The parabolic groupoid

Write $\mathcal{F}$ for the category of finite subsets of a fixed countable set.

Let $\mathrm{Mon}(\mathcal{F})_n$ be the set of all strings of subset inclusions

$$F: \ F_1 \subset F_2 \subset \cdots \subset F_n.$$

Say that such a string is a *formal flag* of length n.

$\mathrm{Mon}(\mathcal{F})_n$ is the set of n-simplices of a simplicial set $\mathrm{Mon}(\mathcal{F})$. Write $F_0 = \emptyset$ for each formal flag, and let $\theta : \mathbf{m} \to \mathbf{n}$ be an ordinal number morphism. The formal flag $\theta^*(F)$ (of length m) is the sequence of inclusions

$$\theta^* F: \ F_{\theta(1)} - F_{\theta(0)} \subset F_{\theta(2)} - F_{\theta(0)} \subset \cdots \subset F_{\theta(m)} - F_{\theta(0)}.$$

Write $\mathcal{O}(Y)$ for the ring of functions of a scheme Y. Then $\mathcal{O}_S$ is the Zariski sheaf of rings on $\mathrm{Sch}\,|_S$ which is defined by associating the ring $\mathcal{O}(Y)$ to each S-scheme $Y \to S$.

Write $\mathbf{Mod}(S)$ for the category of sheaves of $\mathcal{O}_S$-modules on $\mathrm{Sch}\,|_S$. Then the assignment

$$T \to S \mapsto \mathbf{Mod}(T)$$

defines a presheaf of categories $\mathbf{Mod}$ on $\mathrm{Sch}\,|_S$.

You can, if you like, assume that everything is affine: $S = \mathrm{Sp}(R)$ for some commutative unitary ring R and an S-scheme $T \to S$ is defined by an R-algebra $R \to R'$. In this case, the sheaf of rings $\mathcal{O}_S$ is the functor which takes an algebra $R \to R'$ to the ring R'.

Every finite set F determines a free $\mathcal{O}_S$-module $\mathcal{O}_S(F)$, and every function $F \to F'$ induces a morphism $\mathcal{O}_S(F) \to \mathcal{O}_S(F')$. It follows that there is a functor

$$\mathcal{O}_S : \mathcal{F} \to \mathbf{Mod}(S)$$

taking values in $\mathcal{O}_S$-modules, and a corresponding morphism of presheaves of categories

$$\mathcal{O}_S : \Gamma^* \mathcal{F} \to \mathbf{Mod}.$$

A morphism $\alpha : F \to F'$ of formal flags is a collection of $\mathcal{O}_S$-module homomorphisms

$$\alpha_k : \mathcal{O}_S(F_k) \to \mathcal{O}_S(F'_k),\ 1 \leq k \leq n,$$

such that the diagram

$$\begin{array}{ccccccc} \mathcal{O}_S(F_1) & \longrightarrow & \mathcal{O}_S(F_2) & \longrightarrow \cdots \longrightarrow & \mathcal{O}_S(F_n) \\ \alpha_1 \downarrow & & \downarrow \alpha_2 & & \downarrow \alpha_n \\ \mathcal{O}_S(F'_1) & \longrightarrow & \mathcal{O}_S(F'_2) & \longrightarrow \cdots \longrightarrow & \mathcal{O}_S(F'_n) \end{array}$$

commutes. The formal flags of length n and their homomorphisms form an additive category, which will be denoted by $\mathbf{Fl}_n(S)$. Write $\mathbf{Par}_n(S)$ for the groupoid of the formal flags of length n and their isomorphisms.

Every formal flag morphism $\alpha : F \to F'$ uniquely induces a formal flag morphism $\theta^*(\alpha) : \theta^* F \to \theta^* F'$ for all ordinal number maps $\theta : \mathbf{m} \to \mathbf{n}$. It follows that the categories $\mathbf{Fl}_n(S)$ form a simplicial category $\mathbf{Fl}(S)$. Similarly, the groupoids $\mathbf{Par}_n(S)$ form a simplicial groupoid $\mathbf{Par}(S)$.

