Presheaves of spectra

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Simplicial presheaves

A simplicial presheaf is a diagram of simplicial sets

$$X : \mathcal{C}^{op} \to s\mathbf{Set}$$

defined on a category $\mathcal{C}$ having a Grothendieck topology.

Examples:

1) $op|_T$ = open subsets of a topological space T
2) $et|_S$ = étale maps $U \to S$ for a scheme S
3) $(Sch|_S)_{et}$ = "big" étale site for a scheme S
4) $(Sch|)_{Zar}$, $(Sch|_S)_{Nis}$, $(Sch|_S)_{fl}$: big sites with Zariski, Nisnevich, flat topologies ...
5) index category I with no topology: a family $U \to V$ for V is covering if and only if it contains the identity $V \to V$.

Local weak equivalences

$s\,\mathrm{Pre} = s\,\mathrm{Pre}(\mathcal{C})$ is the category of simplicial presheaves on $\mathcal{C}$, with natural transformations between them as morphisms.

A *cofibration* $A \to B$ of $s\,\mathrm{Pre}$ is a monomorphism.

$f : X \to Y$ is a *local weak equivalence* (Joyal) if

1) $f_* : \tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ is a sheaf isomorphism , and
2 the diagrams

$$\begin{array}{ccc} \bigsqcup_{y \in X_0(U)} \pi_n(X(U), y) & \quad & \tilde{\pi}_n X \xrightarrow{f_*} \tilde{\pi}_n Y \\ \downarrow & & \downarrow \qquad\quad \downarrow \\ \bigsqcup_{y \in X_0(U)} * & & \tilde{X}_0 \xrightarrow[f_*]{} \tilde{Y}_0 \end{array}$$

are pullbacks in the sheaf category for $n \geq 1$.

In the presence of stalks $X \mapsto X_x$: f is a local weak equivalence iff all maps $f_x : X_x \to Y_x$ are weak equivalences of simplicial sets.

Injective model structure

A map $p : X \to Y$ is an *injective fibration* if it has the RLP wrt all cofibrations which are local weak equivs. (ie. *trivial* cofibrations).

Theorem 1.

The category $s\,\mathrm{Pre}$*, with the cofibrations, local weak equivalences and injective fibrations, satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*

This is the injective model structure for $s\,\mathrm{Pre}(\mathcal{C})$.

There is a corresponding model structure for simplicial sheaves, due to Joyal, which is Quillen equivalent.

Bounded monomorphisms

A Grothendieck site $\mathcal{C}$ is implicitly a small category, so that there is a set of morphisms $\mathrm{Mor}(\mathcal{C})$.

Choose an infinite (regular) cardinal α such that $|\,\mathrm{Mor}(\mathcal{C})| < \alpha$.

B is *α-bounded* if $|B_n(U)| < \alpha$ for all $n \geq 0$, $U \in \mathcal{C}$. A cofibration (monomorphism) $A \to B$ is α-bounded if B is α-bounded.

The existence of the injective model structure for $s\,\mathrm{Pre}(\mathcal{C})$ follows from the *bounded monomorphism condition* [2]:

Lemma 2.

Suppose that $i : X \to Y$ is a trivial cofibration and that $A \subset Y$ is an α-bounded subobject. Then there is an α-bounded subobject $B \subset Y$, such that $A \subset B$ and $B \cap X \to B$ is a local weak equivalence.

Pointed simplicial presheaves

A *pointed simplicial presheaf* is a morphism $* \to X$ in the simplicial presheaf category, ie. a simplicial presheaf X with a "global" choice of base point. $*$ = one-point presheaf (terminal object).
A *morphism* of pointed simplicial presheaves is a commutative diagram

$$\begin{array}{ccc} & * & \\ \swarrow & & \searrow \\ X & \xrightarrow[f]{} & Y \end{array} \tag{1}$$

Common notations for the category: $s_* \operatorname{Pre}$, $*/s\operatorname{Pre}$.

Corollary 3.

There is a proper closed simplicial model structure on $s_* \operatorname{Pre}(\mathcal{C})$ for which a morphism (1) is a weak equivalence (resp. cofibration, fibration) if and only if the map $f : X \to Y$ is a local weak equivalence (resp. cofibration, injective fibration) of simplicial presheaves. This model structure is cofibrantly generated.

Warning

The map

$$\begin{array}{ccc} & * & \\ \swarrow & & \searrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

of pointed simplicial presheaves is a weak equivalence if and only if the map $f : X \to Y$ is a local weak equivalence of simplicial presheaves, ie. induces isomorphisms of sheaves of all homotopy groups for **all** local choices of base points.

