Stack cohomology J.F. Jardine

Grothendieck sites

A **Grothendieck site** is a small category $\mathcal{C}$ equipped with a **Grothendieck topology**. The topology can be viewed as a collection of covering families $V_i \to U$ (sets of morphisms) satisfying a short list of axioms:

1) covering families are closed under "pullback"

2) if all pullbacks of a family of maps S along a covering family R are covering, then S is covering.
3) $1_U : U \to U$ is covering.

It follows that every isomorphism is covering. If $R \subset S$ and R is covering, then S is covering.

Examples:

1) the category op_T of open subsets of a topological space T, with inclusions.

2) $S =$ a scheme and $\mathrm{Sch}\,|_S$ is the category of S-schemes $U \to S$ which are locally of finite type.
a) The **flat topology** on $\mathrm{Sch}\,|_S$ is generated by families $\phi_i : V_i \to U$ of flat morphisms (of S-schemes) such that $\cup_i \phi_i(V_i) = U$ (ie. the family is faithfully flat). The site is denoted by $(\mathrm{Sch}\,|_S)_{fl}$

b) The **étale topology** on $\mathrm{Sch}\,|_S$ is generated by families $\phi_i : V_i \to U$ of étale morphisms such that $\cup\phi_i(V_i) = U$. Every étale morphism is flat, so every étale covering family is a flat covering family. The site is the (big) étale site $(\mathrm{Sch}\,|_S)_{et}$.

c) The **Zariski topology** on $\mathrm{Sch}\,|_S$ is generated by open coverings $V_i \subset U$ in the usual sense. The site is the big Zariski site $(\mathrm{Sch}\,|_S)_{Zar}$.

I could go on and on: there are many flavours of geometric topologies on $\mathrm{Sch}\,|_S$. The Nisnevich topology is a particular favourite.

3) Every small category I has a trivial topology, which is effectively no topology at all. The covering families are precisely the isomorphisms. This is sometimes called the **chaotic topology** on I.

A **presheaf** F on a site $\mathcal{C}$ is a (contravariant) functor $F : \mathcal{C}^{op} \to \mathbf{Set}$.

F is just a presheaf of sets. One talks about presheaves taking values in anything by tacking on appropriate adjectival cluster, eg. a presheaf of groups is a contravariant functor taking values in groups, a presheaf of groupoids H is a contravariant functor taking valuses in groupoids, a simplicial presheaf takes values in simplicial sets, a presheaf of (symmetric) spectra takes values in (symmetric) spectra.

A presheaf F is said to be a **sheaf** if every covering family $V_i \to U$ determines an isomorphism

$$F(U) \cong \varprojlim_i F(V_i).$$

There are sheaves of groups, sheaves of groupoids, simplicial sheaves, and so on. There is also an associated sheaf functor $F \mapsto \tilde{F}$ which is left adjoint to the inclusion of sheaves in presheaves — this functor is defined by putting in all the required limits twice. The associated sheaf functor also preserves finite limits.

Examples

0) Write $U = \hom(\ ,U)$ for the presheaf represented by an object $U \in \mathcal{C}$.

1) If F is a presheaf on the site op_T of open subsets of a topological space T (or on the Zariski site of a scheme S), then F is a sheaf if and only if the diagram

$$F(U) \to \prod_i F(V_i) \rightrightarrows \prod_{i,j} F(V_i \cap V_j)$$

is a coequalizer, for every open covering $V_i \subset U$.

2) Presheaves and sheaves coincide for the chaotic topology.

3) The chaotic topology is the only topology on the one object, one morphism category $*$. The sheaf category for this site is the category **Set** of sets.

4) Suppose that K is a field and consider the category $fet|_K$ of finite étale maps $X \to \mathrm{Sp}(K)$. Then $X = \sqcup_{i=1}^n \mathrm{Sp}(L_i)$ where L_i/K is a finite separable extension field. The category $fet|_K$ has an étale topology and a Nisnevich topology. Sheaves for the étale topology on $fet|_K$ can be identified with discrete modules over the absolute Galois group of K, and this is the home of all Galois cohomology theory. A presheaf F on $fet|_K$ is a sheaf for the Nisnevich topology if and only if all canonical maps

$$F(\bigsqcup_i \mathrm{Sp}(L_i)) \to \prod_i F(L_i)$$

are isomorphisms ("additivity"). This is why Nisnevich called his topology the "completely decomposed topology".

