Stacks and homotopy theory

Rick Jardine

Univ. of Western Ontario

February 8, 2017

The Borel construction

First appeared during seminar at IAS 1958-59: [1], 1960.

$G \times M \to M$: Lie group G, manifold M. $EG =$ contractible space with free G-action, then

$$EG \times_G M := (EG \times M)/G \ \ \text{(diagonal action)}$$

defines the **Borel construction** for the G-space M ($= M_G$ in [1]).

The G-equiv. maps $M \to *$, $EG \to *$ a standard natural picture

$$\begin{tikzcd} M \arrow[r] & EG \times_G M \arrow[r, "\pi"] \arrow[d, "p"] & BG = EG/G \\ & M/G & \end{tikzcd}$$

Horizontal row is fibre sequence. p may not be a homotopy equiv. $EG \times_G M$ is the space of **homotopy coinvariants**.

Translation groupoid

Suppose $H \times F \to F$ is action of a discrete group H on a set F.

Swan (1982): the **translation groupoid** $E_H F$ has objects $x \in F$ and morphisms $g : x \to g \cdot x$. All morphisms are invertible.

Each category C has a **nerve** BC. BC is a simplicial set with n-simplices BC_n consisting of strings of morphisms

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \to a_{n-1} \xrightarrow{f_n} a_n$$

Simplicial structure def. by composition and insertion of identities.

Example: $B(E_H F)_n = H^{\times n} \times F$ consists of strings

$$x_0 \xrightarrow{g_1} g_1 \cdot x_0 \xrightarrow{g_2} \dots \xrightarrow{g_n} (g_n \cdots g_1) \cdot x_0.$$

Homotopy properties

FACT: Every natural trans. of functors $f, g : C \to D$ induces a homotopy $BC \times \Delta^1 \to BD$.

The groupoid $E_H H$ corr. to $H \times H \to H$ has initial object e, with $e \xrightarrow{h} h$, so $B(E_H H) =: EH$ is contractible.

There is an isomorphism

$$B(E_H F) \cong EH \times_H F = (EH \times F)/F \ \text{(diagonal action)}$$

There is a natural picture

$$\begin{tikzcd} F \arrow[r] & EH \times_H F \arrow[r, "\pi"] \arrow[d, "p"] & BH \\ & F/H & \end{tikzcd}$$

in simplicial sets. Horiz. row is fibre sequence, and p may not be a weak equivalence.

More properties

1) $E_H F = \sqcup_{F_i \in F/H} E_H F_i$, and
2) $E_H F_i \simeq H_x$ as groupoids, $x \in F_i$ (homework)

so

$$EH \times_H F \simeq \sqcup_{[x] \in F/H} BH_x.$$

Moral: The map

$$p : EH \times_H F \simeq \sqcup_{[x] \in F/H} BH_x \to \sqcup_{[x] \in F/H} * = F/H$$

is a weak equivalence if and only if H acts freely on F.

The construction $EG \times_G X$ generalizes to simplicial groups G acting on simplicial sets X — captures the topological const.

Fact: $X \to Y$ G-equivariant weak equiv. Then $EG \times_G X \to EG \times_G Y$ is a weak equivalence (formal nonsense).

Remark: $X/G \to Y/G$ may not be a weak equivalence. Example: $EG \to *$.

EG is contractible and G acts freely on EG.

Apply the free abelian group functor $F \mapsto \mathbb{Z}(F)$ ...

$\mathbb{Z}(EG)$ is a simplicial abelian group with associated chain complex $\mathbb{Z}(EG)$, having boundaries

$$\mathbb{Z}(EG)_n \xrightarrow{\sum_{i=0}^{n} (-1)^i d_i} \mathbb{Z}(EG)_{n-1}.$$

Then $\mathbb{Z}(EG) \to \mathbb{Z}[0]$ is a homology isomorphism, so $\mathbb{Z}(EG)$ is a G-free resolution of the trivial G-module $\mathbb{Z}$.

Then

$$H_n(G, M) = \mathrm{Tor}_n(\mathbb{Z}, M) = H_n(\mathbb{Z}(EG) \otimes_G M).$$

$EG \times_G X$ is the non-abelian version of $\mathbb{Z}(EG) \otimes_G M$. (Cartan-Eilenberg, 1956; also Eilenberg-Mac Lane, 1946?).

$H_*(EG \times_G X, A)$ is one of the flavours of equivariant homology theory for a G-space X.

Principal bundles

The best (and first) examples of stacks are categories of principal G-bundles, in topology and geometry.

G sheaf of groups: a G-**bundle** (G-**torsor**) is a sheaf F with action $G \times F \to F$, which is free (principal) and locally transitive.