A similar observation applies to the model structure for pointed simplicial sets, which is a special case.

This is a common source of errors.

Theorem 4.

There is a model structure on $s\,\mathrm{Pre}_{\mathbb{Z}}$ (presheaves of simplicial abelian groups, for which a map $A \to B$ is a weak equivalence (resp. fibration) if the underlying map $u(A) \to u(B)$ of simplicial presheaves is a local weak equivalence (resp. injective fibration). There is a Quillen adjunction

$$\mathbb{Z} : s\,\mathrm{Pre} \leftrightarrows s\,\mathrm{Pre}_{\mathbb{Z}} : u.$$

Theorem 5.

Suppose that A is a sheaf of abelian groups. There is a natural isomorphism

$$[*, K(A, n)] \cong H^n(\mathcal{C}, A).$$

"Proof" of Theorem 5.

$$K(A,n) = \Gamma(A[-n]),$$

where chain complex $A[-n]$ is A concentrated in degree n and

$$\Gamma : Ch_+ \leftrightarrows s\mathbf{Ab} : N$$

is the Dold-Kan correspondence.

If $A \to J$ is an injective resolution of A, then $A[-n] \to \tau(J[-n])$ is a local weak equivalence (ie. quasi-isomorphism), where τ is good truncation in degree 0 (preserves H_*-isomorphisms).

The chain complex object $\tau(J[-n])$ satisfies descent, so that $[\mathbb{Z}(*), \tau(J[-n])]$ is identified with chain homotopy classes of maps $\mathbb{Z}(*) \to J[-n]$. □

Corollary 6.

Suppose that A is an abelian sheaf, with injective fibrant model $j : K(A, n) \to GK(A, n)$. Then there are isomorphisms

$$\pi_k(GK(A,n)(U)) \cong \begin{cases} H^{n-k}(U,A) & \text{if } 0 \leq k \leq n, \text{ and} \\ 0 & \text{if } k > n. \end{cases}$$

For a simplicial presheaf X, define *cohomology* of X by

$$H^n(X,A) = [X, K(A,n)],\ \ n \geq 0.$$

Example: $G =$ algebraic group, defined over a field k. BG represents a simplical sheaf on $(Sch|_k)_{et}$, and there is an iso.

$$H^n(BG,A) \cong H^n(et|_{BG}, A),$$

where $et|_{BG}$ is the étale site fibred over BG (classical definition).

A *presheaf of spectra* E on a site $\mathcal{C}$ is a functor

$$E : \mathcal{C}^{op} \to \mathbf{Spt}.$$

Alternatively, E consists of pointed simplicial presheaves E^n, $n \geq 0$, together with *bonding maps* $\sigma : S^1 \wedge E^n \to E^{n+1}$.

A *morphism* $\alpha : E \to F$ is a natural transformation of functors. Alternatively, α consists of pointed simplicial presheaf maps $\alpha : E^n \to F^n$, $n \geq 0$, such that the diagrams

$$\begin{array}{ccc} S^1 \wedge E^n & \xrightarrow{\sigma} & E^{n+1} \\ {\scriptstyle S^1\wedge\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ S^1 \wedge F^n & \xrightarrow[\sigma]{} & F^{n+1} \end{array}$$

commute.

$\mathbf{Spt}(\mathcal{C})$ denotes the category of presheaves of spectra on $\mathcal{C}$.

Examples

1) The *sphere spectrum object* S is the constant presheaf of spectra associated to the ordinary sphere spectrum. It consists of the iterated smash powers

$$S^n = S^1 \wedge \cdots \wedge S^1, \;\; n \geq 1,$$

with $S^0 = \partial\Delta^1 = \{0,1\}$, pointed by 0.

2) $K =$ a pointed simplicial presheaf: the *suspension spectrum* $\Sigma^\infty K = S \wedge K$ is defined by $(\Sigma^\infty K)^n = S^n \wedge K$.

3) [$\mathbb{Z}_\bullet(K) = \mathbb{Z}(K)/\mathbb{Z}(*)$, *reduced* simplicial abelian presheaf]

$A =$ simplicial abelian presheaf. The *Eilenberg-Mac Lane spectrum* $H(A)$ consists of pointed simplicial presheaves underlying the objects

$$H(A)^n = \mathbb{Z}_\bullet(S^n) \otimes A, \;\; n \geq 0,$$

with canonical maps

$$S^1 \wedge \mathbb{Z}_\bullet(S^n) \otimes A \xrightarrow{\gamma} \mathbb{Z}_\bullet(S^{n+1}) \otimes A.$$

Chain complexes

$H(A)$ is a suspension spectrum object in simplicial abelian presheaves. The sphere spectrum for that category is $\mathbb{Z}_\bullet(S)$.

