} \to \dots \to y \\ & {\scriptstyle d_2\sigma}\searrow & \Downarrow & \nearrow{\scriptstyle d_0\sigma} & \\ & & x_i & & \end{array}$$

All data in $P_2(K)$ is finite for a finite simplicial complex K.

Remark: $\exists$ algorithm to construct $P_2(K)$ and compute $P(K)$.

Computations

The necklace: $L \subset \Delta^{40}$ is the subcomplex



This is 20 copies of the complex $\partial\Delta^2$ glued together. There are 2^{20} morphisms in $P(L)(0,40)$.

The listing of morphisms of $P(L)$ consumes 2 GB of disk.

Remarks: 1) The size of the path category $P(L)$ grows exponentially with L ("exponential complexity").

2) There are no parallel versions of the code — no general patching algorithms.

3) Work so far: to compute $P(K)(x,y)$, find $\{x,y\} \subset L \subset K$ such that $P(L) \to P(K)$ is fully faithful.

The **path category** $P(X)$ of a simplicial set X is the **fundamental category** $\tau_1(X)$ from Joyal's theory of quasicategories.

Joyal's **quasicategory model structure** on $s\mathbf{Set}$ is constructed, by the methods of Cisinski's thesis, by formally inverting the **inner horn inclusions** $\Lambda^n_k \subset \Delta^n$, $k \neq 0, n$, and the inclusion

$$0 : \Delta^0 \to B(\pi(\Delta^1)) =: I.$$

$I =$ interval object. $\pi(\Delta^1)$ is a trivial groupoid with two objects.

A **quasicategory** is a simplicial set X which has the right lifting property with respect to all inner horn inclusions $\Lambda^n_k \subset \Delta^n$.

Lemma 4 (magic).

Quasicategories are the fibrant objects for the quasicategory model structure.

Weak equivalences

There is a triv. cof. $j : X \to LX$ s.t. LX is a quasicategory, built from inner horn inclusions.

A map $X \to Y$ is a **quasicategory equivalence** (categorical equivalence) if the induced map $LX \to LY$ is an I-homotopy equivalence.

I-**homotopy**, $I = B\pi(\Delta^1)$:

$$\begin{array}{ccc} X & & \\ {\scriptstyle 0}\downarrow & \searrow^{f} & \\ X \times I & \longrightarrow & Y \\ {\scriptstyle 1}\uparrow & \nearrow_{g} & \\ X & & \end{array}$$

Cofibrations for quasicategory structure are monomorphisms.

Joyal calls fibrations for this structure pseudo-fibrations.

Path category, again

Lemma 5.

Every inner horn inclusion $\Lambda^n_k \to \Delta^n$ induces an isomorphism $P(\Lambda^n_k) \cong P(\Delta^n)$

Corollary 6.

Every quasicategory equivalence $X \to Y$ induces an **equivalence** *of categories $P(X) \simeq P(Y)$.*

Consequence of Van Kampen theorem: P is left adjoint to nerve, hence takes pushouts to pushouts, so $P(X) \cong P(LX)$.

Remark: Equivalences of categories are rare in CS applications.

A 1-simplex $\alpha : \Delta^1 \to X$ is **invertible** if the map $[\alpha] : d_1\alpha \to d_0\alpha$ is invertible in $P(X)$.

Fact: If Y is a Kan complex, every 1-simplex of Y is invertible.

Theorem 7 (Joyal).

X = quasicategory. X is a Kan complex if and only if $P(X)$ is a groupoid.

$J(X) \subset X$ is the subcomplex whose simplices $\sigma : \Delta^n \to X$ have 1-skeleta $\mathrm{sk}_1 \Delta^n \subset \Delta^n \xrightarrow{\sigma} X$ consisting of invertible 1-simplices.

$J(X)$ is the **core** of X.

Corollary 8.

$J(X)$ is the maximal Kan subcomplex of a quasicategory X.

Interesting part: $J(X)$ is a Kan complex.

Examples

1) If C is a category, then BC is a quasicategory, since $P(\Lambda^n_k) \cong P(\Delta^n)$ for inner horns $\Lambda^n_k \subset \Delta^n$.

$$J(BC) = B(\mathrm{Iso}(C)).$$

2) Every Kan complex is a quasicategory.

$f : X \to Y$ of Kan complexes is a quasicategory equivalence if and only if it is a standard weak equivalence.

$g : Z \to W$ quasicategory equiv of quasicategories. Then $g_* : J(Z) \to J(W)$ is an ordinary weak equiv of Kan complexes.

Facts: 1) $X =$ quasicategory. Then $[*, X]_q \cong \pi_0 J(X)$.

2) $C =$ category. Then $[*, BC]_q \cong \pi_0 B(\mathrm{Iso}(C)) = \pi_0 \mathrm{Iso}(C)$.

3) $P =$ poset. Then $\mathrm{Ob}(P) \cong [*, BP]_q$. eg. $\Delta^n_0 \cong [*, \Delta^n]_q$.

