Ring spectra: first steps

Rick Jardine

Univ. of Western Ontario

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Freudenthal suspension theorem

The dawn of time for stable homotopy theory is the Freudenthal suspension theorem (1938):

Theorem 1.

X is an n-connected pointed space, $n \geq 0$. The map $\eta : X \to \Omega(S^1 \wedge X)$ induces a map $\Sigma : \pi_i(X) \to \pi_i(\Omega(S^1 \wedge X))$, which is an iso if $i \leq 2n$ and epi if $i = 2n+1$.

Example: The *suspension homomorphism*

$$\Sigma : \pi_i(S^n) \to \pi_i(\Omega(S^{n+1})) \cong \pi_{i+1}(S^{n+1})$$

is an iso if $i \leq 2(n-1)$ and is an epi if $i = 2n-1$.

The maps $\Sigma : \pi_{n+k}(S^n) \to \pi_{n+1+k}(S^{n+1})$ are isomorphisms (i.e. the groups "stabilize") for $n \geq k+2$.

Spheres

The colimit

$$\pi_k(S) := \varinjlim_n \pi_k(S^{n+k})$$

is the k^{th} *stable homotopy group* of the spheres (*stable* k*-stem*).

1) $\pi_i(S^n) = 0$ if $i < n$ by simp. approx., so $\pi_k(S) = 0$ for $k < 0$.

2) $\pi_0(S) = \mathbb{Z} \cong \pi_1(S^1) \cong \pi_2(S^2) \cong \pi_3(S^3) \cong \dots$

3) $\pi_1(S) = \mathbb{Z}/2$:

$$\pi_3(S^2) \twoheadrightarrow \pi_4(S^3) \cong \pi_5(S^4) \cong \dots$$

$\pi_3(S^2) \cong \mathbb{Z}$ is generated by the Hopf map $\eta : S^3 \to S^2$, aka. the principal $\mathbb{C}^*$-bundle

$$\mathbb{C}^2 - \{0\} \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}.$$

The kernel of the map $\pi_3(S^2) \to \pi_4(S^3)$ is generated by 2η.

Spectra

The 1978 paper of Bousfield-Friedlander is the beginning of the modern period of stable homotopy theory and its applications.

Compare with the Adams "blue book" of 1974. Both sources agree on what a spectrum is (sort of):

A **spectrum** E is a sequence of pointed spaces E^n, $n \geq 0$, (really, pointed simplicial sets), with pointed maps (**bonding maps**)

$$\sigma : S^1 \wedge E^n \to E^{n+1},\ n \geq 0.$$

$$S^1 \wedge E^n = \frac{S^1 \times E^n}{S^1 \vee E^n}$$

is the **smash product** (tensor product in pointed spaces).

$S^1 = \Delta^1/\partial\Delta^1$ is the **simplicial circle**.

Category of spectra

A **map** of spectra $f : E \to F$ is a family of pointed (simplicial set) maps $f : E^n \to F^n$, $n \geq 0$, such that the diagrams

$$\begin{array}{ccc} S^1 \wedge E^n & \xrightarrow{\sigma} & E^{n+1} \\ {\scriptstyle S^1 \wedge f} \downarrow & & \downarrow {\scriptstyle f} \\ S^1 \wedge F^n & \xrightarrow[\sigma]{} & F^{n+1} \end{array}$$

commute. This is the Bousfield-Friedlander definition, not at all the Adams definition — it is much simpler.

We have objects and morphisms of the category **Spt** of spectra.

Remark: The pointed simplicial sets (combinatorial) version of **Spt** has all of the benefits of pointed sets: it is complete and cocomplete, and shows up in many algebraic settings.

Examples

1) The **sphere spectrum** S is the sequence of spaces

$$S^0, S^1, S^1 \wedge S^1, \dots, S^{\wedge n}, \dots$$

(iterated smash powers), with the isomorphisms

$$\sigma : S^1 \wedge S^{\wedge k} \xrightarrow{\cong} S^{\wedge (k+1)}.$$

2) Suppose that K is a pointed space. The **suspension spectrum** $\Sigma^\infty K = S \wedge K$ is the list of spaces

$$K, S^1 \wedge K, S^{\wedge 2} \wedge K, \dots$$

with the isomorphisms

$$\sigma : S^1 \wedge (S^{\wedge k} \wedge K) \xrightarrow{\cong} S^{\wedge (k+1)} \wedge K.$$

$K \mapsto \Sigma^\infty K$ is left adjoint to the "level 0" functor $E \mapsto E^0$.

