Galois groups and groupoids, and pro homotopy types

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Galois groups

Suppose that K is a field with algebraic closure $\overline{K}$.

Consider all subfields (subobjects)

$$K \subset L \subset \overline{K}$$

which are finitely generated and Galois over K.

Each such L has the form

$$L = K\langle\alpha_1, \dots, \alpha_k\rangle = K(\alpha_1, \dots, \alpha_n)$$

for a finite list of elements $\alpha_i \in \overline{K}$.

K is the fixed subfield of the group of automorphisms $G = G(L/K)$ of L over K.

Galois extensions

Fin_K = all finite Galois extensions L/K in $\overline{K}$ forms a filtered system under inclusion of subobjects inside $\overline{K}$.

The system of inclusions $L \subset \overline{K}$ is part of the structure, and is a *geometric base point* for the system.

The absolute Galois group

Suppose given inclusions $K \subset L_1 \subset L_2$ in Fin_K.

Every automorphism θ of L_2/K moves roots of polynomials, and restricts to an automorphism θ_* of L_1/K. There is a diagram

$$\begin{tikzcd} & L_1 \arrow[r] \arrow[d, "\theta_*"] & L_2 \arrow[d, "\theta"] \\ K \arrow[ur] \arrow[dr] & & \\ & L_1 \arrow[r] & L_2 \end{tikzcd}$$

and $\theta \mapsto \theta_*$ defines a homomorphism $G(L_2/K) \to G(L_1/K)$.

The assignment $L \mapsto G(L/K)$ defines a contravariant functor

$$\Omega_K : \mathrm{Fin}_K^{op} \to \mathbf{Grp}.$$

Ω_K is the *absolute Galois group* of K. It is a *pro-group*, in fact a *pro-finite group*.

Pro-objects

A *pro-object* X in a category $\mathbf{E}$ is a (contravariant) functor

$$X : I^{op} \to \mathbf{E},$$

which is defined on a small filtered category I.

You can apply functors to pro-objects in one category to get pro-objects in another.

Example: Every group G has a classifying space BG which has G as a fundamental group and no other homotopy groups.

Composing the absolute Galois group with the classifying space functor, ie.

$$\mathrm{Fin}_K^{op} \xrightarrow{\Omega_K} \mathbf{Grp} \xrightarrow{B} s\mathbf{Set}$$

defines a pro-object $L \mapsto BG(L/K)$ in spaces.

This is the motivating example of an *étale homotopy type* (of the field K).

Galois cohomology — naive

Each space Y has cohomology groups $H^n(Y, A)$, and the assignment

$$\begin{aligned} H^n(\Omega_K, A) &= \varinjlim_{L/K} H^n(BG(L/K), A) \\ &= \varinjlim_{L/K} [BG(L/K), K(A, n)] \\ &= \varinjlim_{L/K} \pi_0 \mathbf{hom}(BG(L/K), K(A, n)) \end{aligned}$$

(filtered colimit, defined on Fin_K) defines a *Galois cohomology group* of K, with *constant coefficients* in the abelian group A.

$\mathbf{hom}(BG(L/K), K(A, n))$ is the function complex of maps from $BG(L/K)$ to $K(A, n)$, and $K(A, n)$ is an Eilenberg-Mac Lane space.

Separable extensions

Sep_K is the category of finite separable extensions L/K, with all field homomorphisms between them.

A morphism of Sep_K is a commutative diagram of field homomorphisms

$$\begin{tikzcd} L_1 \arrow[rr, "\theta"] & & L_2 \\ & K \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Example: If L/K is a finite Galois extension, the members of the Galois group $G(L/K)$ are in Sep_K but not in Fin_K.

Every finite separable field extension F/K represents a covariant functor

$$\mathrm{Sp}(F) : \mathrm{Sep}_K \to \mathbf{Set},$$

with

$$\mathrm{Sp}(F)(N) = \hom_K(F, N).$$

L/K is finite Galois, with Galois group $G = G(L/K)$.