$G - \mathbf{tors}$ is the category of G-bundles and G-equiv maps, actually a groupoid — see below.

Basic definition: G = sheaf of groups

$$\pi_0(G - \mathbf{tors}) = \{\text{Iso classes of } G\text{-torsors}\} =: H^1(\mathcal{E}, G)$$

defines **non-abelian** H^1 with coeffs in G, where $\mathcal{E}$ is the underlying category of sheaves.

Repackaging the definition

The simplicial sheaf $EG \times_G F$ is defined in sections by

$$(EG \times_G F)(U) = EG(U) \times_{G(U)} F(U).$$

($G(U)$-action on set $F(U)$, eg. U open subset of top. space)

locally transitive: $\pi_0(EG \times_G F) = G/F$ has trivial associated sheaf.

free: all stabilizer subgroups $G(U)_x$ for $G(U) \times F(U) \to F(U)$ are trivial.

$G(U)_x = \pi_1(EG(U) \times_{G(U)} F(U), x)$ is trivial for all $x \in F(U)$ so $EG \times_G F \to F/G$ is (sectionwise) weak equivalence

Put these together: F is a G-**bundle** (G-**torsor**) iff $EG \times_G F \to *$ is a stalkwise (local) weak equivalence.

$* =$ one-pt (terminal) sheaf.

Examples

1) L/k finite Galois extension with Galois group G.

$EG \times_G Sp(L) \to *$ is a local weak equiv for the étale topology (looks like EG locally), so $Sp(L)$ is a G-bundle (G-torsor) for the étale topology on $\mathrm{Sp}(K)$.

2) $P =$ projective module on a ring R of rank n:

P is locally free of rank n (Zariski topology), or a vector bundle of rank n over $\mathrm{Sp}(R)$.

$\mathrm{Iso}(P_n)$ is the groupoid of isomorphisms of vector bundles of rank n over R.

$Gl_n(R)$ is the group of automorphisms of R^n. There is an isomorphism

$$\pi_0(Gl_n - \mathbf{tors}) \xrightarrow{\cong} \pi_0(\mathrm{Iso}(P_n)).$$

More examples

3) Suppose A is an $n \times n$ invertible symmetric matrix (non-deg sym bil form of rank n) over a field K ($char(K) \neq 2$).

For the étale topology, A is locally trivial: there is an invertible $n \times n$ matrix defined on L/K (finite Galois extension) such that $B^t AB = I_n$.

These things "are" the O_n-torsors for the etale topology on K. There is an isomorphism

$$H^1_{et}(K, O_n) \cong \{\text{iso. classes of non-deg symm. bil. forms}/K \text{ of rank } n\}.$$

O_n is the standard orthogonal group, the group of automorphisms of the trivial form of rank n.

Fact: Every morphism $F \to F'$ of G-torsors is an isomorphism.

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\simeq"'] & EG \times_G F \arrow[r, "\pi"] \arrow[d] & BG \arrow[d, "1"] \\ F' \arrow[r] & EG \times_G F' \arrow[r, "\pi"'] & BG \end{tikzcd}$$

F, F' are simplicial sheaves as well as just sheaves

$F \to F'$ stalkwise equivalence since "total spaces" are contractible, so is an isomorphism of sheaves.

Cocycles

A **cocycle** (from $*$ to BG) is a picture ("span")

$$* \xleftarrow{\simeq} U \to BG.$$

A morphism of cocycles is a picture

$$\begin{tikzcd} & U \arrow[dl, "\simeq"'] \arrow[dd] \arrow[dr] & \\ * & & BG \\ & U' \arrow[ul, "\simeq"] \arrow[ur] & \end{tikzcd}$$

The category of cocyles is $h(*, BG)$.

Examples: 1) standard cocycles $* \xleftarrow{\simeq} \check{C}(U) \to BG$ defined on Čech resolutions for coverings

2) "twisted" cocycles $* \xleftarrow{\simeq} EG \times_G \mathrm{Sp}(L) \to BH$ in algebraic groups H, for the étale topology on K.

Theorem ("Hammock localization") There is an isomorphism

$$\pi_0 h(*, BG) \cong [*, BG]$$

$[*, BG]$ is morphisms in homotopy category of simplicial sheaves.

1) Every G-torsor F determines a **canonical cocycle**

$$* \xleftarrow{\simeq} EG \times_G F \to BG$$

2) Every cocycle $* \xleftarrow{\simeq} U \to BG$ determines a "pullback" torsor $\pi_0(EG \times_{BG} U)$ by pullback over $EG \to BG$.

Theorem: The canonical cocycle and pullback functors

$$h(*, BG) \leftrightarrows G - \textbf{tors}$$

are adjoint, so that there are isos

$$H^1(\mathcal{E}, G) = \pi_0(G - \textbf{tors}) \cong \pi_0 h(*, BG) \cong [*, BG].$$

This is the **homotopy classification** of G-torsors.