Shift operators

Suppose that n is some integer. Any spectrum E has a **shift** $E[n]$, where

$$E[n]^k = \begin{cases} E^{n+k} & \text{if } n+k \geq 0, \text{ and} \\ * & \text{otherwise.} \end{cases}$$

Here, $*$ is the one-point space.

For the sphere spectrum S, $S[1] = S \wedge S^1$. The iterated shift $S[1][-1]$ is the list of spaces

$$*, S^1, S^{\wedge 2}, \dots$$

There is a canonical map (**stabilization map**)

$$(S \wedge S^1)[-1] = S[1][-1] \to S$$

which consists of isomorphisms in levels $k \geq 1$ — ignores S^0.

Stable homotopy groups

$E =$ a spectrum. The **suspension homomorphism** is the composite

$$\Sigma : \pi_k(E^n) \xrightarrow{\sigma_*} \pi_k(\Omega E^{n+1}) \cong \pi_{k+1}(E^{n+1}).$$

$\sigma_* : E^n \to \Omega E^{n+1}$ is the adjoint of $\sigma : S^1 \wedge E^n \to E^{n+1}$.

The k^{th} **stable homotopy group** $\pi_k E$ ($k \in \mathbb{Z}$) is the colimit of the system

$$\dots \xrightarrow{\Sigma} \pi_{n+k} E^n \xrightarrow{\Sigma} \pi_{n+1+k} E^{n+1} \xrightarrow{\Sigma} \dots$$

A map $E \to F$ of spectra induces homomorphisms

$$\pi_k E \to \pi_k F, \ \ k \in \mathbb{Z}.$$

Stable homotopy theory

A map $E \to F$ of spectra is a **stable equivalence** if it induces isomorphisms $\pi_k E \xrightarrow{\cong} \pi_k F$ for all $k \in \mathbb{Z}$.

A map $A \to B$ of spectra is a **cofibration** if

1) the map $A^0 \to B^0$ is a monomorphism (inclusion of CW-complexes), and
2) all induced maps $(S^1 \wedge B^n) \cup_{S^1 \wedge A^n} A^{n+1} \to B^{n+1}$ are cofibrations.

Theorem 2 (Bousfield-Friedlander).

The stable equivalences and cofibrations give the category **Spt** *of spectra the structure of a Quillen model category.*

This structure is proper, closed, simplicial, cofibrantly generated.

The homotopy category Ho(**Spt**) is “the” stable category.

Names of things

1) A spectrum E is **cofibrant** if the map $* \to E$ is a cofibration. This means that E^0 is cofibrant (CW-complex), and all maps $\sigma : S^1 \wedge E^n \to E^{n+1}$ are cofibrations. Also called a CW-**spectrum**.
Examples: All suspension spectra (including S) are cofibrant.

2) **Stable fibrations** are defined by the right lifting property with respect to trivial cofibrations in the model structure.

A spectrum X is **stably fibrant** if and only if all X^n are fibrant and all maps $\sigma_* : X^n \to \Omega X^{n+1}$ are weak equivalences. X is an Ω-**spectrum** in the old language.

Example: The sphere spectrum S is **not** stably fibrant (Hopf invariant).

$$QS^0 = \mathbb{Z} \times QS^0_0 \simeq \mathbb{Z} \times B\Sigma^\infty_+.$$

$B\Sigma^\infty_+$ is an H-space with the integral homology of $B\Sigma^\infty$ (Barratt-Priddy-Quillen theorem).

Chain complexes

"Spectra are like chain complexes and chain complexes are spectra."

You can shift chain complexes just like spectra.

C = a chain complex. The **good truncation** $\mathrm{Tr}_0\, C$ has

$$\mathrm{Tr}_0\, C_k = \begin{cases} C_k & \text{if } k > 0, \\ \ker(C_0 \xrightarrow{\partial} C_{-1}) & \text{if } k = 0, \\ 0 & \text{if } k < 0. \end{cases}$$

$\mathrm{Tr}_0(C[-n])$ is an "ordinary" complex, and there are chain maps

$$\mathrm{Tr}_0(C[-n])[-1] \to \mathrm{Tr}_0(C[-n-1]).$$

This is a **spectrum object** in ordinary chain complexes. There is an obvious category $\mathbf{Spt}(Ch_+)$ of such things.

