The discrete cohomology of algebraic groups

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$k =$ algebraically closed field,
ℓ prime: $(\ell, char(k)) = 1$

Here's a list of polynomial rings:

$k[t_0, t_1, \ldots, t_n]/(\sum t_i - 1) \cong k[t_1, \ldots, t_n], n \geq 0.$

Simplicial structure:

$$d_i(x_i) = 0, \;\; s_j(x_j) = x_j + x_{j+1}$$

Simplicial k-algebra: k_*

Cosimplicial affine variety: $\mathbb{A}^*$

$$\mathbb{A}^n = Sp(k[t_0, t_1, \ldots, t_n]/(\sum t_i - 1)), n \geq 0.$$

Question (Snaith): $G =$algebraic group. Can $BG(k_*)$ be used to calculate the cohomology $H^*(BG(k), \mathbb{Z}/\ell)$ of the discrete group $G(k)$?

$H^*(BG(k), \mathbb{Z}/\ell)$ is "discrete cohomology" of algebraic group G.

Example: $G = Gl_n(k) = (n \times n)$ invertible matrices with entries in k.

$$H^*(BGl_n(k), \mathbb{Z}/\ell) = ? \text{ (still unknown)}$$

Obvious thing to try: inclusion of vertices k in simplicial algebra k_* induces map of simplicial sets

$$BGl_n(k) \to BGl_n(k_*)$$

Question: Is the induced map

$$H^*(BGl_n(k_*), \mathbb{Z}/\ell) \to H^*(BGl_n(k), \mathbb{Z}/\ell)$$

an isomorphism?

Problem: don't know enough about $BG(k_*)$ or $G(k_*)$.

Facts:

1) $\pi_0 G(k_*) = 0$ if and only if $G(k)$ is generated by unipotent elements

2) $\pi_1 G(k_*) \cong K_2(k)$ for all but a finite list of simple algebraic groups.

$E_{i,j}(a)$ elementary transformation matrix in $Sl_n(k)$: there is an algebraic path $\mathbb{A}^1 \to Sl_n$ (1-simplex in $Sl_n(k_*)$) defined by

$$t \mapsto E_{i,j}(ta)$$

This is a path from $E_{i,j}(a)$ to the identity I.

Isomorphism conjectures

These relate discrete cohomology $H^*(BG(k), \mathbb{Z}/\ell)$ to étale cohomology $H^*_{et}(BG, \mathbb{Z}/\ell)$ for the simplicial scheme BG.

H = group: $BH_n = H^{\times n}$ or strings of composable arrows of length n, with faces defined by composition (product) and degeneracies defined by insertion of identities.

G = algebraic group: $BG_n = G^{\times n}$ (affine variety) with faces and degeneracies defined analogously.

Any set X determines a k-variety $\Gamma^* X = \bigsqcup_X Sp(k)$, and the set of k-points $Y(k)$ of a k-scheme Y determine a canonical map

$$\epsilon : \Gamma^* Y(k) = \bigsqcup_{Y(k)} Sp(k) \to Y.$$

There are canonical maps

$$\epsilon : \Gamma^* BG(k) \to BG.$$

There is an isomorphism

$$H^*_{et}(\Gamma^* BG(k), \mathbb{Z}/\ell) \cong H^*(BG(k), \mathbb{Z}/\ell)$$

$\epsilon : \Gamma^* BG(k) \to BG$ induces a comparison map

$$\epsilon^* : H^*_{et}(BG, \mathbb{Z}/\ell) \to H^*(BG(k), \mathbb{Z}/\ell)$$

Quillen (1974): $H^*(BGl(k), \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_1, c_2, \ldots]$

Milnor (1983) "Isomorphism conjecture": ϵ^* is an isomorphism when $k = \mathbb{C}$, G is reductive group defined over $\mathbb{C}$

Friedlander (1984) "Generalized isomorphism conjecture" ϵ^* is an isomorphism for all reductive groups G defined over all algebraically closed fields k.