G acts on the right *on the functor* $\mathrm{Sp}(L)$: $\theta \cdot g$ is the composite form

$$L \xrightarrow{g} L \xrightarrow{\theta} N.$$

There's a groupoid $E_G\,\mathrm{Sp}(L)(N)$ here, given by diagrams

$$\begin{tikzcd} L \arrow[d, "g"'] \arrow[dr, "\theta"] & \\ L \arrow[r, "\gamma"'] & N \end{tikzcd}$$

Such pictures are covariant in N, and so there is a functor

$$E_G\,\mathrm{Sp}(L) : \mathrm{Sep}_K \to \mathbf{Gpd}$$

This is an étale sheaf of groupoids on K, defined by the finite Galois extension L/K. Étale sheaves on K are ...

The Borel construction

The categories $E_G \operatorname{Sp}(L)(N)$ are groupoids. We apply the classifying space functor $B : \mathbf{cat} \to s\mathbf{Set}$, and the composite

$$\mathrm{Sep}_K \xrightarrow{E_G \operatorname{Sp}(L)} \mathbf{cat} \xrightarrow{B} s\mathbf{Set}$$

has a special name, ie.

$$EG \times_G \operatorname{Sp}(L) := B(E_G \operatorname{Sp}(L)).$$

This is the *Borel construction* for the action of the Galois group G on the functor (étale sheaf, or scheme) $\operatorname{Sp}(L)$.

If C is a small category, the *classifying space* BC is the simplicial set with n-simplices given by strings of morphisms

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} a_n$$

of length n in C. A functor $F : C \to D$ induces a simplicial set map $F : BC \to BD$, by mapping strings to strings.

If $L_1 \subset L_2$ are in Fin_K, with Galois groups G_1 and G_2 respectively, then there are diagrams

$$\begin{tikzcd} L_1 \arrow[r] \arrow[d, "g_*"'] & L_2 \arrow[d, "g"'] \arrow[dr, "\theta"] & \\ L_1 \arrow[r] & L_2 \arrow[r, "\gamma"'] & N \end{tikzcd}$$

defined by restriction for each $g \in G_2$, natural in N. These functors therefore define natural transformations

$$E_{G_2}\,\mathrm{Sp}(L_2) \to E_{G_1}\,\mathrm{Sp}(L_1), \quad EG_2 \times_{G_2} \mathrm{Sp}(L_2) \to EG_1 \times_{G_1} \mathrm{Sp}(L_1)$$

which together which define a functor

$$\mathbf{\Omega}_K : \mathrm{Fin}_K^{op} \to \mathbf{Gpd}^{\mathrm{Sep}_K}.$$

and there is an induced functor

$$B\mathbf{\Omega}_K : \mathrm{Fin}_K^{op} \to s\mathbf{Set}^{\mathrm{Sep}_K}.$$

This is the *absolute Galois groupoid* of the field K.

$\mathbf{\Omega}_K$ is a pro-object in sheaves of groupoids (or sheaves of simplicial sets) on the étale site $et|_K$ for the field K.

There is an obvious functor

$$E_G \operatorname{Sp}(L)(N) \to G$$

(G is a groupoid with one object) that strips off the information about N, and which is also natural in L. It follows that there is a natural transformation of functors

$$\pi : EG \times_G \operatorname{Sp}(L) \to BG$$

which respects restriction, and therefore defines a morphism

$$B\mathbf{\Omega}_K \to B\Omega_K$$

from the absolute Galois groupoid to the absolute Galois group, where the latter is identified with a pro-object of constant sheaves.

Galois cohomology

An abelian sheaf for the étale topology on K is an abelian group-valued functor

$$A : \mathrm{Sep}_K \to \mathbf{Ab}$$

which respects fixed points of Galois groups.

