T-spectra

Rick Jardine

University of Western Ontario

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$T =$ is a pointed simplicial presheaf on a site $\mathcal{C}$.

A *T-spectrum* X consists of pointed simplicial presheaves X^n, $n \geq 0$ and bonding maps $\sigma : T \wedge X^n \to X^{n+1}$, $n \geq 0$.

A *map of T-spectra* $g : X \to Y$ consists of pointed simplicial presheaf maps $g : X^n \to Y^n$, such that the diagrams

$$\begin{array}{ccc} T \wedge X^n & \xrightarrow{\sigma} & X^{n+1} \\ {\scriptstyle T\wedge g}\downarrow & & \downarrow{\scriptstyle g} \\ T \wedge Y^n & \xrightarrow[\sigma]{} & Y^{n+1} \end{array}$$

commute.

$\mathrm{Spt}_T(\mathcal{C})$ is the category of T-spectra.

Examples

1) Presheaves of spectra are S^1-spectra.

2) Suppose that k is a perfect field and let $(Sm|_k)_{Nis}$ be the category of smooth k-schemes, equipped with the Nisnevich topology. There are various choices:

$$T = \mathbb{P}^1,\ S^1 \wedge \mathbb{G}_m,\ \mathbb{A}^1/(\mathbb{A}^1 - \{0\}),\ [\mathbb{G}_m = \mathbb{A}^1 - \{0\}]$$

All of these isomorphic in the motivic homotopy category, and give equivalent (motivic) homotopy theories of T-spectra, on account of

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array} \quad \text{and } \mathbb{A}^1 \simeq *.$$

T is often called the *Tate object*.

Examples and constructions

1) K = pointed simp. presheaf. The *T-suspension spectrum* $\Sigma_T^\infty K$ consists of the list

$$K,\ T \wedge K,\ T \wedge T \wedge K,\ \dots\ T^{\wedge n} \wedge K,\ \dots$$

2) $S_T = \Sigma_T^\infty S^0$ is the *sphere T-spectrum*. $\Sigma_T^\infty K = S_T \wedge K$.

3) X a T-spectrum, $n \in \mathbb{Z}$: the shifted object $X[n]$ is defined by

$$X[n]^k = \begin{cases} X^{k+n} & \text{if } k+n \geq 0, \\ * & \text{otherwise.} \end{cases}$$

If $n \geq 0$, $K \mapsto \Sigma_T^\infty K[-n]$ is left adjoint to $X \mapsto X^n$.

4) The isomorphisms $T^{\wedge(k-1)} \wedge T \cong T^k$, $k > 0$, define the *stabilization map*

$$(S_T \wedge T)[-1] \to S_T$$

Strict model structure

The *strict weak equivalences* (resp. *strict fibrations*) are those maps $X \to Y$ of T-spectra for which all $X^n \to Y^n$ are local weak equivalences (resp. injective fibrations).

A map $i : A \to B$ is a *cofibration* if $A^0 \to B^0$ is a cofibration, along with all maps

$$(T \wedge B^n) \cup_{T \wedge A^n} A^{n+1} \to B^{n+1}.$$

Lemma 1.

The category $\mathrm{Spt}_T(\mathcal{C})$, *together with the strict weak equivalences, strict fibrations and cofibrations as defined above, satisfies the axioms for a proper closed simplicial model structure. This model structure is cofibrantly generated.*

$$\mathbf{hom}(X, Y)_n = \{X \wedge \Delta^n_+ \to Y\}.$$

T-loops

T-loops of a pointed simplicial presheaf Y:

$$\Omega_T(Y) = \mathbf{Hom}(T, Y)$$

where

$$\mathbf{Hom}(T, Y)(U) = \mathbf{hom}(T \wedge U_+, Y)$$

defines $\Omega_T(Y)$ as a pointed simplicial presheaf. There is a nat. iso.

$$\hom(T \wedge A, Y) \cong \hom(A, \Omega_T(Y)).$$

Exercise: $\Omega_T(Y)$ is injective fibrant if Y is injective fibrant.

A T-spectrum X consists of pointed simplicial presheaf and (adjoint) bonding maps $X^n \to \Omega_T(X^{n+1})$.

Warning: $X \mapsto \Omega_T X$ may not preserve local weak equivalences of presheaves of Kan complexes.