Why?

$H^*_{et}(BG, \mathbb{Z}/\ell)$ is computable by base change results (Friedlander-Parshall) and comparison with cohomology of topological group $G(\mathbb{C})$.

Examples:

$H^*_{et}(BGl_n, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_1, \ldots, c_n]$

$H^*_{et}(BGl, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_1, \ldots]$

$H^*_{et}(BO_n, \mathbb{Z}/2) \cong \mathbb{Z}/2[HW_1, \ldots, HW_n]$

Outcomes:

1) Any inclusion $k \subset L$ of alg. closed fields would induce isomorphism

$$H^*(BG(k), \mathbb{Z}/\ell) \cong H^*(BG(L), \mathbb{Z}/\ell).$$

2) $K_*(k, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\beta]$ (Suslin)

Sheaf theory

Any k-variety X represents a sheaf $X = \hom(\ \ , X)$ defined on the smooth étale site $(Sm|_k)_{et}$.

The simplicial scheme BG represents a simplicial sheaf BG for any algebraic group G.

Simplicial sheaves (resp. simplicial presheaves) have homotopy theory: $X \to Y$ is a (local) weak equivalence iff all $X_x \to Y_x$ is weak equiv. of simplicial sets for all $x \in U/k$

Simplicial sheaves X have homology sheaves $\tilde{H}_n(X, \mathbb{Z}/\ell)$.

Fact: $H^n_{et}(X, A) \cong [X, K(A, n)]$.

1) $f : X \to Y$ local w.e. then $\tilde{H}_*(X) \cong \tilde{H}_*(Y)$.

2) f $\tilde{H}_*$-iso then $H^*_{et}(Y, \mathbb{Z}/\ell) \cong H^*_{et}(X, \mathbb{Z}/\ell)$

Rigidity program

There is a canonical map $\epsilon : \Gamma^* BG(k) \to BG$; $\Gamma^* BG(k)$ is constant simplicial sheaf.

$H^*_{et}(\Gamma^* BG(k), \mathbb{Z}/\ell) \cong H^*(BG(k), \mathbb{Z}/\ell)$

Show that ϵ induces

$$\tilde{H}_*(\Gamma^* BG(k), \mathbb{Z}/\ell) \cong \tilde{H}_*(BG, \mathbb{Z}/\ell)$$

ie. show that $\tilde{H}_*$ sheaves for BG are constant.

Stalkwise version: show that ϵ induces

$$H_*(BG(k), \mathbb{Z}/\ell) \cong H_*(BG(\mathcal{O}_x^{sh}), \mathbb{Z}/\ell), \;\; x \in U/k$$

Gabber rigidity theorem: $x \in U/k$ closed

$$H_*(BGl(k), \mathbb{Z}/\ell) \cong H_*(BGl(\mathcal{O}_x^h), \mathbb{Z}/\ell)$$

Cor: $H^*(BGl(k), \mathbb{Z}/\ell) \cong H^*_{et}(BGl, \mathbb{Z}/\ell)$.

Question: Are the étale homology sheaves $\tilde{H}_*(BG, \mathbb{Z}/\ell)$ constant for an arbitrary reductive group G?

Examples: Gl, Sl, $O = \varinjlim_n O_n$, Sp.

Conjecture also known for any torus, any additive group, any solvable group.

Non-example: (Friedlander-Mislin)

$k = \overline{\mathbb{F}}_p$ where $(p, \ell) = 1$. Then

$$H^*_{et}(BG, \mathbb{Z}/\ell) \cong H^*(BG(k), \mathbb{Z}/\ell)$$

Proof uses Lang isomorphism: $G/G(\mathbb{F}_q) \cong G$. There has never been a rigidity argument for this result.

What's been tried:

1) Stability results:

The map $H_p(BGl_n(k),\mathbb{Z}/\ell) \to H_p(BGl_{n+1}(k),\mathbb{Z}/\ell)$ is an isomorphism for $p \leq n$.