Every (unbounded) chain complex object D determines a presheaf of spectra $\Gamma(D)$:

$$\Gamma(D)^n = \Gamma(\tau(D[-n])).$$

Here,

1) $D[-n]_k = D_{k-n}$ is D shifted up n times
2) τ is good truncation in degree 0, and
3) $\Gamma : Ch_+ \leftrightarrows s\mathbf{Ab} : N$ is the Dold-Kan correspondence.

The canonical maps

$$\tau(D[-n])[-1] \to \tau(D[-n-1])$$

induce the bonding maps — use the natural equivalence

$$N(\mathbb{Z}_\bullet(S^1) \otimes A) \overset{\simeq}{\leftrightarrows} (NA)[-1]$$

in normalized chains for simplicial abelian groups A.

The strict model structure

Strict weak equivalences (resp. *strict fibrations*) are those maps $\alpha : E \to F$ of presheaves of spectra for which all maps $E^n \to F^n$ are local weak equivalences (resp. injective fibrations) of pointed simplicial presheaves.

A map $i : A \to B$ is a *cofibration* if $A^0 \to B^0$ and all

$$(S^1 \wedge B^n) \cup_{S^1 \wedge A^n} A^{n+1} \to B^{n+1}$$

are cofibrations of pointed simplicial presheaves.

The proof of the following is an exercise:

Lemma 7.

The category $\mathbf{Spt}(\mathcal{C})$*, together with the strict weak equivalences, strict fibrations and cofibrations as defined above, satisfies the axioms for a proper closed simplicial model structure. This model structure is cofibrantly generated.*

The stable model structure

A map $\alpha : E \to F$ of $\mathbf{Spt}(\mathcal{C})$ is a *stable equivalence* if it induces an isomorphism

$$\tilde{\pi}_k E \xrightarrow{\cong} \tilde{\pi}_k F$$

of sheaves of stable homotopy groups for all $k \in \mathbb{Z}$.

A map $p : X \to Y$ is a *stable fibration* if it has the RLP wrt all maps which are cofibrations and stable equivalences.

Theorem 8.

The category $\mathbf{Spt}(\mathcal{C})$, *with the cofibrations, stable equivalences and stable fibrations as defined above, satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*

This is the *stable model structure* for $\mathbf{Spt}(\mathcal{C})$.

"Proof" of Theorem 8

The function complex $\mathbf{hom}(X, Y)$ for the simplicial model structure of the Theorem has n-simplices given by the maps

$$X \wedge \Delta^n_+ \to Y$$

The following are the basic steps in the proof of Theorem 8:

Lemma 9.

A map $p : X \to Y$ is a trivial stable fibration if and only if it is a trivial strict fibration.

Lemma 10.

Suppose that $i : E \to F$ is a monomorphism and a stable equivalence of $\mathbf{Spt}(\mathcal{C})$, and suppose that $A \subset F$ is an α-bounded subobject. Then there is an α-bounded subobject $B \subset F$ such that $A \subset B$ and $B \cap E \to B$ is a stable equivalence.

Stably fibrant models

The stable model structure of Theorem 8 is cofibrantly generated, so there is a natural stably fibrant model $\eta : E \to LE$.

A map $X \to Y$ of stably fibrant objects is a stable equivalence if and only if it is a strict equivalence. Thus a map $E \to F$ is a stable equivalence if and only if $LE \to LF$ is a strict equivalence.

Lemma 11.

X is stably fibrant if and only if X is strictly fibrant and all $X^n \to \Omega X^{n+1}$ are local weak equivalences.

If X is stably fibrant, then the maps

$$\mathbf{hom}(S \wedge U_+[-n], X) \to \mathbf{hom}(S \wedge U_+ \wedge S^1[-n-1], X)$$

are weak equivalences for all $n \geq 0$ and all $U \in \mathcal{C}$, so all $X^n \to \Omega X^{n+1}$ are sectionwise weak equivalences.