Localization

$\mathcal{F}$ = a set of cofibrations $\mathcal{F}$ of T-spectra. Suppose that

1) A is cofibrant for all morphisms $A \to B$ in $\mathcal{F}$.
2) $\mathcal{F}$ includes a set of generators for the trivial cofibrations of the strict structure.
3) $\mathcal{F}$ is closed under the formation of the maps

$$(T \wedge A) \cup_{S \wedge A} (S \wedge B) \to T \wedge B$$

with $S \to T$ in $\mathcal{F}$ and $A \to B$ a generating cofibration for the strict structure.

Form the natural map $X \to L_{\mathcal{F}}X$ such that $L_{\mathcal{F}}X \to *$ has the RLP wrt all maps of $\mathcal{F}$.

A map $f : X \to Y$ is an $\mathcal{F}$-equivalence if $L_{\mathcal{F}}X \to L_{\mathcal{F}}Y$ is a strict equivalence. *$\mathcal{F}$-fibrations* are defined by a RLP wrt to cofibrations which are $\mathcal{F}$-equivalences.

Theorem 2.

The category $\mathrm{Spt}_T(\mathcal{C})$*, with the cofibrations,* $\mathcal{F}$*-equivalences and* $\mathcal{F}$*-fibrations, has the structure of a left proper closed simplicial model category. This model structure is cofibrantly generated.*

Suppose that $f : A \to B$ is a cofibration of simplicial presheaves. The *f-local model structure* on $s\,\mathrm{Pre}(\mathcal{C})$ is cofibrantly generated.

Suppose that $\mathcal{F}$ generated by the set of maps:

1) $\Sigma_T^\infty C_+[-n] \to \Sigma_T^\infty D_+[-n]$, where $C \to D$ is a set of generators for the trivial cofibrations of the f-local structure,
2) cofibrant replacements of $(S_T \wedge T)[-n-1] \to S_T[-n]$, $n \geq 0$.

The *f-local stable model structure* on $\mathrm{Spt}_T(\mathcal{C})$ is the $\mathcal{F}$-local model structure given by Theorem 2, for this set $\mathcal{F}$.

The weak equivalences are *stable f-equivalences* and the fibrations are *stable f-fibrations*.

$(Sm|_S)_{Nis}$ is the smooth Nisnevich site of a decent scheme S (usually a perfect field).

$f : * \to \mathbb{A}^1$ is a rational point of the affine line $\mathbb{A}^1 = \mathbb{A}^1 \times S$ over S, usually the 0-section.

The f-local model structure on $s\,\mathrm{Pre}((Sm|_S)_{Nis})$ is the *motivic* (or $\mathbb{A}^1$*-local*) model structure.

Let $T = S^1 \wedge \mathbb{G}_m$. The f-local stable model structure on $\mathrm{Spt}_T((Sm|_S)_{Nis})$ is the *motivic stable model structure*. The corresponding homotopy category is the *motivic stable category*.

Remark: The original construction of the motivic stable model category used the methods of Bousfield and Friedlander [1], with Nisnevich descent. The localization approach first appeared in a paper of Hovey, and was an idea Jeff Smith.

Stable f-fibrant models

Lemma 3.

X is stable f-fibrant if and only if all X^n are f-fibrant and all maps $X^n \to \Omega_T X^{n+1}$ are sectionwise equivalences.

$$\Omega_T^\infty X^n = \varinjlim \Omega_T^k X^{n+k}$$

defines a T-spectrum $\Omega_T^\infty X$ and a natural map $X \to \Omega_T^\infty X$. We have the composite

$$X \xrightarrow{j} FX \to \Omega_T^\infty FX \xrightarrow{j} F\Omega_T^\infty FX =: Q_T X$$

defining $\eta : X \to Q_T X$. $j : Y \to FY$ is strict f-fibrant model. This construction might fail: it is not clear that the composite

$$\Omega_T^\infty FX^n \xrightarrow{\cong} \Omega_T \Omega_T^\infty X^{n+1} \xrightarrow{\Omega_T j} \Omega_T(F\Omega_T^\infty FX^{n+1})$$

is an f-equivalence. This requires a compactness condition on T.

A1: T is *compact up to f-equivalence* if the composite

$$\varinjlim_s \Omega_T X_s \to \Omega_T(\varinjlim_s X_s) \xrightarrow{\Omega_T j} \Omega_T F(\varinjlim_s X_s)$$

is an f-equivalence for every inductive system $s \mapsto X_s$ of f-fibrant pointed simplicial presheaves.