Recall $\Delta^n = B\mathbf{n}$, $\mathbf{n} = \{0, 1, \dots, n\}$.

Simplicial complexes

Lemma 9.

If $K \subset \Delta^N$, then $K_0 \cong [*, K]_q$.

Proof: $j : X \to LX$ is an isomorphism on vertices.

Suppose $i : K \subset \Delta^N$ is an isomorphism on vertices.

Form the diagram

$$\begin{array}{ccc} K & \xrightarrow{i} & \Delta^N \\ {\scriptstyle j}\downarrow & \nearrow & \\ LK & & \end{array}$$

$[*, LK]_q$ is I-htpy classes of maps $* \to LK$.

If 2 vertices $x, y : * \to LK$ are I-homotopic, they are equal in Δ^N so $x = y$ in LK.

Thus $LK_0 \to [*, LK]_q$ is a bijection, so $K_0 \to [*, K]_q$ is a bijection.

Theorem 10.

$K = \Delta^n, n \geq 0$ or $\partial\Delta^n, n \geq 1$. $f : X \to Y$ map of quasicategories. Then f is a quasicategory weak equivalence if and only if all functors

$$\pi(J\mathbf{hom}(K, X)) \to \pi(J\mathbf{hom}(K, Y))$$

are equivalences of groupoids

Corollary 11.

$g : Z \to W$ map of Kan complexes. Then g is a standard weak equivalence if and only if all functors

$$g_* : \pi(\mathbf{hom}(\partial\Delta^n, Z)) \to \pi(\mathbf{hom}(\partial\Delta^n, W)), \;\; n \geq 1,$$

are equivalences of groupoids.

Classical local structure

$C = op|_T$ is the category of open subsets of a topological space T, so that the sheaf category $\mathrm{Shv}(C)$ has enough points (stalks).

If F is a sheaf or presheaf on C, write

$$p^*F = \bigsqcup_{x \in T} F_x$$

for the collection of all stalks.

There are (injective) model structures on $s\,\mathrm{Shv}(C)$, $s\,\mathrm{Pre}(C)$ s.t. **cofibrations** are monomorphisms, and $f : X \to Y$ is a **local weak equiv.** iff $p^*X \to p^*Y$ is a weak equiv. of simplicial sets.

The inclusion and associated sheaf functors form a Quillen equivalence

$$L^2 : s\,\mathrm{Pre}(C) \leftrightarrows s\,\mathrm{Shv}(C) : i.$$

Local quasicategory structure

Theorem 12 (Meadows [4]).

There is a left proper model structure on $s\,\mathrm{Pre}(C)$ *with cofibrations* $=$ *monomorphisms, and s.t.* $f : X \to Y$ *is a weak equiv. iff* $p^*X \to p^*Y$ *is a quasicategory weak equiv of simplicial sets.*

Remarks: 1) There is a corresponding model structure for $s\,\mathrm{Shv}(C)$ with a Quillen equivalence

$$L^2 : s\,\mathrm{Pre}(C) \leftrightarrows s\,\mathrm{Shv}(C) : i.$$

2) These are special cases of a general result which applies to all Grothendieck sites C, where p^* is a Boolean localization.

3) Weak equivalences for the theory are **local quasicategory equivalences**.

About the proof

The proof is **not** a localization argument.

Use a natural sectionwise quasicategory equivalence $j : X \to LX$, with LX a presheaf of quasicategories.

Show that $p^* j : p^* X \to p^* LX$ is a quasicategory equivalence: the map

$$\Lambda^n_k \times U \to \Delta^n \times U$$

is a local quasicategory equivalence for all $0 < k < n$ and all $U \subset T$ open.

Y = presheaf of quasicategories. Then $p^* J(Y) \cong J(p^* Y)$, because $J(Y)$ is defined by pullbacks:

$$\begin{array}{ccc} J(Y)_n & \longrightarrow & Y_n \\ \downarrow & & \downarrow \\ \prod_{\mathbf{1} \subset \mathbf{n}} \mathrm{Inv}(Y) & \to & \prod_{\mathbf{1} \subset \mathbf{n}} Y_1 \end{array} \qquad \begin{array}{ccc} \mathrm{Inv}(Y) & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ \mathrm{Mor}(\mathrm{Iso}(P(Y))) & \to & \mathrm{Mor}(P(Y)) \end{array}$$

More on the proof

Lemma 13.

Suppose $K = \Delta^n, \partial\Delta^n$. $f : X \to Y$ map of presheaves of quasicategories is a local quasicategory equivalence if and only if all

$$\pi(J(\mathbf{hom}(K,X)) \to \pi(J(\mathbf{hom}(K,Y))$$

are local weak equivs of presheaves of groupoids.

Lemma 14 (Bounded cofibration).