There are isomorphisms

$$\begin{aligned} H^n_{Gal}(\Omega_K, A) &\cong \varinjlim_{L \in \mathrm{Fin}_K} \pi(EG \times_G \mathrm{Sp}(L), K(A,n)) \\ &\cong \varinjlim_{L \in \mathrm{Fin}_K} \pi_0 \mathbf{hom}(EG \times_G \mathrm{Sp}(L), K(A,n)) \\ &\cong [*, K(A,n)], \end{aligned}$$

where the last thing is morphisms in a homotopy category of simplicial sheaves (or presheaves) for the étale topology on K.

Non-abelian cohomology

There is a similar story for non-abelian H^1. Here's the first example:

Suppose that O_n is the algebraic group of automorphisms of the trivial non-degenerate symmetric bilinear form over K. There are isomorphisms

$$\begin{aligned} H^1_{Gal}(\Omega_K, O_n) &\cong \varinjlim_{L \in \mathrm{Fin}_K} \pi(EG \times_G \mathrm{Sp}(L), BO_n) \\ &\cong \varinjlim_{L \in \mathrm{Fin}_K} \pi_0 \mathbf{hom}(EG \times_G \mathrm{Sp}(L), BO_n) \\ &\cong [*, BO_n]. \end{aligned}$$

$H^1_{Gal}(\Omega_K, O_n)$, aka. the set of isomorphism classes of O_n-torsors, is identified with isomorphism classes of non-degenerate symmetric bilinear forms of rank n over K, by a classical argument.

Étale homotopy types

The object $EG \times_G \mathrm{Sp}(L)$ is represented by a simplicial scheme, with n-simplices

$$\bigsqcup_{G^{\times n}} \mathrm{Sp}(L).$$

The scheme $\mathrm{Sp}(L)$ has one connected component (in the Zariski topology), so the simplicial set of connected components of $EG \times_G \mathrm{Sp}(L)$ is BG.

An étale homotopy type for a scheme X is constructed by taking a cofinal family of representable hypercovers $U \to X$, which produces the pro-object $U \mapsto \pi_0(U)$ in simplicial sets.

Hypercovers

The natural transformation $EG \times_G \mathrm{Sp}(L) \to *$ is a hypercover, because all simplicial sets

$$EG \times_G \mathrm{Sp}(L)(N)$$

are nerves of groupoids (hence Kan complexes), and are contractible if non-empty (in which case $EG \times_G \mathrm{Sp}(L)(N)$ is a copy of EG up to isomorphism).

Simplicial sheaves X, Y are functors $\mathrm{Sep}_K \to s\mathbf{Set}$, and a natural transformation $p : X \to Y$ is a *hypercover* if the induced map

$$\varinjlim_{L \in \mathrm{Fin}_K} X(L) \to \varinjlim_{L \in \mathrm{Fin}_K} Y(L)$$

is a trivial fibration of simplicial sets, or has the right lifting property wrt. all inclusions $\partial\Delta^n \subset \Delta^n$.

In old days (eg. [1]), hypercovers $X \to *$ were represented by simplicial schemes.

Pro-categories

Every pro-object $X : I^{op} \to \mathbf{E}$ represents a functor

$$X_* : \mathbf{E} \to \mathbf{Set},$$

which is defined by

$$Z \mapsto \varinjlim_{i \in I} \hom(X(i), Z) = X_*(Z)$$

(filtered colimit — covariant in I), where the morphisms are in $\mathbf{E}$.

If $Y : J^{op} \to \mathbf{E}$ is another pro-object, the collection of all natural transformations

$$f : Y_* \to X_*.$$

is the set of the *pro-maps* $X \to Y$.

We now have objects and morphisms for the category pro-$\mathbf{E}$ of pro-objects in $\mathbf{E}$.

This is Grothendieck's original description of the pro-category.

1) An object Z of $\mathbf{E}$ is a pro-object $Z : * \to \mathbf{E}$.

A pro-map $X \to Z$ is a member of the set

$$\varinjlim_{i \in I} \hom(X(i), Z).$$

A pro-map $Z \to X$ is a member of the set

$$\varprojlim_{i \in I} \hom(Z, X(i)).$$

The proofs amount to tricks with the Yoneda Lemma.

