Pointed torsors and Galois groups

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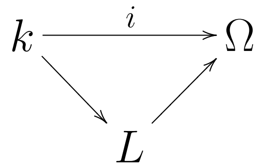
Galois theory

Suppose that k is a field, and that $i : k \to \Omega$ is a fixed imbedding in an algebraically closed field. I also write $i : \mathrm{Sp}(\Omega) \to \mathrm{Sp}(k)$ for the corresponding geometric point.

I write Gal/k for the category of finite Galois extensions



of k in Ω, with the obvious maps between them. For the experts, this is the category of étale neighbourhoods of the geometric point $i : \mathrm{Sp}(\Omega) \to \mathrm{Sp}(k)$ which happen to consist of finite Galois extensions. This category is filtered.

The absolute Galois group $G(k)$ is the profinite group $L \mapsto Gal(L/k)$ for $L \in Gal/k$.

The (finite) *étale site* $et|_k$ for k consists of all finite products

$$\prod_{i=1}^{n} L_i,$$

where each L_i is a finite separable extension of k. Equivalently the category consists of all finite disjoint unions

$$\mathrm{Sp}(L_1) \sqcup \cdots \sqcup \mathrm{Sp}(L_n)$$

of k-schemes, where again L_i/k is a finite separable extension.

A *presheaf* on $et|_k$ is a contravariant functor

$$F : (et|_k)^{op} \to \mathbf{Set}$$

(contravariant on k-schemes or covariant on k-algebras). We say that F is a *sheaf* if

1) F is additive in the sense that the map

$$F(\sqcup_i \mathrm{Sp}(L_i)) \to \prod_i F(\mathrm{Sp}(L_i)$$

is an isomorphism, and

2) if L'/L is a finite Galois extension with Galois group G, then the canonical map

$$F(\mathrm{Sp}(L)) \to F(\mathrm{Sp}(L'))^G$$

is a bijection.

Example: Every k-scheme represents a sheaf on $et|_k$, so that every algebraic group H defined over k represents a sheaf of groups H.

GL_n is the sheaf of groups defined by $\mathrm{Sp}(L) \mapsto Gl_n(L)$.

For a sheaf or presheaf F on the étale site $et|_k$, the *canonical stalk* i^*F is the set defined by

$$i^*F = \varinjlim_{L \in Gal/k} F(L).$$

This is the unique stalk for the finite étale site on $\mathrm{Sp}(k)$.

If L/k is a finite separable extension of k, then there is an imbedding of pro-groups $G(L) \to G(k)$ (restrict to all Galois extensions of k which contain L).

The set i^*F is a discrete $G(k)$-module. The assignment

$$L \mapsto (i^*F)^{G(L)}$$

defines a sheaf $\tilde{F}$ on $et|_k$. There is a canonical map $\eta : F \to \tilde{F}$ which is an isomorphism if F is a sheaf; otherwise, it's the *associated sheaf map* and $\tilde{F}$ is the *associated sheaf*.

$\mathbf{Shv}(et|_k)$ is the category of sheaves on $et|_k$. We'll stick to sheaves most of the time in what follows.

Simplicial sheaves and cocycles

The base change functor (canonical stalk) i^* is exact and reflects exactness: a map $f : F \to F'$ is an isomorphism (resp. monic, epi) if and only if the function $i^*f : i^*F \to i^*F'$ is an isomorphisms (resp. monic, epi).

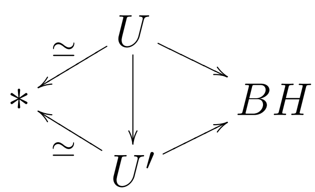
In particular, a simplicial sheaf map $f : X \to Y$ on $et|_k$ is a local weak equivalence if and only if $i^*f : i^*X \to i^*Y$ is a weak equivalence of simplicial sets, so the injective model structure for the simplicial sheaf category $s\mathbf{Shv}(et|_k)$ is relatively simple to describe: cofibrations are just monomorphisms. I will restrict attention to simplicial sheaves in what follows.

Suppose that H is an algebraic group which is defined over k, and let it represent a sheaf of groups on $et|_k$, with associated classifying object BH.

The **cocycle category** $h(*, BH)$ has objects consisting of diagrams of simplicial sheaf morphisms

$$* \xleftarrow{\simeq} U \to BG,$$

and morphisms given by diagrams



Theorem 1. *There is an isomorphism*

$$\phi : \pi_0 B(h(*, BH)) \xrightarrow{\cong} [*, BH]$$

where $[X, Y]$ is morphisms $X \to Y$ in the homotopy category of simplicial sheaves.

The function ϕ takes a cocycle $* \xleftarrow[\simeq]{f} U \xrightarrow{g} BH$ to the composite $g \cdot f^{-1}$ in the homotopy category.