Suppose now that $\mathcal{C}$ is your favourite site. A **presheaf of categories** A on $\mathcal{C}$ is a functor $A : \mathcal{C}^{op} \to \mathbf{cat}$ taking values in small categories. In other words A assigns to each $U \in \mathcal{C}$ a small category $A(U)$ and to each morphism $\phi : V \to U$ a functor $\phi^* : A(U) \to A(V)$ such that identities go to identity functors and composition go to composition of functors. This is a "strict" object — it's rigidly functorial.

The topology on $\mathcal{C}$ determines a Grothendieck topology on the Grothendieck construction $\mathcal{C}/A$. This is called the **site fibred over the presheaf of categories** A.

The objects of $\mathcal{C}/A$ are pairs (U, x) where x is an object of $A(U)$. Objects can be identified with presheaf morphisms $x : U \to \mathrm{Ob}(A)$, hence the notation $\mathcal{C}/A$.

The morphisms $(V, y) \to (U, x)$ of $\mathcal{C}/A$ are pairs (α, f) consisting of a morphism $\alpha : V \to U$ of $\mathcal{C}$ and a morphism $f : y \to \alpha^*(x)$ of $A(V)$. The composition of the morphisms

$$(W, z) \xrightarrow{(\gamma, g)} (V, y) \xrightarrow{(\alpha, f)} (U, x)$$

is the pair $(\alpha\gamma, \gamma^*(f)g)$, where $\gamma^*(f)g$ is the composite

$$z \xrightarrow{g} \gamma^*(y) \xrightarrow{\gamma^*(f)} \gamma^*\alpha^*(x) = (\alpha\gamma)^*(x).$$

Every morphism $\alpha : V \to U$ and $x : U \to \mathrm{Ob}(\mathcal{C})$

together determine a morphism $(\alpha, 1) : (V, \alpha^*(x)) \to (U, x)$ in $\mathcal{C}/U$. The covering families of the topology for $\mathcal{C}/U$ are generated by the families of maps

$$(\alpha_i, 1) : (V_i, \alpha_i^*(x)) \to (U, x)$$

associated to covering families $\alpha_i : V_i \to U$ in $\mathcal{C}$.

Examples

1) A **stack** on $\mathcal{C}$ is "traditionally" a sheaf of groupoids G on $\mathcal{C}$ which satisfies an effective descent condition. The site $\mathcal{C}/G$ fibred over G is the Grothendieck site which is used to define stack cohomology: if F is an abelian sheaf on $\mathcal{C}/G$ then the cohomology $H^n(G, F)$ of G with coefficients in F is the sheaf cohomology group $H^n(\mathcal{C}/G, F)$.

2) Suppose that $Y : I \to \mathrm{Pre}(\mathcal{C})$ is a diagram taking values in presheaves on $\mathcal{C}$ (to fix ideas, you can suppose that Y is represented by a simplicial scheme). In particular, Y consists of set-valued functors

$$Y(U) : I \to \mathrm{Pre}(\mathcal{C}) \xrightarrow{U\text{-sections}} \mathbf{Set}$$

which fit together in the obvious way. Each $Y(U)$ has a translation category (homotopy colimit) $E_I Y(U)$ and assembling them gives a presheaf of categories $E_I Y$.

The objects of $\mathcal{C}/E_I Y$ are the presheaf morphisms $x : U \to Y_i$ and the morphisms are the commutative diagrams

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & U \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle x} \\ Y_j & \xrightarrow[\theta_*]{} & Y_i \end{array}$$

The covering families of $\mathcal{C}/E_I Y$ are generated by families

$$\begin{array}{ccc} V_i & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle x} \\ Y_i & \xrightarrow[1]{} & Y_i \end{array}$$

induced by covering familes $V_i \to U$ of $\mathcal{C}$. This is a slight generalization of the usual description of a site fibred over a diagram of sheaves (eg. site fibred over a simplicial scheme) — it's a ubiquitous construction.

3) Even the special case of a site $\mathcal{C}/X$ fibred over a presheaf ($I = *$, $X : * \to \mathrm{Pre}(\mathcal{C})$) has content. If $V = \hom(\ ,V)$ is the presheaf represented by $V \in \mathcal{C}$, then $\mathcal{C}/V$ is the usual site of objects over V. $\mathcal{C}/Y$ for a sheaf Y is a standard construction in topos theory.