2) All natural transformations $X \to Y$ of I-diagrams (I filtered) define pro-maps.

Cofinal functors

There is an intuitive sense of what cofinality means:

$2\mathbb{N}$ is a cofinal subcategory of $\mathbb{N}$ — every number is less than an even number.

There is a standard description of what it means for a functor $\phi : I \to J$ between filtered categories to be cofinal: the slice categories j/ϕ are filtered for all $j \in J$.

This is equivalent to the assertion that the slice categories j/ϕ are contractible for all $j \in J$. ($\Rightarrow$ Quillen's Theorem A)

If $\phi : I \to J$ is a functor between filtered categories and $Y : J^{op} \to \mathbf{E}$ is a pro-object, then restriction along ϕ defines a pro-map $Y \cdot \phi \to Y$.

Fact: The map $Y \cdot \phi \to Y$ is an **isomorphism** in the pro-category if ϕ is cofinal.

Pro-objects in simplicial sets

Theorem: Every finite diagram in the pro-category can be replaced up to isomorphism by a diagram of natural transformations [1].

Example: A pro-map is a monomorphism in the pro-category if it is isomorphic to a natural transformation which is a sectionwise monomorphism [3].

This is the starting point for a model structure (or homotopy theory) for pro-s**Set** of pro-objects of Edwards and Hastings [2].

The *cofibrations* for their theory (the EH model structure) are the monomorphisms of pro-s**Set**.

A pro-map $f : X \to Y$ is a *weak equivalence* if and only if it induces weak equivalences

$$\varinjlim_{j \in J} \mathbf{hom}(Y(j), Z) \to \varinjlim_{i \in I} \mathbf{hom}(X(i), Z)$$

for all Kan complexes (aka. fibrant objects) Z.

Pro-weak equivalences

The EH structure is not the model structure for pro-simplicial sets that has calculational interest.

For that, one localizes using the Postnikov tower functor, so that a pro-map $X \to Y$ is a *pro-weak equivalence* if and only if the induced map $P_* X \to P_* Y$ is a weak equivalence for the EH-structure.

The localization step is a formality.

It amounts to changing focus to pro-simplicial sets which have only finitely many non-trivial homotopy groups in each section.

Pro-objects in simplicial presheaves

The EH model structure and the localization via Postnikov towers have analogues for pro-objects in simplicial presheaves [3].

Cofibrations are monomorphisms, and weak equivalences are those maps which induce weak equivalences

$$\varinjlim_{j \in J} \mathbf{hom}(Y(j), Z) \to \varinjlim_{i \in I} \mathbf{hom}(X(i), Z)$$

for all injective fibrant objects Z.

We can again localize at the Postnikov tower construction to produce a pro-homotopy theory for simplicial presheaves (or sheaves).

This theory is the right setting for resolving the finite descent vs. Galois cohomological descent question that used to afflict algebraic K-theorists [5].

Generalized pro-objects

Let's talk about small diagrams $X : I \to s\mathbf{Set}$, where the index category I is *not necessarily filtered*.

Given another such thing $Y : J \to s\mathbf{Set}$, what's the right way to describe a "pro-morphism" $Y \to X$?

Suppose that Z is fixed. The colimit

$$\varinjlim_{i \in I} \hom(X(i), Z)$$

is the set of path components of the category $T_X(Z)$, having maps

$$\begin{tikzcd} X(i) \arrow[dr] \arrow[dd, "\alpha_*"'] & \\ & Z \\ X(i') \arrow[ur] & \end{tikzcd}$$

Any map $Z \to W$ induces a functor $T_X(Z) \to T_X(W)$, and we have a functor

$$T_X : s\mathbf{Set} \to \mathbf{cat}.$$

Morphisms

Suppose that $Y : J \to s\mathbf{Set}$ is another small diagram. What is a natural transformation $T_X \to T_Y$ of functors $s\mathbf{Set} \to \mathbf{cat}$?