Theorem 4.

Suppose that T is compact up to f-equivalence. Then the natural map $X \to Q_T X$ is a stable f-fibrant model for all T-spectra X.

Fact: The family of objects which is compact up to f-equivalence is closed under finite smash products and f-equivalence.

Example: All finite pointed simplicial sets are compact for the injective model structure on $s\,\mathrm{Pre}(\mathcal{C})$.

Nisnevich descent

Suppose that $s \mapsto X_s$ is an inductive diagram of motivic fibrant simplicial presheaves. The Nisnevich fibrant model

$$j : \varinjlim_s X_s \to G(\varinjlim_s X_s)$$

is a sectionwise equivalence, by the Nisnevich descent theorem.

A Nisnevich fibrant simplicial presheaf Z is motivic fibrant if all $X(U) \to X(U \times \mathbb{A}^1)$ are weak equivalences. This condition is preserved by filtered colimits and sectionwise equivalences, so $G(\varinjlim_s X_s)$ is motivic fibrant.

Lemma 5.

1) *Every pointed scheme is compact up to motivic equivalence.*
2) *Every finite pointed simplicial set is compact up to motivic equivalence.*

Inductive colimit descent

A2 The f-local model structure satisfies *inductive colimit descent* if, given an inductive system $s \mapsto Z_s$ of f-fibrant simplicial presheaves, an f-fibrant model

$$j : \varinjlim_s Z_s \to F(\varinjlim_s Z_s)$$

is a local weak equivalence.

Example: The motivic model structure on $(Sm|_S)_{Nis}$ satisfies this property, again by Nisnevich descent.

Fact: If **A2** is satisfied, then every finite pointed simplicial set K is compact up to f-equivalence, because Ω_K preserves local weak equivalences of locally fibrant objects.

Fact: If T is compact up to f-equivalence and **A2** holds, then $S^1 \wedge T$ is compact up to f-equivalence.

A3: Say that T is *cycle trivial* if $x_1 \wedge x_2 \wedge x_3 \mapsto x_2 \wedge x_3 \wedge x_1$ is the identity on $T^{\wedge 3}$ in the f-local homotopy category.

Examples: 1) S^1 is cycle trivial, everywhere
2) (Voevodsky) : $\mathbb{P}^1$ is cycle trivial in the motivic model structure on $(Sm|_S)_{Nis}$.
3) If S and T are cycle trivial, then $S \wedge T$ is cycle trivial.

Theorem 6.

Suppose that T is compact up to f-equivalence and is cycle trivial. Then the composite

$$X \to \mathbf{Hom}(T, X \wedge T) \xrightarrow{\eta_*} \mathbf{Hom}(T, Q_T(X \wedge T))$$

is a stable f-equivalence for all T-spectra X.

Consequence: T-suspensions and T-loops form a Quillen equivalence $\mathrm{Spt}_T(\mathcal{C}) \leftrightarrows \mathrm{Spt}_T(\mathcal{C})$.

Fake suspensions

The *fake suspension* $\Sigma_T X$: $\Sigma_T X^n = T \wedge X^n$ with bonding maps $T \wedge \sigma_T : T \wedge T \wedge X^n \to T \wedge X^{n+1}$.

Lemma 7.

There is a natural stable f-equivalence $\sigma : \Sigma_T X \to X[1]$.

Lemma 8.

T compact up to f-equivalence and cycle trivial. There is natural stable equivalence $\Sigma_T X \simeq X \wedge T$.

The bonding maps for $\Sigma_T X$ and $X \wedge T$ differ by a twist $\tau : T^{\wedge 2} \xrightarrow{\cong} T^{\wedge 2}$. These objects restrict to equivalent $T^{\wedge 2}$-spectra, by the cycle triviality.

Corollary 9.

Same assumptions as Lemma 8, X strictly f-fibrant. There are stable f-equivalences $\mathbf{hom}(T, X) \simeq \Omega_T X \simeq X[-1]$.

S and T are compact up to f-equivalence. An (S,T)-*bispectrum* Y is an T-spectrum object in S-spectra.