$H^p(BGl_n(k),\mathbb{Z}/\ell) \cong H^p(BGl(k),\mathbb{Z}/\ell)$ for $p \leq n$.

$H^p_{et}(BGl_n,\mathbb{Z}/\ell) \cong H^p_{et}(BGl,\mathbb{Z}/\ell)$ for $p \leq 2n$.

$H^p(BGl_n(k),\mathbb{Z}/\ell) \cong H^p_{et}(BGl_n,\mathbb{Z}/\ell)$ for $p \leq n$.

There are corresponding stablity results for Sl_n, O_n, Sp_n.

2) Show that the presheaf of spectra

$$H(G, \ell) = H(\mathbb{Z}/\ell) \wedge BG_+$$

is constant up to local stable equivalence for the étale topology.

Suslin-Voevodsky rigidity theorem

$\mathcal{F}$ = presheaf of ℓ-torsion abelian groups on $Sch|_k$ which admits transfers for finite surjective morphisms and satisfies $\mathcal{F}(U\times\mathbb{A}^1) \cong \mathcal{F}(U)$. If $x \in X$ is a closed point on a smooth k-scheme X, then $\mathcal{F}(\mathcal{O}_x^h) \cong \mathcal{F}(k)$.

Example: $\mathcal{F} = K_*(\ ,\mathbb{Z}/\ell)$

Need to show that étale sheaves $\pi_* H(G, \ell)$ admit transfers and $H(G, \ell)(U) \simeq H(G, \ell)(U \times \mathbb{A}^1)$ for all smooth U/k.

Transfers

An abelian presheaf $\mathcal{F}$ admits transfers if for all finite surjective $p : X \to S$ with X reduced, irreducible, S irreducible and normal there is homomorphism

$$Tr_{X/S} : \mathcal{F}(X) \to \mathcal{F}(S)$$

such that

1) If p is isomorphism then $Tr_{X/S} \cdot p^* = 1$

2) Given closed irreducible regular $V \subset S$, if W_i are components of $p^{-1}(V)$ with "multiplicities" n_i then the following commutes:

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "Tr_{X/S}"] \arrow[d] & \mathcal{F}(S) \arrow[d] \\ \bigoplus \mathcal{F}(W_i) \arrow[r, "\sum n_i Tr_{W_i/V}"'] & \mathcal{F}(V) \end{tikzcd}$$

Cycles:

S normal integral: $z_0^c(X)(S) = C_0(X \times S/S)$ is free abelian group generated by closed integral subschemes $Z \subset X \times S$ such that the composite $Z \subset X \times S \to S$ is finite and surjective.

If $S' \to S$ is map of normal schemes and $S' \times_S Z$ is a union of irreducible subsets Z_i with multiplicities m_i then $Z \mapsto \sum m_i Z_i$ defines $z_0^c X(S) \to z_0^c X(S')$.

$\mathbb{Z}_{qfh}(X)$ is the free abelian sheaf on X for the qfh topology: coverings are quasi-finite maps which are universally surjective, eg. finite surjective morphisms.

Any abelian sheaf for qfh topology has transfers.

There are morphisms $z_0^c(X)(S) \to \mathbb{Z}_{qfh}(X)(S)$:

Z maps to image of $p : X \times S \to X$ under composite

$$\mathbb{Z}_{qfh}(X)(X \times S) \to \mathbb{Z}_{qfh}(X)(Z) \xrightarrow{Tr_{Z/S}} \mathbb{Z}_{qfh}(X)(S)$$

This defines a qfh sheaf isomorphism $z_0^c(X) \cong \mathbb{Z}_{qfh}(X)$ (up to inverting $char(k)$) with inverse defined by graphs.

X smooth k-variety:

$\mathbb{Z}_{qfh}(X) \otimes \mathbb{Z}/\ell$ is a presheaf with transfers.