The definition of cocycle and the Theorem are very special cases: one could talk about cocycles

$$X \xleftarrow{\simeq} U \to Y$$

for arbitrary simplicial sheaves X and Y, and then there is a bijection

$$\pi_0 Bh(X, Y) \cong [X, Y].$$

Torsors

An H**-torsor** is a sheaf F together with a group action $H \times F \to F$ such that the Borel construction $EH \times_H F$ is contractible in the simplicial sheaf category.

A morphism $f : F \to F'$ of H-torsors is an H-equivariant map. Any such morphism must be an isomorphism, by the comparison of fibre sequences

$$\begin{array}{ccccc} F & \longrightarrow & EH \times_H F & \longrightarrow & BH \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle 1} \\ F' & \longrightarrow & EH \times_H F' & \longrightarrow & BH \end{array}$$

The map f must be a weak equivalence of simplicial sheaves, and hence an isomorphism of sheaves.

$H-\mathbf{tors}$ will denote the groupoid of H-torsors.

Every H-torsor F has an associated canonical cocycle

$$* \xleftarrow{\simeq} EH \times_H F \to BH.$$

The canonical cocycle functor

$$EH\times_H ? : H - \mathbf{tors} \to h(*, BH)$$

has a left adjoint, and therefore induces a homotopy equivalence

$$B(H - \mathbf{tors}) \simeq B(h(*, BH)).$$

The H-torsor associated to a cocycle

$$* \xleftarrow{\simeq} U \xrightarrow{f} BH$$

is the homotopy fibre $F(f)$ of the map $U \xrightarrow{f} BH$ over the global base point.

Computing in π_0 gives the following:

Theorem 2. *There is a natural isomorphism*

$$H^1_{et}(k, H) \cong [*, BH].$$

This theorem (and proof technique) holds in much greater generality: non-abelian H^1 with coefficients in a sheaf of groupoids G is isomorphic to $[*, BG]$ in any Grothendieck topos.

Example: Suppose given an action $H \times M \to M$ on a sheaf M, and suppose given a cocycle

$$* \xleftarrow{\simeq} U \to EH \times_H M.$$

Then computing homotopy fibres for the maps

$$U \to EH \times_H M \to BH$$

gives an H-torsor P and a G-equivariant map $P \to M$. This construction is again left adjoint to the canonical cocycle construction, and gives an isomorphism

$$\pi_0[M/G] \xrightarrow{\cong} \pi_0 Bh(*, EH \times_H M) \cong [*, EH \times_H M],$$

where $[M/G]$ is the quotient stack. From a homotopy theoretic point of view, the Borel construction $EH \times_H M$ **is** the quotient stack.

Pointed torsors

Let's return to Galois theory.

Here are some new definitions (which are again special cases):

1) A **pointed cocycle** is a cocycle

$$* \underset{\simeq}{\xleftarrow{g}} U \xrightarrow{f} BH,$$

together with a choice of basepoint $x : * \to i^*U$ for the simplicial set i^*U, and a morphism of pointed cocycles is a map of cocycles which preserves base points on the stalk level. Write $h(*, BH)_*$ for the category of pointed cocycles.

2) A **pointed torsor** is an H-torsor F together with a distinguished element (trivialization) $x \in i^*F$. A morphism of pointed torsors is a morphism of torsors which respects base points on the stalk level. Write $H - \mathbf{tors}_*$ for the groupoid of pointed torsors.

There are two lemmas:

Lemma 3. *The simplicial set $B(H - \mathbf{tors}_*)$ is the homotopy fibre over the trivial torsor i^*H of the inverse image morphism*

$$i^* : B(H - \mathbf{tors}) \to B(i^*H - \mathbf{tors}) \simeq B(i^*H).$$

Remark 4. If G is a plain old group in the set category, the notion of G-torsor still makes sense: it's a set X with G-action such that $EG \times_G X$ is contractible. Every element $x \in X$ induces a trivialization $i_x : G \xrightarrow{\cong} X$, so that all G-torsors are isomorphic. The group of automorphisms of the trivial torsor G is a copy of G itself, so that there is a weak equivalence

$$B(G - \mathbf{tors}) \simeq BG.$$

Lemma 5. *The canonical cocycle adjunction restricts to an adjoint pair of functors*

$$F(?) : h(*, BH)_* \leftrightarrows H - \mathbf{tors}_* : EH \times_H ?$$

so that there is a homotopy equivalence

$$Bh(*, BH)_* \simeq B(H - \mathbf{tors}_*).$$

The moral of this equivalence is that we can use cocycle techniques to understand the groupoid of pointed torsors, which is the homotopy fibre of i^*, by Lemma 3.