Y consists of pointed simplicial presheaves $Y^{p,q}$, $p,q \geq 0$ and bonding maps $\sigma_S : S \wedge Y^{p,q} \to Y^{p+1,q}$, $\sigma_T : Y^{p,q} \wedge T \to Y^{p,q+1}$, such that the following commute:

$$\begin{array}{ccc} Y^{p+1,q} \wedge T & \xrightarrow{\sigma_T} & Y^{p+1,q+1} \\ {\scriptstyle \sigma_S \wedge T} \uparrow & & \uparrow {\scriptstyle \sigma_S} \\ S \wedge Y^{p,q} \wedge T & \xrightarrow[S \wedge \sigma_T]{} & S \wedge Y^{p,q+1} \end{array}$$

Y defines a *diagonal* $(S \wedge T)$-spectrum $d(Y)$ with $d(Y)^p = Y^{p,p}$ and bonding maps

$$S \wedge T \wedge Y^{p,p} \xrightarrow{S \wedge \tau} S \wedge Y^{p,p} \wedge T \xrightarrow{S \wedge \sigma_T} S \wedge Y^{p,p+1} \xrightarrow{\sigma_S} Y^{p+1,p+1}.$$

Every $(S \wedge T)$-spectrum X has an (S, T)-bispectrum $X^{*,*}$ such that $d(X^{*,*}) \cong X$.

$$\begin{array}{ccc} X^0 \wedge T^2 \dashrightarrow X^1 \wedge T & & X^2 \\ X^0 \wedge T & X^1 & S \wedge X^1 \\ X^0 & S \wedge X^0 & S^2 \wedge X^0 \end{array}$$

Example:

$$S \wedge X^0 \wedge T^2 \xrightarrow[\cong]{S \wedge \tau \wedge T} S \wedge T \wedge X^0 \wedge T \xrightarrow{\sigma \wedge T} X^1 \wedge T$$

Lemma 10.

S, T compact up to f-equivalence, $g : X \to Y$ a map of (S, T)-bispectra. If $g : X^{*,q} \to Y^{*,q}$ is a stable f-equivalence of S-spectra for all q, then $g : d(X) \to d(Y)$ is a stable f-equivalence of $(S \wedge T)$-spectra.

$(S^1 \wedge T)$-spectra

Corollary 11.

T compact up to f-equivalence and the f-local model structure satisfies inductive colimit descent, $g : X \to Y$ a map of $(S^1 \wedge T)$-bispectra such that all $g : X^{*,q} \to Y^{*,q}$ are stable f-equivalences of presheaves of spectra. Then $g_* : d(X) \to d(Y)$ is a stable f-equivalence of $(S^1 \wedge T)$-spectra.

Consequences

1) $p : X \to Y$ strict fibration of $(S^1 \wedge T)$-spectra with fibre F. Then $X/F \to Y$ is a stable f-equivalence.
2) $A \to B$ a monomorphism of $(S^1 \wedge T)$-spectra such that $B \to B/A$ has strict homotopy fibre F. Then there is a stable f-equivalence $A \to F$.
3) The f-local stable model structure on $\mathrm{Spt}_{S^1 \wedge T}(\mathcal{C})$ is proper.
4) The canonical map $X \vee Y \to X \times Y$ is a stable f-equivalence of $(S^1 \wedge T)$-spectra.

Stable homotopy groups

An $(S^1 \wedge T)$-spectrum X has bigraded presheaves of stable homotopy groups:

$$\pi_{s,t}X(U) := \varinjlim_{n \geq 0} [S^{n+s} \wedge T^{n+t} \wedge U_+, X^n], \ U \in \mathcal{C}, s, t \in \mathbb{Z}.$$

$[\ ,\ ]$ is morphisms in f-local homotopy category of pointed simplicial presheaves. s is the *degree*, t is the *weight*.

There are sheaf isomorphisms $\tilde{\pi}_{k-n,-n}X \cong \tilde{\pi}_k Q_{S^1 \wedge T} X^n$.

Lemma 12.

$g : X \to Y$ is a stable f-equivalence of $(S^1 \wedge T)$-spectra if and only if all $\tilde{\pi}_{s,t}X \to \tilde{\pi}_{s,t}Y$ are isomorphisms of sheaves of groups.

Motivic case: use presheaf isomorphisms, by Nisnevich descent.