"The derived category of $\mathbf{Spt}(Ch_+)$ is equivalent to the derived category of unbounded complexes."

Simplicial abelian groups

The Dold-Kan correspondence

$$N : s\mathbf{Ab} \leftrightarrows Ch_+ : \Gamma$$

is an equivalence of categories, and there is a natural equivalence

$$S^1 \otimes \Gamma(D) \xrightarrow{\simeq} \Gamma(D[-1])$$

for ordinary chain complexes D. $[S^1 \otimes \Gamma(D) := \tilde{\mathbb{Z}}(S^1) \otimes \Gamma(D)]$

A **spectrum object** A in $s\mathbf{Ab}$ consists of simplicial abelian groups A^n, $n \geq 0$, with homomorphisms

$$S^1 \otimes A^n \to A^{n+1}, \ n \geq 0.$$

The Dold-Kan correspondence extends to an equivalence of categories:

$$N : \mathbf{Spt}(s\mathbf{Ab}) \leftrightarrows \mathbf{Spt}(Ch_+) : \Gamma$$

Hurewicz map

There is a natural map

$$S^1 \wedge \Gamma(D) \xrightarrow{\gamma} S^1 \otimes \Gamma(D)$$

on the level of underlying pointed simplicial sets.

The forgetful/free abelian group adjunction extends to an adjuction

$$\tilde{\mathbb{Z}} : \mathbf{Spt} \leftrightarrows \mathbf{Spt}(s\mathbf{Ab}) : u$$

The canonical map $E \to u(\tilde{\mathbb{Z}}(E))$ is the **Hurewicz map**, for a spectrum E.

General story: $C =$ chain complex. Play with truncations and shifts, apply the Dold-Kan correspondence, giving a spectrum

$$\Gamma(\mathrm{Tr}_0(C[-n])),\ \ n \geq 0,$$

in $s\mathbf{Ab}$, and then a spectrum by forgetting structure.

Eilenberg-Mac Lane spectra

$A =$ abelian group. Applying the constructions above to the chain complex $A(0)$ (A concentrated in degree 0) produces the **Eilenberg-Mac Lane spectrum** $H(A)$:

$$H(A)^n = \Gamma(A(n)) \simeq K(A,n), \;\; n \geq 0.$$

Alternatively, set

$$H(A)^n = \tilde{\mathbb{Z}}(S^{\wedge n}) \otimes A \cong (S^1)^{\otimes n} \otimes A,$$

together with the isomorphisms

$$\sigma : S^1 \otimes (S^1)^{\otimes n} \otimes A \cong (S^1)^{\otimes (n+1)} \otimes A.$$

This is the **suspension object** for the (simplicial) abelian group A.

Eilenberg-Mac Lane spaces represent singular cohomology:

$$H^n(X,A) \cong [X, K(A,n)] \cong [\Sigma^\infty(X), H(A)[-n]].$$

Cup products

Suppose that A and B are abelian groups. The isomorphism

$$((S^1)^{\otimes n} \otimes A) \otimes ((S^1)^{\otimes m} \otimes B) \cong (S^1)^{\otimes (m+n)} \otimes (A \otimes B)$$

defines maps

$$K(A,n) \otimes K(B,m) \xrightarrow{\cup} K(A \otimes B, n+m).$$

These maps induce the cup product maps

$$H^n(X,A) \otimes H^m(X,B) \xrightarrow{\cup} H^{n+m}(X, A \otimes B).$$

The cup product is induced by a smash product pairing

$$H(A) \wedge H(B) \to H(A \otimes B)$$

in the stable category.

Naive smash products

The spectra E and F have a naturally associated **bispectrum**, consisting of all spaces $E^p \wedge F^q$ and bonding maps

$$\sigma_E : S^1 \wedge E^p \wedge F^q \xrightarrow{\sigma \wedge F} E^{p+1} \wedge F^q$$

and

$$\sigma_F : S^1 \wedge E^p \wedge F^q \xrightarrow{\tau \wedge F} E^p \wedge S^1 \wedge F^q \xrightarrow{E \wedge \sigma} E^p \wedge F^{q+1}.$$

The map $\tau : S^1 \wedge E^p \xrightarrow{\cong} E^p \wedge S^1$ flips smash factors.