The simplicial category $\mathbf{Fl}(S)$ is the global sections object of a presheaf of simplicial categories defined on $Sch|_S$, which is denoted by $\mathbf{Fl}$ (transition functors defined by restriction). Similarly, the simplicial groupoid $\mathbf{Par}(S)$ is the global sections object of a simplicial presheaf of groupoids $\mathbf{Par}$ on the category of S-schemes. I say that $\mathbf{Par}$ is the *parabolic groupoid.*

Write $\mathrm{Ar}(\mathbf{n})$ for the category of arrows $(i,j) : i \leq j$ in the ordinal number $\mathbf{n}$. Let M be an exact category, and recall that Waldhausen's category $S_n M$ is defined to have objects consisting of all functors $P : \mathrm{Ar}(\mathbf{n}) \to M$, such that

1) $P(i,i) = 0$ for all i, and

2) all sequences

$$0 \to P(i,j) \to P(i,k) \to P(j,k) \to 0$$

are exact for $i \leq j \leq k$.

The morphisms of $S_n M$ are the natural transformations between diagrams in M. The categories $S_n(M)$ form a simplicial category $S_\bullet M$, and the simplicial set of objects $s_\bullet M = \mathrm{Ob}(S_\bullet M)$ is Waldhausen's $s_\bullet$-construction. Recall that there are natural weak equivalences

1) $s_\bullet M \simeq BQ(M)$, and
2) $s_\bullet M \simeq B \operatorname{Iso} S_\bullet M$, where $\operatorname{Iso} S_\bullet M$ is the simplicial groupoid of isomorphisms in $S_\bullet M$.

Let $\mathcal{P}(S)$ denote the full subcategory of the category of $\mathcal{O}_S$-modules which consists of $\mathcal{O}_S$-modules which are locally free of finite rank. This is the category of (big site) vector bundles on S. It is global sections of a presheaf of categories $\mathcal{P}$ which is defined on $\mathrm{Sch}\,|_S$.

Suppose that

$$F : \; F_1 \subset \cdots \subset F_n$$

is a formal flag of length n. Then F determines an object $P(F) \in S_n \mathcal{P}(S)$ with

$$P(F)(i,j) = \mathcal{O}_S(F_j - F_i).$$

If $(i,j) \leq (k,l)$ is an arrow morphism, then the induced map

$$\mathcal{O}_S(F_j - F_i) \to \mathcal{O}_S(F_l - F_k)$$

is the composite

$$\mathcal{O}_S(F_j - F_i) \to \mathcal{O}_S(F_j - F_k) \to \mathcal{O}_S(F_l - F_k).$$

The assignments $F \mapsto P(F)$ define a morphism of simplicial categories

$$P : \mathbf{Fl}(S) \to S_\bullet\mathcal{P}(S) \subset S_\bullet(\mathbf{Mod}(S)).$$

This morphism P restricts to a simplicial groupoid morphism

$$P : \mathbf{Par}(S) \to \operatorname{Iso} S_\bullet\mathcal{P}(S),$$

and this latter morphism is global sections of a morphism

$$P : \mathbf{Par} \to \operatorname{Iso} S_\bullet\mathcal{P},$$

of presheaves of simplicial groupoids on $Sch|_S$.

Here's the main result:

Theorem 1. *The morphism $P : \mathbf{Par}_n \to \operatorname{Iso} S_n\mathcal{P}$ of presheaves of groupoids is sectionwise weakly equivalent to the Zariski stack completion $\mathbf{Par}_n \to St(\mathbf{Par}_n)$, for each $n \geq 0$.*

Now I've got some explaining to do:

1) There is a model structure for presheaves of groupoids on $\mathrm{Sch}\,|_S$, for which a morphism $G \to H$ is a weak equivalence (respectively fibration) if and only if the induced map of simplicial presheaves $BG \to BH$ is a local weak equivalence (respectively global fibration). This model structure is a special case of one defined for presheaves of groupoids on arbitrary small Grothendieck sites. In this particular case, the weak equivalences are easier to define: $BG \to BH$ is a local weak equivalence if and only if all simplicial set maps $BG_x \to BH_x$ in stalks ($x \in T$, T an S-scheme) are weak equivalences.

2) Stacks in this setup are presheaves of groupoids G which *satisfy descent*: this means that any fibrant model $G \to H$ (weak equivalence with H fibrant) induces weak equivalences $G(T) \to H(T)$ in each section. The "stack completion" for a groupoid G is therefore nothing more than a fibrant model.

The moral is that stacks are homotopy types of presheaves of groupoids, in a given local model structure.

3) Theorem 1 therefore asserts that the map

$$P : \mathbf{Par}_n \to \mathrm{Iso}\, S_n\mathcal{P}$$

is a local weak equivalence, and that the presheaf of groupoids $\mathrm{Iso}\, S_n\mathcal{P}$ satisfies descent.