Long exact sequences

Any strict f- fibre sequence $F \to X \to Y$ of strictly f-fibrant (S^1, T)-bispectra determines strict f-fibre sequences

$$\Omega_T^{t+n} F^n \to \Omega_T^{t+n} X^n \to \Omega_T^{t+n} Y^n$$

of presheaves of spectra, and long exact sequences

$$\cdots \to \pi_s \Omega_T^{t+n} F^n \to \pi_s \Omega_T^{t+n} X^n \to \pi_s \Omega_T^{t+n} Y^n \to \pi_{s-1} \Omega_T^{t+n} F^n \to \dots$$

in presheaves of stable homotopy groups. Taking filtered colimits in n gives long exact sequences

$$\cdots \to \tilde{\pi}_{s,t} F \to \tilde{\pi}_{s,t} X \to \tilde{\pi}_{s,t} Y \to \tilde{\pi}_{s-1,t} F \to \dots \quad \text{for each } t.$$

Have corresponding long exact sequences for strict f-fibre sequences and cofibre sequences of $(S^1 \wedge T)$-spectra.

Corollary 13.

There are natural isomorphisms $\tilde{\pi}_{s+1,t}(X \wedge S^1) \cong \tilde{\pi}_{s,t} X$.

Postnikov sections

E = a spectrum. The n^{th} *Postnikov section* is a functorial map $E \to P_n E$ such that $\pi_s E \to \pi_s P_n E$ is an isomorphism if $s \leq n$ and $\pi_s P_n E = 0$ for $s > n$.

The *Postnikov tower* can be constructed from the filtered colimits

$$P_n E \xrightarrow{\simeq} P_n P_{n+1} E \xrightarrow{\simeq} P_n P_{n+1} P_{n+2} E \xrightarrow{\simeq} \dots$$

The homotopy fibre $f_{n+1}E$ of the map $E \to P_n E$ is the *n-connected cover*. E is *connective* if $P_{-1}E \simeq *$ or $F_0 E \xrightarrow{\simeq} E$.

$$\dots \to f_2 E \to f_1 E \to f_0 E = E$$

is the *slice filtration* of a connective spectrum E. The homotopy cofibre $s_n E$ of $f_{n+1}E \to f_n E$ is the n^{th} *slice* of E.

$$s_n E \simeq H(\pi_n E)[-n].$$

Construction of Postnikov sections

Formally invert the maps $* \to \Sigma^\infty(S^q)[-r]$, where $q - r > n$ in the stable model structure.

Z is fibrant for this localized model structure if and only if Z is stably fibrant and all spaces

$$\mathbf{hom}(\Sigma^\infty S^q[-r], Z) \simeq \Omega^q Z^r$$

are contractible for $q - r > n$. Equivalently $\pi_s Z = 0$ for $s > n$.

Construct the fibrant model $E \to LE$ by killing stable homotopy group elements with cofibre sequences

$$\Sigma^\infty(S^q)[-r] \to E \to E'$$

Then $\pi_s E \to \pi_s LE$ is an isomorphism for $s \leq n$ and $\pi_s LE = 0$ for $s > n$.

Set $P_n E = LE$.

Suppose that all $U \in \mathcal{C}$ are compact up to f-equivalence (eg. motivic category).

In the stable f-local structure, invert the maps

$$* \to \Sigma^{\infty}_{S^1 \wedge T}(S^s \wedge T^t \wedge U_+)[-n], \;\; U \in \mathcal{C}, \;\; s,t \geq n, \;\; t-n > q.$$

Fibrant model: kill the groups $\pi_{s,t}Z(U)$ with $t > q$.

There is a strict f-local fibre sequence

$$f_{q+1}Z \to Z \xrightarrow{j} L_q Z =: s_{<q}Z$$

where j is the fibrant model for the localization.

Then $\pi_{s,t}(f_{q+1}Z) \cong \pi_{s,t}Z$ and $\pi_{s,t}(L_q Z) = 0$ for $t > q$.

More slices

The q^{th} *slice* s_qZ is defined by the cofibre sequence

$$f_{q+1}Z \to f_qZ \to s_qZ,$$

for T-connective objects Z.

Z is T-*connective* if $f_0Z \to Z$ is stable f-equivalence.

More elephants

Symmetric T-spectra

Motivic symmetric spectra

$\mathbb{G}_m$-spectra

Motives: effective motives, Voevodsky's big category of motives

Motivic cohomology theories

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