Symmetric bilinear forms, again

Every non-deg symm bilinear form β over K (char$\neq 2$) determines a homotopy class of maps $[\beta] : * \to BO_n$ for the etale topology, and an induced map

$$H^*_{Gal}(K, \mathbb{Z}/2)[HW_1, \dots, HW_n] \cong H^*_{et}(BO_n, \mathbb{Z}/2) \xrightarrow{\beta^*} H^*_{Gal}(K, \mathbb{Z}/2).$$

$\deg(HW_i) = i$, like Stiefel-Whitney classes.

$HW_i \mapsto \beta^*(HW_i) = HW_i(\beta)$, higher Hasse-Witt invariants of β (formerly Delzant Stiefel-Whitney classes).

Examples $HW_1(\beta) = det(\beta)$, $HW_2(\beta) =$ Hasse-Witt invariant.

This where the homotopy classification of torsors and the homotopy theoretic approach to stacks began (1989).

Stacks

Origins: Grothendieck "effective descent" (1959); Giraud "champ" (1966, 1971); Deligne-Mumford "stack" (1969)

Most compact definition: a **stack** is a sheaf of groupoids H for which the simplicial sheaf BH **satisfies descent**

ie. there is a fibrant model $BH \to Z$ which is a sectionwise equivalence, ie. all $BH(U) \to Z(U)$ are weak equivs.

Alternative: A **stack** is a sheaf of groupoids H which satisfies **effective descent**, ie. any covering $R \subset hom(\ ,U)$ induces an equivalence of groupoids

$$H(U) \to \varprojlim_{\phi : V \to U} H(V).$$

NB: H is a *sheaf* of groupoids, so only need show that

$$\pi_0 H(U) \to \pi_0(\varprojlim_{\phi : V \to U} H(V))$$

is surjective.

“Sheaf” of torsors

We have defined $G-\textbf{tors}$ only in global sections.

If X = scheme and G = alg. group, we have $G-\textbf{tors}/X$ for any decent topology on X.

Given $f:Y\to X$, inverse image $f^*:\mathrm{Shv}/X\to\mathrm{Shv}/Y$ is exact, hence induces a functor $f^*:G-\textbf{tors}/X\to G-\textbf{tors}/Y$.

$U\subset X\ \mapsto G-\textbf{tors}/U$ is only a pseudo-functor (in groupoids): there are natural isomorphisms

$$(\beta\alpha)^*\xrightarrow{\cong}\alpha^*\beta^*,\ \ \eta:1\xrightarrow{\cong}1^*$$

which satisfy standard coherence conditions.

Grothendieck construction

Suppose $i \mapsto G(i)$ is a pseudo-functor in groupoids on $i \in I$.

The **Grothendieck construction** $E_I G$ has objects (i, x) with $x \in G(i)$, and morphisms $(\alpha, f) : (i, x) \to (j, y)$ with $\alpha : i \to j$ in I and $f : \alpha_* x \to y$ in $G(j)$.

The composite

$$(i, x) \xrightarrow{(\alpha, f)} (j, y) \xrightarrow{(\beta, g)} (k, z)$$

is defined by $\beta\alpha$ and the composite

$$(\beta\alpha)_*(x) \xrightarrow[\cong]{\omega} \beta_*\alpha_*(x) \xrightarrow{\beta_* f} \beta_*(y) \xrightarrow{g} z.$$

There is a **canonical functor**

$$\pi : E_I G \to I \text{ with } (i, x) \mapsto i.$$

Homotopy and old times

The slice categories π/i have objects $\pi(j,y) \to i$, and define a functor $I \to \mathbf{Cat}$ with $i \mapsto \pi/i$.

There is a homotopy equivalence of categories $\pi/i \to G(i)$ defined by flowing objects into $G(i)$.

By applying fundamental groupoid, we have equivalences of groupoids $G(\pi/i) \xrightarrow{\simeq} G(i)$.

The assignment $i \mapsto G(\pi/i)$ defines a functor in groupoids, sectionwise equivalent to the pseudo-functor $i \mapsto G(i)$.

Remark: Effective descent was originally defined for the pseudo-functor $U \mapsto G - \mathbf{tors}/U$, with a description equivalent to that given above for the equivalent diagram (sheaf) of groupoids $U \mapsto G(\pi/U)$.

Homotopy theory

There is a homotopy theory for sheaves of groupoids (Joyal-Tierney, 1990; Hollander, 2008), for which $G \to H$ is a weak equivalence (resp. fibration) if the induced map $BG \to BH$ is a local weak equivalence (resp. fibration) of simplicial sheaves.