4) Why would you care? The Theorem implies that the bisimplicial presheaf $B\mathbf{Par}$ is a geometric model for the K-theory presheaf $\mathbf{K}^1$ up to Zariski local weak equivalence.

NB: Schechtman "proved" a result which is equivalent to the Theorem in [1] (1987). People understood that this result gave a geometric model for K-theory at the time, but nobody ever really came to terms with either Schechtman's proof or his model for the object $B\mathbf{Par}$.

How would you prove such a thing?

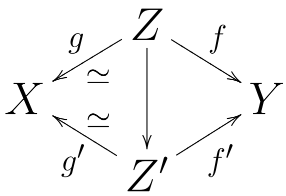
The claim that P is a local weak equivalence is essentially obvious, for the Zariski topology.

There are various models for the associated stack, which arise from cocycle theory:

I) Given simplicial presheaves (or lots of other things) X, Y, the *cocycle category* $H(X, Y)$ has as objects all pictures

$$X \xleftarrow{g} Z \xrightarrow{f} Y$$

where g is a local weak equivalence. A morphism in $H(X,Y)$ is a commutative diagram



The assignment $(g,f) \mapsto fg^{-1}$ defines a function

$$\psi : \pi_0 H(X,Y) \to [X,Y]$$

taking values in morphisms in the homotopy category of simplicial presheaves (or sheaves).

Theorem 2. *The function ψ is a bijection.*

Remark: Cocycle categories have appeared before, in the Dwyer-Kan theory of hammock localizations for arbitrary model categories. They have a theorem like Theorem 2 in that context, provided that Y is fibrant. The point of most applications of Theorem 2 is that Y doesn't need to be fibrant (although most of the time Y is projective fibrant).

II) Suppose that G is a sheaf of groupoids. A G-diagram X consists of functors $X(U) : G(U) \to$ **Sets** which fit together in an obvious way — in particular the sets

$$\bigsqcup_{x \in \mathrm{Ob}(G(U)} X(x)$$

should form a sheaf.

Equivalently, a G-diagram X is a sheaf map $\pi : X \to \mathrm{Ob}(G)$ with an action

$$\begin{array}{ccc} X \times_s \mathrm{Mor}(G) & \xrightarrow{m} & X \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Mor}(G) & \xrightarrow[t]{} & \mathrm{Ob}(G) \end{array}$$

(source map $s = d_1$) which is associative and respects identities.

One can form the homotopy colimit $\underrightarrow{\mathrm{holim}}_G X$ (nerve of translation category) section by section, and there is a canonical simplicial presheaf map $\underrightarrow{\mathrm{holim}}_G X \to BG$.

A (discrete) G-*torsor* is a G-diagram X (of sheaves) such that the canonical map $\underrightarrow{\mathrm{holim}}_G X \to *$ is a weak equivalence.

Remark: If G is a sheaf of groups and Y is a sheaf with G-action then Y is a G-torsor if and only if the map $EG \times_G Y \to *$ is a local weak equivalence (ie G acts freely and locally transitively). The definition of G-torsor for a sheaf of groupoids G is a direct generalization of this.

A map $X \to Y$ of G-torsors is a natural transformation, or equivalently it's a sheaf map

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & \mathrm{Ob}(G) & \end{array}$$

fibred over $\mathrm{Ob}(G)$ which respects the actions.

If X is a G-diagram then the diagram

$$\begin{array}{ccc} X & \longrightarrow & \underrightarrow{\mathrm{holim}}_G X \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \mathrm{Ob}(G) & \longrightarrow & BG \end{array}$$

is homotopy cartesian (Quillen's Theorem B), and so every map $X \to Y$ of G-torsors is an isomorphism fibred over $\mathrm{Ob}(G)$.

$G - \mathbf{tors}$ = the category of G-torsors. This category is a groupoid.

Here's a construction:

Suppose that $* \xleftarrow{\simeq} Y \to BG$ is a cocycle, and form the G-diagram $\mathrm{pb}(Y)$ by the pullbacks

$$\begin{array}{ccc} \mathrm{pb}(Y)(U)_x & \longrightarrow & Y(U) \\ \downarrow & & \downarrow \\ BG(U)/x & \longrightarrow & BG(U) \end{array}$$

Then

- By formal nonsense there is a weak equivalence

$$\underrightarrow{\mathrm{holim}}_G \,\mathrm{pb}(Y) \xrightarrow{\simeq} Y \simeq *$$

- The map $\mathrm{pb}(Y) \to \tilde{\pi}_0\,\mathrm{pb}(Y)$ is a local weak equivalence of diagrams, since $\mathrm{pb}(Y)$ is made up of loop spaces of BG (locally).