Here's a candidate for the smash product:

$$E^0 \wedge F^0 \stackrel{\sigma_E}{\dashrightarrow} E^1 \wedge F^0 \stackrel{\sigma_F}{\dashrightarrow} E^1 \wedge F^1 \stackrel{\sigma_E}{\dashrightarrow} \dots$$

Here's another:

$$E^0 \wedge F^0 \stackrel{\sigma_F}{\dashrightarrow} E^0 \wedge F^1 \stackrel{\sigma_E}{\dashrightarrow} E^1 \wedge F^1 \stackrel{\sigma_F}{\dashrightarrow} \dots$$

The criminal τ

The composites

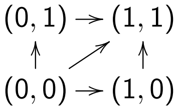
$$S^1 \wedge S^1 \wedge E^p \wedge F^p \xrightarrow{S^1 \wedge \Sigma_E} S^1 \wedge E^{p+1} \wedge F^p \xrightarrow{\sigma_F} E^{p+1} \wedge F^{p+1}$$

and

$$S^1 \wedge S^1 \wedge E^p \wedge F^p \xrightarrow{S^1 \wedge \Sigma_F} S^1 \wedge E^p \wedge F^{p+1} \xrightarrow{\sigma_E} E^{p+1} \wedge F^{p+1}$$

differ by the automorphism $\tau : S^1 \wedge S^1 \xrightarrow{\cong} S^1 \wedge S^1$, which has degree -1, and is therefore non-trivial.

$S^1 \wedge S^1 \cong \frac{\Delta^1 \times \Delta^1}{\partial(\Delta^1 \times \Delta^1)}$, with top cells σ_1, σ_2. $\tau(\sigma_1 - \sigma_2) = \sigma_2 - \sigma_1$.



One solution

Do the interchange operation twice, to obtain two maps

$$(S^1)^{\wedge 4} \wedge E^p \wedge F^p \to E^{p+4} \wedge F^{p+4}$$

that differ by the map $\tau \wedge \tau : S^4 \to S^4$, which has degree $(-1)(-1) = 1$ and is therefore the identity homotopically.

You end up with homotopically equivalent "S^4-spectra"

$$E^0 \wedge F^0 \dashrightarrow E^4 \wedge F^4 \dashrightarrow E^8 \wedge F^8 \dashrightarrow \dots$$

so that the naive smash products are naturally stably equivalent.

Such constructions become **very** awkward when dealing with higher order smash products.

One wants to deal with all of the permutation actions on the spheres S^n, $n \geq 0$, at one go.

Symmetric spectra

There have been multiple solutions to the symmetry problem. The one with the most impact in applications is the Hovey-Shipley-Smith theory of symmetric spectra (2000).

A **symmetric spectrum** X is a spectrum with actions

$$\Sigma_q \times X^q \to X^q, \ \ q \geq 0,$$

such that all iterated bonding maps

$$(S^1)^{\wedge p} \wedge X^q \to X^{p+q}$$

are equivariant for $\Sigma_p \times \Sigma_q \subset \Sigma_{p+q}$ (block matrices).

A **map** of symmetric spectra $f : X \to Y$ is a map of spectra which respects the symmetric group actions.

The category of symmetric spectra is denoted by $\mathbf{Spt}^{\Sigma}$.

Examples

1) All suspension spectra, including the sphere spectrum S.

2) The Eilenberg-Mac Lane spectra $H(A)$: $\theta \in \Sigma_n$ acts on

$$H(A)^n = (S^1)^{\otimes n} \otimes A \cong ((S^1)^{\wedge n}) \otimes A$$

by flipping smash (or tensor) factors.

3) The algebraic K-theory spectrum $K(\mathbf{M})$ for an exact category $\mathbf{M}$ — use the Waldhausen construction.

There are adjoint functors

$$V : \mathbf{Spt} \leftrightarrows \mathbf{Spt}^{\Sigma} : U$$

U is the obvious forgetful functor. V is more complicated, and is defined inductively using the "layer filtration" and shifted suspension spectra.

Cofibrations are defined levelwise. A symmetric spectrum Z is **stably fibrant** if the underlying spectrum $U(Z)$ is stably fibrant (ie. Ω-spectrum).

A map $f : X \to Y$ is a **stable equivalence** if the functions $[Y, Z] \xrightarrow{f^*} [X, Z]$ (for an "injective structure") are bijections for all stably fibrant Z.

Theorem 3 (Hovey-Shipley-Smith).