A stack is a sheaf (or presheaf) of groupoids which satisfies descent in this homotopy theory

Equiv.: G is a stack if every fibrant model $G \to H$ is a sectionwise equivalence.

Every fibrant sheaf of groupoids is a stack (formal nonsense).

Slogan: Stacks are homotopy types of sheaves of groupoids.

Torsors are stacks

Suppose that $j : G \to H$ is a fibrant model (stack completion) for group obect G in sheaves of groupoids. Form the diagram

$$\begin{tikzcd}
BG \arrow[r, "j"] \arrow[d] & BH \arrow[d, "\simeq"] \\
B(G - \mathbf{tors}) \arrow[r] \arrow[d, "\simeq"'] & B(H - \mathbf{tors}) \arrow[d, "\simeq"] \\
B\mathbb{H}(*, BG) \arrow[r, "\simeq"'] & B\mathbb{H}(*, BH)
\end{tikzcd}$$

All displayed weak equivs are sectionwise, $j : BG \to BH$ is a local weak equiv.

So $j : BG \to B(G - \mathbf{tors})$ is a local weak equiv, and $B(G - \mathbf{tors})$ satisfies descent.

Example: quotient stacks

X is a scheme (sheaf) with G-action.

The **quotient stack** $[X/G]$ is the groupoid with objects all G-equivariant maps $P \to X$ with P a G-torsor, and all G-equivariant pictures

$$\begin{tikzcd} P \arrow[dd, "\cong"'] \arrow[dr] & \\ & X \\ Q \arrow[ur] & \end{tikzcd}$$

as morphisms.

Fact: There is an isomorphism

$$\pi_0([X/G]) \cong [*, EG \times_G X].$$

Every $P \to X$ in $[X/G]$ determines a cocycle

$$* \xleftarrow{\simeq} EG \times_G P \to EG \times_G X.$$

Detail: Cocycles to torsors

Given a cocycle $* \xleftarrow{\simeq} U \to EG \times_G X$, pull back over $EG \to BG$ to form $P = \pi_0(EG \times_{BG} U) \to X$ in homotopy fibres:

$$\begin{tikzcd}
EG \times_{BG} U \arrow[r] \arrow[dd] & U \arrow[d] \\
& EG \times_G X \arrow[d, "\pi"] \\
EG \arrow[r, "\pi"'] & BG
\end{tikzcd}$$

$EG \times_{BG} U$ is homotopy fibre of $U \to BG$, so weakly equivalent to

$$P := \tilde{\pi}_0(EG \times_{BG} U).$$

General nonsense: $EG \times_G (EG \times_{BG} U) \to U \simeq *$ is a weak equivalence.

Examples

1) Borel constructions $EG \times_G F = B(E_G F)$ are quotient stacks.

The stack completion is the functor $\phi : E_G F \to [F/G]$, defined in global sections by

$$x \in F \;\mapsto\; G \xrightarrow{x} F$$

ϕ is a local weak equivalence, and there is a sect. equiv.

$$B([F/G]) \simeq B\mathbb{H}(*, EG \times_G F).$$

2) A **gerbe** H is a locally connected stack, ie. $\pi_0 BH$ is the trivial sheaf.

Gerbes are souped up sheaves of groups.

Fact: If H is an ordinary connected groupoid and $x \in \mathrm{Ob}(H)$ then the inclusion functor $H(x,x) \subset H$ is an equivalence of groupoids, so H is a group.

Remarks

1) There are model structures for sheaves of 2-groupoids, presheaves of n-groupoids for $n \geq 2$. The homotopy types are 2-stacks, n-stacks etc. See [3] — other people define them differently.

2) Weak equivalence classes of gerbes (with structure) are classified homotopy theoretically by cocycles in 2-stacks. This is Giraud's non-abelian H^2 [2].

eg. Gerbes locally equivalent to a fixed sheaf of groups H are classified by the 2-groupoid $\mathrm{Aut}(H)$, which has automorphisms of H as 1-cells, and 2-cells given by homotopies.

Opinions: a) Stacks and higher stacks should have geometric content, like groupoids enriched in simplicial sets.

b) Simpson: a stack is a homotopy type of simplicial presheaves (non-abelian Hodge theory).

References

Armand Borel.
Seminar on transformation groups.
With contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais. Annals of Mathematics Studies, No. 46. Princeton University Press, Princeton, N.J., 1960.

Jean Giraud.
Cohomologie non abélienne.
Springer-Verlag, Berlin, 1971.
Die Grundlehren der mathematischen Wissenschaften, Band 179.

J.F. Jardine.
Local Homotopy Theory.
Springer Monographs in Mathematics. Springer-Verlag, New York, 2015.