It follows that there are weak equivalences

$$\underrightarrow{\mathrm{holim}}_G \,\tilde{\pi}_0\,\mathrm{pb}(Y) \xleftarrow{\simeq} \underrightarrow{\mathrm{holim}}_G \;\mathrm{pb}(Y) \xrightarrow{\simeq} Y \simeq *$$

and so the diagram $\tilde{\pi}_0\,\mathrm{pb}(Y)$ is an G-torsor.

If X is a G-torsor then $* \xleftarrow{\simeq} \underrightarrow{\mathrm{holim}}_G X \to BG$ is a cocycle.

Even more is true: the functors

$$\tilde{\pi}_0\,\mathrm{pb} : H(*, BG) \leftrightarrows G - \mathbf{tors} : \underrightarrow{\mathrm{holim}}$$

is an adjoint pair so that there is a weak equivalence

$$BH(*, BG) \simeq B(G - \mathbf{Tors}).$$

We therefore have

Theorem 3. *There are natural bijections*

$$\{\textit{iso. classes of } G\textit{-torsors}\} = \pi_0(G - \mathbf{tors}) \cong \pi_0 H(*, BG) \cong [*, BG].$$

The groupoid $G - \mathbf{tors}$ is global sections of a presheaf of groupoids $G - \mathbf{Tors}$. In effect, all G-diagrams X restrict to $G|_U$-diagrams $X|_U$ on $\mathcal{C}/U$ for all $U \in \mathcal{C}$, as do all local weak equivalences.

The category $H(*, BG)$ is global sections of a presheaf of categories $\mathbf{H}(*, BG)$, and there is a functor

$$i : G \to \mathbf{H}(*, BG)$$

defined by taking $x \in \mathrm{Ob}(G)$ to the cocycle

$$* \xleftarrow{\simeq} B(G/x) \to BG$$

The functor i induces a map $j : G \to G - \mathbf{Tors}$ of presheaves of groupoids, which is defined by $j(x) = G(\ , x)$.

Theorem 4. *The simplicial presheaf* $B\mathbf{H}(*, BG)$ *satisfies descent and* $i : BG \to B\mathbf{H}(*, BG)$ *is a local weak equivalence.*

Proof: Show that $G \mapsto BH(*, BG)$ preserves local weak equivalences and that i is a $\tilde{\pi}_0$-epimorphism and j is a $\tilde{\pi}_1$-isomorphism if G is a stack.

Remark: This means that $G - \mathbf{Tors}$ is a model for the stack associated to G, but so is $\mathbf{H}(*, BG)$.

Now how to prove Theorem 1?

Suppose that $* \xleftarrow{\simeq} H \xrightarrow{f} \mathbf{Par}_n$ is a cocycle in groupoids on $\mathrm{Sch}\,|_S$ and form the composite

$$H \xrightarrow{f} \mathbf{Par}_n \xrightarrow{P} \mathrm{Iso}\, S_n\mathcal{P} \subset \mathrm{Iso}\, S_n(\mathbf{Mod}).$$

Taking colimit of this functor determines an object $L(f)$ of $S_n(\mathbf{Mod})(S)$ which is locally free in each node (i,j) since $H \to *$ is a local weak equivalence. The assignment $f \mapsto L(f)$ therefore determines a functor

$$L : H(*, B\mathbf{Par}_n) \to \mathrm{Iso}\, S_n\mathcal{P}(S).$$

One shows that the composite

$$\mathbf{Par}_n - \mathbf{tors} \to H(*, B\mathbf{Par}_n) \xrightarrow{L} \mathrm{Iso}\, S_n\mathcal{P}_n(S)$$

is a weak equivalence of groupoids.

This is just a souped up version of the classical weak equivalence

$$Gl_n - \mathbf{tors} \to H(*, BGl_n) \to \mathrm{Iso}\,\mathcal{P}_n(S)$$

taking values in vector bundles of rank n. The key points are that every vector bundle is trivialized along some cocycle, and that

$$\mathrm{Aut}(Gl_n) \cong \mathrm{Aut}(\mathcal{O}_S^n).$$

References

[1] V. V. Schechtman. On the delooping of Chern character and Adams operations. In *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, volume 1289 of *Lecture Notes in Math.*, pages 265–319. Springer, Berlin, 1987.