There is a **model structure** on sheaves of groupoids (Joyal-Tierney), for which a morphism $G \to H$ is a weak equivalence (resp. fibration) if and only if the induced map $BG \to BH$ is a local (or stalkwise) weak equivalence (respectively a global fibration). A sheaf of groupoids G is a stack if and only if it is sectionwise equivalent to its fibrant model. Thus stacks can be identified with homotopy types, even of presheaves of groupoids (Hollander).

The overall point of this talk is that sheaf (and generalized sheaf) cohomology of a site fibred over a presheaf of groupoids G is an invariant of the homotopy type of G. In particular, all cohomology theories arising from a stack H can be computed on the site $\mathcal{C}/G$ fibred over a presheaf of groupoids G which is locally weakly equivalent to H.

Example: generalized cohomology theories for the stack of formal group laws (for the flat topology) can be computed on the site arising from an explicit representing object.

How do you see this?

A) Suppose that B is a presheaf of categories. A B**-diagram** in simplicial presheaves consists of a simplicial presheaf map $\pi_X : X \to \mathrm{Ob}(B)$ together with an action ("multiplication")

$$\begin{array}{ccc} X \times_s \mathrm{Mor}(B) & \xrightarrow{m} & X \\ \downarrow & & \downarrow \pi_X \\ \mathrm{Mor}(B) & \xrightarrow[t]{} & \mathrm{Ob}(B) \end{array}$$

by the morphisms $\mathrm{Mor}(B)$ of B which respects composition and identities in the obvious way. A morphism of B-diagrams is a morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle \pi_X} \searrow \quad \swarrow {\scriptstyle \pi_Y} & \\ & \mathrm{Ob}(B) & \end{array}$$

of simplicial presheaves fibred over $\mathrm{Ob}(B)$ which respects multiplication by morphisms. Write $s\,\mathrm{Pre}(\mathcal{C})^B$ for the corresponding category.

Every simplicial presheaf $Y : (x, U) \mapsto Y(x, U)$ on the fibred site $\mathcal{C}/A$ determines an A^{op}-diagram

$$\bigsqcup_{x \in \mathrm{Ob}(A)(U)} Y(x, U) \to \mathrm{Ob}(A)(U).$$

This assignment determines an equivalence of categories

$$s\,\mathrm{Pre}(\mathcal{C}/A) \simeq s\,\mathrm{Pre}(\mathcal{C})^{A^{op}}.$$

The various model structures for simplicial presheaves on $\mathcal{C}/A$ therefore induce model structures for A^{op}-diagrams in simplicial presheaves.

B) Write

$$O(Y)(U) = \bigsqcup_{x \in \mathrm{Ob}(A)(U)} Y(x, U).$$

Then one can show that the functor $Y \mapsto O(Y)$ preserves and reflects cofibrations and local weak equivalences. It follows that, for the model structure on A^{op}-diagrams induced from the standard (injective) local model structure on $s\,\mathrm{Pre}(\mathcal{C}/A)$, a map

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle \pi_X} \searrow \quad \swarrow {\scriptstyle \pi_Y} & \\ & \mathrm{Ob}(A) & \end{array}$$

of A^{op}-diagrams is a weak equivalence (respectively cofibration) if and only if the simplicial presheaf map $f : X \to Y$ is a local weak equivalence (respectively monomorphism). This is the "injective" model structure for A^{op}-diagrams.

There is a "projective" model structure for A^{op}-diagrams for which a map f as above is a fibration if and only if the simplicial presheaf map $f : X \to Y$ is a global fibration. The projective model structure has the same weak equivalences as the injective structure.

To see this, you need to know the following:

Lemma: Suppose that T is a presheaf, and that $Z \to T$ is a map of simplicial presheaves. Then the functor sending $X \to T$ to $Z \times_T X$ preserves weak equivalences of $s\operatorname{Pre}(\mathcal{C})/T$.

Remark: There are projective and injective model structures for I-diagrams of simplicial sets which you are used to. In particular $p : X \to Y$ is a projective fibration of I-diagrams if and only if all maps $X_i \to Y_i$ are Kan fibrations. This is equivalent to the assertion that the induced map

$$\bigsqcup_{i \in \operatorname{Ob}(I)} X_i \to \bigsqcup_{i \in \operatorname{Ob}(I)} Y_i$$

is a fibration of $\mathbf{S}/\operatorname{Ob}(I)$. The injective model structure for I-diagrams has cofibrations defined pointwise, and a map $A \to B$ is an injective cofibration if and only if the map

$$\bigsqcup_{i \in \operatorname{Ob}(I)} A_i \to \bigsqcup_{i \in \operatorname{Ob}(I)} B_i$$

is a cofibration of $\mathbf{S}/\operatorname{Ob}(I)$. The model structures that we are using in the enriched context here are direct analogues of these.