The classical model

Y strictly fibrant: $(\Omega^\infty Y)^n$ is the filtered colimit of the system

$$Y^n \xrightarrow{\sigma} \Omega Y^{n+1} \xrightarrow{\Omega\sigma} \Omega^2 Y^{n+2} \xrightarrow{\Omega^2\sigma} \dots$$

The objects $(\Omega^\infty Y)^n$ form a presheaf of spectra $\Omega^\infty Y$, and there is a natural map of spectra

$$\nu : Y \to \Omega^\infty Y,$$

which is a stable equivalence. $\Omega^\infty Y$ may not be strictly fibrant.

The composite

$$\eta : X \xrightarrow{j} FX \xrightarrow{\nu} \Omega^\infty FX \xrightarrow{j} F\Omega^\infty FX =: QX$$

is a natural stably fibrant model for a presheaf of spectra X.

Bousfield-Friedlander localization

The map $\eta : X \to QX$ is the "classical" stably fibrant model.

Here are some basic facts:

A4: The functor Q preserves strict equivalences.

A5: The maps $\eta, Q\eta : QX \to QQX$ are strict equivalences.

A6: Q-equivalences are preserved by pullback along stable fibrations and pushout along cofibrations.

Theorem 8 is a formal consequence of the existence of the model $\eta : X \to QX$, and this list of statements.

This is the Bousfield-Friedlander method of proof for the existence of the stable model structure on the category of spectra, and was the original argument for Theorem 8.

Sheaf cohomology, revisited

A = abelian sheaf on $\mathcal{C}$, with corresponding Eilenberg-Mac Lane spectrum $H(A)$, and strictly fibrant model $j : H(A) \to FH(A)$.

$$\begin{array}{ccc} H(A)^n & \xrightarrow[\simeq]{\sigma} & \Omega H(A)^{n+1} \\ {\scriptstyle j}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \Omega j} \\ FH(A)^n & \xrightarrow[\sigma]{} & \Omega FH(A)^{n+1} \end{array}$$

Ωj is a local weak equivalence(!), so $\sigma : FH(A)^n \to \Omega FH(A)^{n+1}$ is a local weak equivalence, and $FH(A)$ is stably fibrant.

Corollary 6 implies the following:

Corollary 12.

$$\pi_k FH(A)(U) \cong \begin{cases} H^{-k}(U, A) & \text{if } k \leq 0, \text{ and} \\ 0 & \text{if } k > 0. \end{cases}$$

Étale cohomology

1) Suppose that S is a scheme and let A be an abelian sheaf for an étale site over S. Take a strict fibrant model $j : H(A) \to FH(A)$ in presheaves of spectra for the étale topology. Then there are isomorphisms

$$\pi_k FH(A)(S) \cong \begin{cases} H_{et}^{-k}(S, A) & \text{if } k \leq 0, \text{ and} \\ 0 & \text{if } k > 0. \end{cases}$$

2) Suppose that $S = Sp(k)$ for a field, and let G be the absolute Galois group for k. If A is an abelian sheaf for the étale topology on k and $j : H(A) \to FH(A)$ is a strictly fibrant model, then

$$\pi_p FH(A)(k) \cong \begin{cases} H^{-p}(G, A) & \text{if } p \leq 0, \text{ and} \\ 0 & \text{if } p > 0. \end{cases}$$

Here, $H^*(G, A)$ is Galois cohomology.

Profinite groups

The absolute Galois group G of a field k is the pro-group of Galois groups $G(L/k)$ for the finite Galois extensions L/k inside k_{sep}.

$H = \{H_i\}$ = a profinite group. Write also $H = \varprojlim H_i$.

$H - \mathbf{Set}_{df}$ is the site of finite discrete H-sets, ie. finite H-sets F for which the action $H \times F \to F$ factors through some action $H_i \times F \to F$, with the H-equivariant maps between them. The coverings are surjective maps. The associated topos is the *classifying topos* for H.

A presheaf F on this site is a sheaf if all maps

$$F(\sqcup H_i/N_i) \to \prod F(H_i)^{N_i}$$

are isomorphisms.

A *generalized Galois cohomology theory* is a cohomology theory represented by a presheaf of spectra E on such a site.

Hypercohomology

$D =$ an unbounded presheaf of chain complexes on $\mathcal{C}$, with associated spectrum object $\Gamma(D)$. Let $j : \Gamma(D) \to F\Gamma(D)$ be a strictly fibrant model.

Then $F\Gamma(D)$ is stably fibrant, and there are isomorphisms

$$\pi_k F\Gamma(D)(U) \cong \mathbb{H}^{-k}(U, D), \;\; k \in \mathbb{Z}, U \in \mathcal{C}.$$

$\mathbb{H}^*(U, D)$ is *hypercohomology*.