1) *The cofibrations and stable equivalences give the category* $\mathbf{Spt}^{\Sigma}$ *of symmetric spectra a Quillen model structure (with all usual adjectives).*
2) *The adjoint functors*

$$V : \mathbf{Spt} \leftrightarrows \mathbf{Spt}^{\Sigma} : U$$

define a Quillen equivalence of (stable) model structures

A **symmetric space** A is a collection of pointed spaces A^n, $n \geq 0$, together with symmetric group actions $\Sigma_n \times A^n \to A^n$.

The **tensor product** $A \otimes B$ of symmetric spaces A, B has

$$(A \otimes B)^n = \bigvee_{p+q=n} \Sigma_n \otimes_{(\Sigma_p \times \Sigma_q)} (A^p \wedge B^q).$$

A **symmetric spectrum** X is a symmetric space, equipped with a map (module structure) $m : S \otimes X \to X$.

The **twist** $\tau : A \otimes B \xrightarrow{\cong} B \otimes A$ is defined by the composites

$$A^p \wedge B^q \xrightarrow{\tau} B^q \wedge A^p \to (B \otimes A)^{q+p} \xrightarrow{c_{p,q}} (B \otimes A)^{p+q}.$$

$c_{p,q} \in \Sigma_{p+q}$ is the shuffle which moves the first p letters past the last q letters.

Smash product

The **smash product** $X \wedge_\Sigma Y$ of symmetric spectra X, Y, is defined in symmetric spaces by the coequalizer

$$S \otimes X \otimes Y \rightrightarrows X \otimes Y \to X \wedge_\Sigma Y.$$

which removes the ambiguity between the S-actions given by $m \otimes Y$ and

$$S \otimes X \otimes Y \xrightarrow{\tau \otimes Y} X \otimes S \otimes Y \xrightarrow{X \otimes m} X \otimes Y.$$

The smash product is symmetric monoidal, and is well behaved homotopically if one of the two arguments is cofibrant.

Ring spectra

A **ring spectrum** is a symmetric spectrum E, equipped with a multiplication map $E \wedge_\Sigma E \to E$.

Examples: 1) The sphere spectrum S.

2) If R is a ring with identity in the usual sense, then the Eilenberg-Mac Lane spectrum $H(R)$ is a ring spectrum.

3) $S =$ a scheme, $\mathbf{P}(S)$ the exact category of vector bundles over S. The tensor product $(V, V') \mapsto V \otimes V'$ gives the K-theory spectrum $K(S) = K(\mathbf{P}(S))$ the structure of a ring spectrum.

The imbeddings $\Sigma_n \subset Gl_n(S)$ (permutation matrices) define a unit homomorphism $S \to K(S)$.

The K-theory spectrum $K(\mathbf{M}(S))$ for the category of coherent sheaves $\mathbf{M}(S)$ is a module over $K(S)$, again via tensor product.

Remarks

1) There are stable model structures on categories $\mathrm{Pre}(\mathbf{Spt})$ of presheaves of spectra and presheaves $\mathrm{Pre}(\mathbf{Spt}^{\Sigma})$ of presheaves of symmetric spectra, such that the functors

$$V : \mathrm{Pre}(\mathbf{Spt}) \leftrightarrows \mathrm{Pre}(\mathbf{Spt}^{\Sigma}) : U$$

define a Quillen equivalence of the respective stable model structures.

2) There are stable homotopy theories for presheaves of chain complexes.

The category $\mathrm{Pre}(\mathbf{Spt}^{\Sigma}(s\mathbf{Ab}))$ has a tensor product (analogous to smash for symmetric spectra) — gives a theory of derived tensor products for presheaves (and sheaves) of unbounded complexes.

Changing parameter objects

T = a pointed simplicial presheaf.

A T-**spectrum** X is a collection of pointed simplicial presheaves X^n, $n \geq 0$, with bonding maps $T \wedge X^n \to X^{n+1}$. There is a corr. definition of symmetric T-spectrum.

The stable model structures are constructed with localization arguments, and have the good properties of spectra and symmetric spectra if T is a suspension and is compact in some sense.

Example: The **sphere spectrum** object S_T is given by the list of "spaces"

$$S^0,\ T,\ T \wedge T,\ T^{\wedge 3}\ \ldots .$$

Voevodsky' motivic stable category is a special case. He uses the parameter object $T = \mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$, all defined over a decent scheme (usually a field), with respect to the Nisnevich topology.

This is the setting for the cup product in motivic cohomology.

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