$S_*\mathbb{Z}_{qfh}(X) \otimes \mathbb{Z}/\ell$ is a simplicial abelian presheaf with transfers

$$S_n\mathbb{Z}_{qfh}(X) \otimes \mathbb{Z}/\ell(U) = \mathbb{Z}_{qfh}(X) \otimes \mathbb{Z}/\ell(U \times \mathbb{A}^n)$$

$H_nS_*\mathbb{Z}_{qfh}(X) \otimes \mathbb{Z}/\ell$ is a presheaf with transfers that satisfies the homotopy property, so is constant for the étale topology.

$\Gamma^*(\mathbb{Z}_{qfh}(X)(\mathbb{A}^*) \otimes \mathbb{Z}/\ell) \to S_*\mathbb{Z}_{qfh}(X) \otimes \mathbb{Z}/\ell$ is a qfh homology sheaf isomorphism.

$H^*_{qfh}(X, \mathbb{Z}/\ell) \cong H^*_{et}(X, \mathbb{Z}/\ell)$

$\mathbb{Z}_{qfh}(X) \to S_*\mathbb{Z}_{qfh}(X)$ induces isomorphism in $H^*_{qfh}(\ ,\mathbb{Z}/\ell)$

Consequences:

$$H^p \operatorname{hom}(\mathbb{Z}_{qfh}(X)(\mathbb{A}^*), \mathbb{Z}/\ell) \cong H^p_{et}(X, \mathbb{Z}/\ell).$$

$$H^p \operatorname{hom}(\mathbb{Z}_{qfh}(BG)(\mathbb{A}^*), \mathbb{Z}/\ell) \cong H^p_{et}(BG, \mathbb{Z}/\ell).$$

and

$$H^p_{et}(BG, \mathbb{Z}/\ell) \cong H^p_{qfh}(BG, \mathbb{Z}/\ell).$$

2 formulations:

1) Show that $\mathbb{Z}(BG)(k) \to \mathbb{Z}_{qfh}(BG)(\mathbb{A}^*)$ induces ordinary $H^*(\ ,\mathbb{Z}/\ell)$ isomorphism.

In fact, $\mathbb{Z}_{qfh}(BG)(k) \cong \mathbb{Z}(BG)(k)$ since $P(k) \cong \tilde{P}(k)$, so there is a spectral sequence (Knudson-Walker, 2004)

$$E_2^{p,q} \Rightarrow H_{et}^{p+q}(BG, \mathbb{Z}/\ell)$$

with

$$E_1^{p,0} = H^p(BG(k), \mathbb{Z}/\ell).$$

NB: $\mathbb{Z}(BG)(\mathbb{A}^*) = \mathbb{Z}BG(k_*)$, in original notation.

2) Show that $\epsilon : \Gamma^* BG(k) \to BG$ induces an isomorphism in mod ℓ qfh homology sheaves.

Enough to show that the qfh homology sheaves $\tilde{H}_*(BG, \mathbb{Z}/\ell)$ have the homotopy property — they already admit transfers.

Downside: qfh topology is weird.

What's been tried (cont.)

3) Ultraproducts:

It's enough to show that for every alg. closed field k there is an alg. closed field L such that $k \subset L$ and such that the comparison map

$$H^*_{et}(BG, \mathbb{Z}/\ell) \to H^*(BG(L), \mathbb{Z}/\ell)$$

is an isomorphism (finite dimension argument).

In any characteristic $\neq \ell$, can find fields L of arbitrarily large cardinality with $L =$ ultraproduct of fields $\overline{\mathbb{F}}_p$ involving "all" primes p.

We know that $H^*_{et}(BG, \mathbb{Z}/\ell) \cong H^*(BG(\overline{\mathbb{F}}_p), \mathbb{Z}/\ell)$ for "all" p, by Friedlander-Mislin.

Ultraproduct construction applies to simplicial sets: ultraproducts are stalks of certain direct image sheaves on $Sp(\prod_{infinite}$ fields)

Show that ultraproduct construction commutes with homology functor in good cases.