If E is a presheaf of spectra with stably fibrant model $j : E \to GE$, then one often sees (Thomason)

$$\mathbb{H}^k(\mathcal{C}, E) := \pi_{-k}(\Gamma_* GE).$$

Here, $\Gamma_* GE = \varprojlim_{U \in \mathcal{C}} \; GE(U)$ is global sections of GE.

Algebraic K-theory

The category $Sch|_k$ of k-schemes can be endowed with the geometric topologies that you like: Zariski, flat, étale, Nisnevich.

There is a presheaf of spectra $K : (Sch|_k)^{op} \to \mathbf{Spt}$ such that $K(U)$ is a model for the algebraic K-theory spectrum of the k-scheme U.

Suppose that $\ell \neq char(k)$ is a prime. Form the cofibre sequence

$$K \xrightarrow{\times \ell} K \to K/\ell$$

$K_p(U, \mathbb{Z}/\ell) := \pi_p K/\ell(U)$ defines the mod ℓ K-theory of U.

The K-theory presheaf and its relatives have stably fibrant models for each of the topologies that we know, eg $j : K \to K_{Zar}$, $j : K \to K_{Nis}$, and $j : K/\ell \to (K/\ell)_{et}$ define Zariski, Nisnevich and étale K-theory.

Descent

If $j : E \to GE$ is a stably fibrant model for a presheaf of spectra E, E *satisfies descent* for U if $E(U) \to GE(U)$ is a stable equivalence.

Fact: All stably fibrant presheaves of spectra Z satisfy descent for all objects U.

1) The map $K(S) \to K_{Zar}(S)$ is a stable equivalence if S is smooth (and Noetherian) over k (Brown-Gersten). Algebraic K-theory satisfies Zariski descent for smooth schemes S.

2) The map $K(S) \to K_{Nis}(S)$ is a stable equivalence if S is smooth (Morel-Voevodsky). Algebraic K-theory satisfies Nisnevich descent for smooth schemes.

3) The original descent theorem of Nisnevich says that K/ℓ satisfies Nisnevich descent (*cdh*-descent) for smooth schemes.

4) Suppose $\ell \neq 2$. The Lichtenbaum-Quillen conjecture says that

$$K_s(k, \mathbb{Z}/\ell) = \pi_s K/\ell(k) \to \pi_s(K/\ell)_{et}(k) =: K_s^{et}(k, \mathbb{Z}/\ell)$$

is an isomorphism for $s \geq d-1$, where d is the Galois cohomological dimension of k with respect to ℓ-torsion sheaves.

4) For more general (ie. non-smooth) schemes S, the Nisnevich K-theory spectrum $K_{Nis}(S)$ has all of the attributes of Thomason-Trobaugh K-theory except for the geometric construction (perfect complexes). The Nisnevich K-groups

$$\pi_p K_{Nis}(S) =: K_p^{Nis}(S)$$

coincide with the Thomason-Trobaugh groups $\pi_p K^{TT}(S)$ for $p \geq 0$.

Descent spectral sequence

Suppose that E is a presheaf of connective spectra. E has a Postnikov tower, which tower has a stably fibrant model

$$
\begin{array}{ccccc}
 & & P_2E & \xrightarrow{j} & GP_2E \\
 & \nearrow & \downarrow & & \downarrow p \\
 & \nearrow & P_1E & \xrightarrow{j} & GP_1E \\
 & & \downarrow & & \downarrow p \\
E & \longrightarrow & P_0E & \xrightarrow[j]{} & GP_0E
\end{array}
$$

There are fibre sequences

$$GH(\pi_nE)[-n](S) \to GP_nE(S) \xrightarrow{p} GP_{n-1}E(S)$$

in sections, and reindexing a Bousfield-Kan spectral sequence gives

$$E_2^{s,t} = H^s(S, \tilde{\pi}_tE) \text{ “}\Rightarrow\text{” } \pi_{t-s}GE(S) = \mathbb{H}^{s-t}(S, E).$$

Elephants in the room

Presheaves of symmetric spectra

Derived categories, symmetric spectrum objects

Elliptic cohomology theories, tmf

Derived schemes

References

J. F. Jardine.
Generalized Étale Cohomology Theories.
Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2010.
Reprint of the 1997 edition.

J.F. Jardine.
Local Homotopy Theory.
Springer Monographs in Mathematics. Springer-Verlag, New York, 2015.