There is a regular cardinal α such that, given a picture

$$\begin{array}{ccc} & & X \\ & & \downarrow i \\ A & \xrightarrow[\subset]{} & Y \end{array}$$

with i a trivial cofibration and $|A| < \alpha$, there is a subobject $A \subset B \subset Y$ s.t. $|B| < \alpha$ and $B \cap X \to B$ is a trivial cofibration.

Rezk's complete Segal model structure on $s^2\mathbf{Set}$ is constructed by localizing the Reedy (injective) structure at the cofibrations

$$G(n) := Str(\Delta^n)\tilde{\times}\Delta^0 \subset \Delta^n\tilde{\times}\Delta^0 =: F(n)$$
$$F(0) = \Delta^0\tilde{\times}\Delta^0 \to B(\pi(\Delta^1))\tilde{\times}\Delta^0 =: I$$

where $Str(\Delta^n)$ is the string of 1-simplices $0 \to 1 \to \dots \to n$ in Δ^n.
I is the "discrete nerve" of $\pi(\Delta^1)$.

There are adjoint functors

$$t_! : s^2\mathbf{Set} \leftrightarrows s\mathbf{Set} : t^!,$$

where $t^!X_{p,q} = \hom(\Delta^p \times B\pi(\Delta^q), X)$ for simp. sets X.

Comparison theorem

Theorem 15 (Joyal, Tierney).

The functors $t_!, t^!$ induce a Quillen equivalence between Rezk's complete Segal model structure on $s^2\mathbf{Set}$ and the quasicategory model structure on $s\mathbf{Set}$.

Local version

Meadows localizes the injective model structure on $s^2\,\mathrm{Pre}(C)$ at the maps

$$G(n) := Str(\Delta^n)\tilde{\times}\Delta^0 \subset \Delta^n\tilde{\times}\Delta^0 =: F(n)$$
$$F(0) = \Delta^0\tilde{\times}\Delta^0 \to B(\pi(\Delta^1))\tilde{\times}\Delta^0 =: I.$$

(identified with constant maps of bisimplicial presheaves) to form the **local complete Segal model structure** on $s^2\,\mathrm{Pre}(C)$.

The Joyal-Tierney theorem bootstraps to a comparison result:

Theorem 16 (Meadows [3]).

There is a Quillen equivalence

$$t_! : s^2\,\mathrm{Pre}(C) \leftrightarrows s\,\mathrm{Pre}(C) : t^!,$$

between the local complete Segal model structure on $s^2\,\mathrm{Pre}(C)$ and the local quasicategory structure on $s\,\mathrm{Pre}(C)$.

More to come

The quasicategory structures and complete Segal model structures both fail to be right proper, and therefore do not have a global theory of cocycles, such as one sees in the injective model structure for $s\,\mathrm{Pre}(C)$.

Bergner's model structure for categories enriched in simplicial sets is right proper, and Meadows expects to produce a right proper local version of that theory.

Descent

Original definition: Y = simp. presheaf. Y **satisfies descent** iff every injective fibrant model $j : Y \to Z$ is a sectionwise equiv.

eg: A sheaf of groupoids G is a stack iff BG satisfies descent.

Theorem 17 (Meadows).

1) *$f : X \to Y$ a morphism of presheaves of quasicategories. f is a local quasicategory equivalence if and only if all simplicial presheaf maps*

$$\mathbf{Jhom}(\Delta^n, X) \to \mathbf{Jhom}(\Delta^n, Y)$$

are local weak equivalences.

2) *X = presheaf of quasicategories. X satisfies descent for the local quasicategory structure if and only if all simplicial presheaves $\mathbf{Jhom}(\Delta^n, X)$ satisfy descent for the injective model structure.*

Constructions

Use $k^! Y$ for quasicategories Y:

$$k^! Y_n = \hom(B\pi(\Delta^n), Y).$$

$\exists$ a natural triv. fibration $k^! Y \to J(Y)$ if Y is a quasicategory.

$$t^! Y_n = k^! \mathbf{hom}(\Delta^n, Y).$$

Proof of quasicategory descent theorem uses Joyal-Tierney comparison theorem and Meadows' local version.

An application

$X =$ presheaf of quasicategories, $j : X \to LX$ be a fibrant model in the local quasicategory structure.

$J(X) \to J(LX)$ is a local weak equivalence. LX satisfies quasicategory descent.

All $J\mathbf{hom}(\Delta^n, LX)$ satisfy descent for the injective model structure, so $J(LX)$ satisfies descent. There are isomorphisms

$$[*, X]_q \cong [*, LX]_q \cong \pi_I(*, LX) \cong \pi_I(*, J(LX)) \cong [*, J(LX)] \cong [*, J(X)].$$

$[\ ,\ ]_q$ is morphisms in local quasicategory homotopy category, $[\ ,\ ]$ is morphisms in local homotopy category.

Example: $A =$ presheaf of categories:

$$[*, BA]_q \cong [*, B\,\mathrm{Iso}(A)] \ \ (\text{non-abelian } H^1).$$

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