Such a thing consists of a functor $\alpha : I \to J$ and a natural transformation $\theta : Y \cdot \alpha \to X$, with a commutative diagram

$$\begin{tikzcd} T_X \arrow[r, "\theta^*"] \arrow[d, "\pi"'] & T_{Y \cdot \alpha} \arrow[r, "i_\alpha"] \arrow[d, "\pi"] & T_Y \arrow[d, "\pi"] \\ I \arrow[r, "1"'] & I \arrow[r, "\alpha"'] & J \end{tikzcd}$$

such that the composite $i_\alpha \cdot \theta^*$ is the transformation $T_X \to T_Y$.

The category of small diagrams $X : I \to s\mathbf{Set}$ and (reversed) natural transformations (α, θ) of slice functors is a Grothendieck construction. This is the generalized pro-category.

Weak equivalences

Any pair (α, θ) (any "pro-map") induces a commutative diagram

$$\begin{tikzcd}
\underrightarrow{\text{holim}}_I \, \mathbf{hom}(X, Z) \arrow[r, "\theta^*"] \arrow[d, "\pi"'] & \underrightarrow{\text{holim}}_I \, \mathbf{hom}(Y \cdot \alpha, Z) \arrow[r, "i_\alpha"] \arrow[d, "\pi"] & \underrightarrow{\text{holim}}_J \, \mathbf{hom}(Y, Z) \arrow[d, "\pi"] \\
BI \arrow[r, "1"'] & BI \arrow[r, "\alpha"'] & BJ
\end{tikzcd}$$

for all Z.

I say that a morphism $(\alpha, \theta) : Y \to X$ is a generalized *EH-equivalence* if

1) the top composite is a weak equivalence for all fibrant Z, and
2) if the map $\alpha : BI \to BJ$ is a weak equivalence.

Remarks

1) This definition "specializes" to Edwards-Hastings weak equivalences of ordinary pro-objects, because all filtered categories are contractible and filtered colimits are homotopy colimits.

2) The definition is motivated, in part, by existing work on homotopy theories of dynamical systems, where one varies dynamical systems and parameter spaces simultaneously. Homotopy colimits correspond to spaces of dynamics [4].

Questions:

What's a cofibration?

What's a fibration?

Is there even a model structure?

How I spent my summer

Here's a relatively straightforward statement:

Proposition: Suppose that I is a small *filtered* category. Then there is a model structure on the category of I-diagrams for which the cofibrations are the sectionwise monomorphisms, and the weak equivalences are the EH-equivalences.

The following is more interesting:

Theorem: Suppose that I is a small category. Then there is a model structure on the category of I-diagrams for which the cofibrations are the sectionwise monomorphisms, and the weak equivalences are the generalized EH-equivalences.

A generalized EH-equivalence is a natural transformation $X \to Y$ of I-diagrams which induces a weak equivalence

$$\underrightarrow{\text{holim}}_{i \in I} \ \mathbf{hom}(Y(i), Z) \to \underrightarrow{\text{holim}}_{i \in I} \ \mathbf{hom}(X(i), Z)$$

for all fibrant Z.

References

[1] M. Artin and B. Mazur.
Etale homotopy.
Lecture Notes in Mathematics, No. 100. Springer-Verlag, Berlin, 1969.

[2] David A. Edwards and Harold M. Hastings.
Čech and Steenrod homotopy theories with applications to geometric topology.
Lecture Notes in Mathematics, Vol. 542. Springer-Verlag, Berlin, 1976.

[3] J. F. Jardine.
Model structures for pro-simplicial presheaves.
J. K-Theory, 7(3):499–525, 2011.

[4] J. F. Jardine.
Homotopy theories of diagrams.
Theory Appl. Categ., 28:No. 11, 269–303, 2013.

[5] J.F. Jardine.
Galois descent criteria.
Preprint, `http://www.math.uwo.ca/~jardine`, 2013.