A *pointed Čech cocycle* is a pointed cocycle

$$* \xleftarrow{\simeq} C(U) \to BH,$$

where $C(U)$ is the Čech resolution associated to some sheaf epimorphism $U \to *$. There is a category $h_{Cech}(*, BH)$ of such things and an obvious inclusion functor

$$j : h_{Cech}(*, BH)_* \subset h(*, BH)_*$$

Lemma 6. *The inclusion functor j induces a weak equivalence*

$$B(h_{Cech}(*, BH)_*) \subset B(h(*, BH)_*).$$

Proof. The fundamental groupoid functor

$$U \mapsto \tilde{\pi} U$$

defines a left adjoint to the inclusion j. In effect, the canonical map

$$B\tilde{\pi}(U) \to C(U_0)$$

is an isomorphism, since $C(U_0)$ is the nerve of the free groupoid on the sheaf of vertices U_0, and $U \simeq *$ (check this stalkwise, or with Boolean localization). □

Among all coverings $U \to *$ we have the epimorphisms $\mathrm{Sp}(L) \to *$ defined by the finite Galois extensions L/k in Ω. There are pointed cocycles

$$* \xleftarrow{\simeq} C(L) \xrightarrow{f} BH$$

with base points

$$e_L \in i^* \mathrm{Sp}(L) = \varinjlim_{N/k} \hom(L, N)$$

where the colimit is indexed over finite Galois extensions N/k in Ω. The element e_L corresponds to the identity field homomorphism 1_L on L. If $j : L \to N$ is a morphism of Galois extensions of k inside Ω, then $j^*(e_N) = e_L$. Thus there is a subcategory $h_{Gal}(*, BH)_*$ of $h_{Cech}(*, BH)_*$ whose objects are the objects above, and which has morphisms

$$\begin{array}{ccccc} & & C(N) & & \\ & \overset{\simeq}{\swarrow} & & \overset{f'}{\searrow} & \\ * & & \downarrow{\scriptstyle j^*} & & BH \\ & \underset{\simeq}{\nwarrow} & & \underset{f}{\nearrow} & \\ & & C(L) & & \end{array}$$

determined by inclusions j of finite Galois extensions of k in Ω.

Lemma 7. *The induced map*

$$B(h_{Gal}(*, BH)_*) \to B(h_{Cech}(*, BH)_*)$$

is a weak equivalence.

Proof. This is a "Quillen Theorem A"-type argument.

Let f be the Čech cocycle

$$* \xleftarrow{\simeq} C(U) \xrightarrow{f} BH,$$

and let j be the functor

$$j : h_{Gal}(*, BH)_* \to h_{Cech}(*, BH)$$

Then one can show that the slice category j/f is non-empty and filtered. The element

$$x \in i^*(U) = \varinjlim_{L} \hom(\mathrm{Sp}(L), U)$$

is represented by a map $v : \mathrm{Sp}(L) \to U$, and $v_*(e_L) = x$. The category j/f is therefore non-empty; it is also a category of representatives for x. Now show that the category of representatives $v : \mathrm{Sp}(L) \to U$ of x is left filtered. $\square$

Corollary 8. *There is an isomorphism*

$$\pi_0 B(h(*, BH)_*) \cong \varinjlim_{L/k} \hom(C(L), BH).$$

Some remarks:

1) All automorphisms of a pointed torsor are trivial, so that the map

$$B(H - \mathbf{tors}_*) \to \pi_0 B(H - \mathbf{tors}_*)$$

is a weak equivalence.

2) Recall that the Čech resolution $C(L)$ is isomorphic to the Borel construction

$$EG(L/k) \times_{G(L/k)} \mathrm{Sp}(L),$$

by Galois theory. This Borel construction is the nerve of the translation groupoid $E_{G(L/k)} \mathrm{Sp}(L)$. The pro object defined by the functor

$$L \mapsto E_{G(L/k)} \mathrm{Sp}(L)$$

is the *absolute Galois groupoid* of k. The Corollary above says that $\pi_0 B(h(*, BH)_*)$ is the set of "representations" of the absolute Galois groupoid in H.

3) If H is a constant group, then a morphism

$$EG(L/k) \times_{G(L/k)} \mathrm{Sp}(L) \to BH$$

can be identified with a group homomorphism

$$G(L/K) \to H,$$

because $BG(L/K)$ is the simplicial set of connected components of the Borel construction. It follows that $\pi_0 B(h(*, BH)_*)$ is the set of representations of the absolute Galois group in H.

This is the first homotopy theoretic invariant for simplicial sheaves or presheaves that I've seen that is represented by the absolute Galois group.

4) The results displayed here actually hold for all profinite groups, and in particular for the Grothendieck fundamental group of a (nice: connected, Noetherian) scheme S (via the finite étale site $fet|_S$ for S). It follows that there is a well-defined Grothendieck fundamental groupoid consisting of the translation groupoids $E_G X$ associated to finite Galois extensions X/S. This pro object represents all pointed torsors $et|_S S$, and the Grothendieck fundamental group represents pointed torsors for constant groups H on $et|_S$.

Recall that a Galois cover $X \to S$ is a finite étale morphism with X connected such that X is a G-torsor for the étale topology for some finite group G. In this case G is the Galois group of the cover X/S. These things are ubiquitous, since every finite étale morphism $Y \to S$ which Y connected can be refined by a Galois cover.

References

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