C) A^{op}-diagrams $Y \to \mathrm{Ob}(A)$ have homotopy colimits

$$\underrightarrow{\mathrm{holim}}_{A^{op}} Y \to BA^{op},$$

(this is the usual construction sectionwise) and the homotopy colimit construction preserves weak equivalences. [You need the Lemma above for the last statement, and in many other places.]

D) Now we know a thing or two about ordinary groupoids G:

1) Suppose $X \to BG$ is a map of simplicial sets, and form the G-diagram $a \mapsto \mathrm{pb}(X)_a$ via the pullback diagrams

$$\begin{array}{ccc} \mathrm{pb}(X)_a & \longrightarrow & X \\ \downarrow & & \downarrow \\ B(G/a) & \longrightarrow & BG \end{array}$$

Then the canonical map

$$\underrightarrow{\mathrm{holim}}_{a}\, \mathrm{pb}(X)_a \to X$$

is a weak equivalence, and there is a natural weak equivalence

$$\mathrm{pb}\, \underrightarrow{\mathrm{holim}}_{a} Y_a \to Y$$

for all G-diagrams Y.

2) The pullback diagram

$$\begin{array}{ccc} \sqcup_{a \in \mathrm{Ob}(G)} Y_a & \longrightarrow & \underrightarrow{\mathrm{holim}}_G Y \\ \downarrow & & \downarrow \\ \mathrm{Ob}(G) & \longrightarrow & BG \end{array}$$

is homotopy cartesian for all G-diagrams Y (Quillen's Theorem B).

It follows that the homotopy colimit functor and the pullback functor (which is left adjoint to homotopy colimit) together determine an equivalence of categories

$$\mathrm{Ho}(\mathbf{S}^G) \simeq \mathrm{Ho}(\mathbf{S}/BG).$$

One can use these results to show

Lemma: There is an equivalence of categories

$$\mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C})^H) \simeq \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C})/BH)$$

for any presheaf of groupoids H on a Grothendieck site $\mathcal{C}$.

Remark: The functors pb and $\underrightarrow{\mathrm{holim}}$ do not form a Quillen equivalence. The homotopy colimit functor has a right adjoint which does determine a Quillen equivalence for the projective structure on $s\,\mathrm{Pre}(\mathcal{C})^H$.

E) There is a natural equivalence $BH^{op} \simeq BH$ for any presheaf of groupoids (really categories) H, through the bisimplicial object with bismplices

$$x_m \to \cdots \to x_0 \to y_0 \to \cdots \to y_n.$$

F) There are corresponding equivalences

$$\mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C})/BH) \simeq \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C})/BH^{op}).$$

Explicitly, if $f : X \to Y$ is a local weak equivalence of simplicial presheaves, then the pullback functor (along f)

$$f_* : s\,\mathrm{Pre}(\mathcal{C})/Y \to s\,\mathrm{Pre}(\mathcal{C})/X$$

and its left adjoint form a Quillen equivalence for the respective "standard" model structures.

Say that a map $G \to H$ is a **weak equivalence** of presheaves of groupoids if it induces a local weak equivalence $BG \to BH$ of simplicial presheaves.

Theorem: Every weak equivalence $f : G \to H$ of presheaves of groupoids induces a Quillen equivalence between the projective model structures on the categories of simplicial presheaves $s\,\mathrm{Pre}(\mathcal{C}/G)$ and $s\,\mathrm{Pre}(\mathcal{C}/H)$, and hence induces an equivalence of homotopy categories

$$\mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/G)) \simeq \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/H)).$$

The functors involved in the Quillen equivalence are restriction along $f : G \to H$ and its left Kan extension.

The Theorem is a homotopy invariance property. Since it exists on the simplicial presheaf level, you expect similar statements for all related theories, such as pointed simplicial presheaves, presheaves of spectra and presheaves of symmetric spectra. I have proved corresponding results in all of these contexts, and I expect many more.